Non-equilibrium statistics for relativistic heavy-ion collisions

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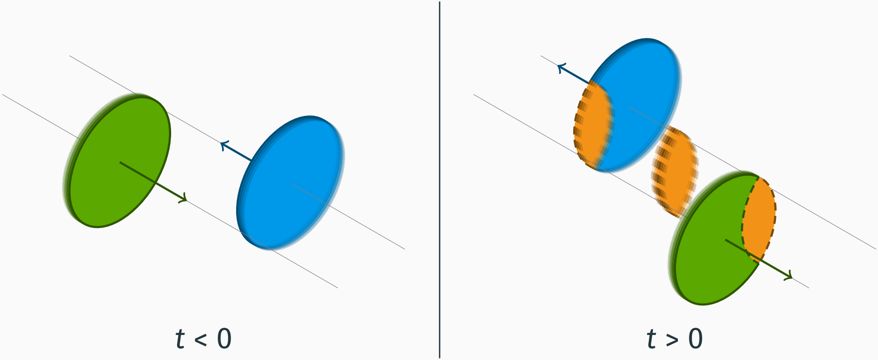
Motivation



Idea: model dynamics of produced particles via diffusion

- heat bath = light partonic d.o.f.
- Brownian particles = charged hadrons, net-protons, ...

Outline

Reminder: Basic stochastics

Random variables

Definition (for our purposes)

A random variable Y is a placeholder which *randomly* takes a value (“realization”) from a set Ω.

discrete

$$p_y \coloneqq \mathbb{P}\{Y = y\}$$

$$\sum_{y \in \Omega} p_y = 1$$

continuous

$$f(y)\,\mathrm{d}y \coloneqq \mathbb{P}\{Y \in [y, y + \mathrm{d}y]\}$$

$$\int_\Omega \mathrm{d}y\, f(y) = 1$$

Stochastic processes and Markov chains

Stochastic process = sequence of random variables:

$$\{Y_t\}_{t\in T} = \left\{Y_{t_0} \to Y_{t_1} \to Y_{t_2} \to \cdots \to Y_{t_N}\right\}$$

Q: Having taken n steps, how likely will we get value y_{n+1} next?

A: $p_{y_{n+1}} = \mathbb{P}\left\{Y_{t_{n+1}} = y_{n+1} \,\middle|\, Y_{t_n} = y_n,\ Y_{t_{n-1}} = y_{n-1},\ \ldots,\ Y_{t_0} = y_0\right\}$

⇨ depends on history of sequence (“path taken”)

Important special case: Markov chains

$p_{y_{n+1}} = \mathbb{P}\left\{Y_{t_{n+1}} = y_{n+1} \,\middle|\, Y_{t_n} = y_n\right\}$ ⇨ no memory of past events

Random walk and diffusion

Example: Simple symmetric random walk in 1D

$$X_{t_{n+1}} = X_{t_n} \pm \Delta x$$

with equal probability $p = ½$ for going left/right

Physically interesting limiting case:

$$t_n = \frac{n}{N}\,, \quad \Delta x = \frac{1}{\sqrt{N}} \quad \overset{N\to\infty}{\Rightarrow}$$

Wiener process $W(t)$
(Brownian motion)

Properties of $\mathrm{d}W(t) := W(t + \mathrm{d}t) - W(t)$:

$$\mathbb{P}\{\mathrm{d}W(t) \in [x, x + \mathrm{d}x]\} = (2\pi\,\mathrm{d}t)^{-½} \exp\left[-\frac{x^2}{2\,\mathrm{d}t}\right] \mathrm{d}x$$

incompatible with SRT!

$$\langle \mathrm{d}W(t) \rangle = 0\,, \qquad \langle \mathrm{d}W(t)\,\mathrm{d}W(s) \rangle = \begin{cases} \mathrm{d}t & \text{for } t = s \\ 0 & \text{otherwise} \end{cases}$$

Relativistic Markov processes

Theorem (Dudley, Hakim, Łopuszański)

Nontrivial Lorentz-invariant Markov processes cannot exist in Minkowski spacetime (t, x)!

Alternatives:

- stochastic processes with memory

$$p_{y_{n+1}} = \mathbb{P}\left\{Y_{t_{n+1}} = y \,\middle|\, Y_{t_n} = y_n, \dots, Y_{t_{n-k+1}} = y_{n-k+1}\right\}$$

$$\updownarrow$$

$$\langle dY(t)\, dY(s)\rangle \neq \begin{cases} dt & \text{for } t = s \\ 0 & \text{otherwise} \end{cases}$$

Not now, maybe later ...

- Markov processes in phase space (x, p)

Langevin dynamics and stochastic differential equations

Idea:

- phenomenological e.o.m. for Brownian particle
- heat bath d.o.f. ⇨ stochastic force

Ansatz:

$$\mathrm{d}X = \frac{P}{(M^2 + P^2)^{1/2}}\,\mathrm{d}t = \frac{P}{E(P)}\,\mathrm{d}t$$

$$\mathrm{d}P = -\alpha(P)\,P\,\mathrm{d}t + [2\Delta(P)]^{1/2} \circledcirc \mathrm{d}W$$

heat bath rest frame

$$x^0 \equiv t\,, \quad p^0 \equiv E$$

$$x^1 \equiv x\,, \quad p^1 \equiv p$$

$$c \equiv 1$$

presence of heat bath ⇨ friction and noise

[Debbasch et al., Dunkel and Hänggi]

Langevin dynamics

Integrate e.o.m. to obtain trajectory of Brownian particle:

$$X(t_2) - X(t_1) = \int_{X(t_1)}^{X(t_2)} \mathrm{d}X(t) = \int_{t_1}^{t_2} \frac{P(t)}{E[P(t)]}\,\mathrm{d}t$$

$$P(t_2) - P(t_1) = \underbrace{-\int_{t_1}^{t_2} \alpha[P(t)]\,P(t)\,\mathrm{d}t}_{\text{Riemann–Stieltjes integral}} + \underbrace{\int_{W(t_1)}^{W(t_2)} \{2\Delta[P(t)]\}^{1/2} \circledcirc \mathrm{d}W(t)}_{\text{stochastic integral}}$$

Stochastic integrals

Consider general stochastic integral:

$$I := \int_{W(t_1)}^{W(t_2)} F[Y(t)] \circledcirc \mathrm{d}W(t)$$

Naïve ansatz: $$I \neq \int_{t_1}^{t_2} F[Y(t)] \underbrace{\frac{\partial W(t)}{\partial t}}_{\text{does not exist!}} \mathrm{d}t$$

Define via sum:

$$I_\circledcirc =: \lim_{N\to\infty} \sum_{k=0}^{N-1} \left\{ \frac{1+\lambda_\circledcirc}{2} F[Y(t_{k+1})] + \frac{1-\lambda_\circledcirc}{2} F[Y(t_k)] \right\} [W(t_{k+1}) - W(t_k)]$$

$$\lambda_\circledcirc = \begin{cases} -1 =: \lambda_{◐} & \text{Itô integral} \quad \text{(pre-point)} \\ 0 =: \lambda_{\bullet} & \text{Stratonovich integral} \quad \text{(mid-point)} \\ +1 =: \lambda_{◑} & \text{Klimontovich integral} \quad \text{(post-point)} \end{cases}$$

Stochastic integrals

Consequences:

- Rules of differential calculus do not necessarily apply!
- Especially: $I_{◐} \neq I_{●} \neq I_{◑}$ in general
- $X(t), P(t)$ depend on choice of $⊚ \in \{◐, ●, ◑\}$

Fokker–Planck equation for phase-space PDF

Phase-space PDF for Brownian particle:

$$f(t; x, p)\, \mathrm{d}x\, \mathrm{d}p \coloneqq \mathbb{P}\big\{(X(t), P(t)) \in [x, x + \mathrm{d}x] \times [p, p + \mathrm{d}p]\big\}$$

Transform e.o.m. into evolution equation for PDF:

$$\Big(\partial_t + \tfrac{p}{E}\partial_x\Big) f_{\LEFTcircle} = \partial_p\Big[\alpha p f_{\LEFTcircle} + \partial_p\big(\Delta f_{\LEFTcircle}\big)\Big] \qquad \text{(pre-point)}$$

$$\Big(\partial_t + \tfrac{p}{E}\partial_x\Big) f_{\bullet} = \partial_p\Big[\alpha p f_{\bullet} + \Delta^{1/2}\, \partial_p\big(\Delta^{1/2} f_{\bullet}\big)\Big] \qquad \text{(mid-point)}$$

$$\Big(\partial_t + \tfrac{p}{E}\partial_x\Big) f_{\RIGHTcircle} = \partial_p\Big[\alpha p f_{\RIGHTcircle} + \Delta\, \partial_p f_{\RIGHTcircle}\Big] \qquad \text{(post-point)}$$

⇨ differ by multiples of $\pm\frac{1}{2}[\partial_p \Delta(p)] f_{\circledcirc}$

⇨ $f_{\LEFTcircle} \neq f_{\bullet} \neq f_{\RIGHTcircle}$ describe different physical systems!

Langevin dynamics – revised ansatz

Idea: friction coefficient with counter-terms

$$\alpha_{\circledcirc}(p) \coloneqq \alpha_{\bullet}(p) + \lambda_{\circledcirc} \frac{1}{2p}\, \partial_p \Delta(p)$$

$\lambda_{\text{◐}}$	$= -1$	(pre)
$\lambda_{\bullet}$	$= 0$	(mid)
$\lambda_{\text{◑}}$	$= +1$	(post)

Revised Langevin equations:

$$\mathrm{d}X = \frac{P}{E(P)}\, \mathrm{d}t\,, \qquad \mathrm{d}P = -\,\alpha_{\circledcirc}(P)\, P\, \mathrm{d}t + [2\Delta(P)]^{1/2} \circledcirc \mathrm{d}W$$

Result: $f_{\text{◐}} = f_{\bullet} = f_{\text{◑}} \eqqcolon f$ with

$$\begin{aligned} \left(\partial_t + \tfrac{p}{E}\partial_x\right) f &= \partial_p\left[\alpha_{\text{◐}}\, p f + \partial_p(\Delta f)\right] \\ &= \partial_p\left[\alpha_{\bullet}\, p f + \Delta^{1/2}\, \partial_p\left(\Delta^{1/2} f\right)\right] \\ &= \partial_p\left[\alpha_{\text{◑}}\, p f + \Delta\, \partial_p f\right] \end{aligned} \quad \Bigg\} \quad \text{equivalent formulations of same physics}$$

[J. Dunkel and P. Hänggi, Phys. Rep. **471**, 1 (2009)]

Role of discretization rule

Discretization ⊚ has to be set, but can be chosen freely.

Advantages:

- ◐ numerical simulations
- ● ordinary differential calculus
- ◑ simple form of fluctuation–dissipation relations

Marginal momentum probability distribution function

In the following: consider only momentum dependence

$$\phi(t;p) \coloneqq \int_{\mathbb{R}} \mathrm{d}x\, f(t;x,p)\,, \qquad \int_{\mathbb{R}} \mathrm{d}p\, \phi(t;p) = 1$$

$$\Big(\partial_t + \underbrace{\tfrac{p}{E}\partial_x}_{\to 0}\Big) f = \underbrace{\partial_p\Big[\alpha_{\circ} p f + \Delta\, \partial_p f\Big]}_{\alpha \equiv \alpha(p),\ \Delta \equiv \Delta(p)} \xrightarrow{\int \mathrm{d}x} \partial_t \phi = \partial_p\Big[\alpha_{\circ} p \phi + \Delta\, \partial_p \phi\Big]$$

Fluctuation–dissipation relations

Kinematic background

Detailed balance

At equilibrium, each microscopic process is equilibrated by its reverse process.

$\Downarrow$

Fluctuation–dissipation theorem

If there is a dissipative process, then a reverse process related to thermal fluctuations exists.

Drag		Brownian motion
motion $\to$ heat	$\leftrightarrow$	heat $\to$ motion
friction α		noise Δ

Asymptotic momentum distribution

$t \to \infty$: Brownian particle in equilibrium state

$$\phi_\infty(p) \coloneqq \lim_{t\to\infty} \phi(t;p) \quad \text{with} \quad 0 = \partial_t \phi_\infty = \partial_p \Big[\alpha_{\text{◑}}\, p \phi_\infty + \Delta\, \partial_p \phi_\infty \Big]$$

Solution:

$$\phi_\infty(p) \propto \exp\left[- \int_*^p \mathrm{d}\tilde{p}\, \frac{\alpha_{\text{◑}}(\tilde{p})}{\Delta(\tilde{p})} \tilde{p} \right]$$

$$\Downarrow$$

Fluctuation–dissipation relation for diffusion

$$\frac{\alpha_{\text{◑}}(p)}{\Delta(p)} = -\frac{1}{p} \partial_p \ln\big[\phi_\infty(p)\big]$$

[Klimontovich]

Example: Ornstein–Uhlenbeck-type processes

NR-limit
$E \to M$

- Stationary isotropic homogeneous heat bath:
  Jüttner distribution

$$\phi_{\mathrm{J}}(p) \propto \exp[-\beta E(p)] \quad \overset{\phi_\infty \overset{!}{=} \phi_{\mathrm{J}}}{\Rightarrow} \quad \frac{\alpha_{\odot}(p)}{\Delta(p)} = \frac{\beta}{E(p)}$$

- Asymptotic spatial diffusion constant

$$D_\infty := \lim_{t\to\infty} \frac{1}{2t}\left\langle [X(t) - X(0)]^2 \right\rangle$$

Constant noise: $\Delta(p) \equiv (\beta b)^{-1}$

$$\alpha_{\odot}(p) \overset{\mathrm{FDR}}{=} \frac{1}{bE(p)}$$

$$D_\infty = \cdots = \frac{b}{\beta}$$

Constant friction: $\alpha_{\odot}(p) \equiv (bM)^{-1}$

$$\Delta(p) \overset{\mathrm{FDR}}{=} \frac{E(p)}{\beta b M}$$

$$D_\infty = \cdots = \frac{b}{\beta} \underbrace{\frac{K_0(\beta M)}{K_1(\beta M)}}_{\leq 1}$$

Mean relaxation behavior

Criteria for physical stochastic processes

A physically meaningful stochastic process should ...

- ... approach the correct equilibrium state

  FDR ✓ $\rightarrow$ fixes 1$^{\text{st}}$ coefficient

- ... reproduce the correct mean relaxation behavior

$$\left\langle \frac{\mathrm{d}P}{\mathrm{d}t} \right\rangle \overset{!}{=} \left\langle \frac{\delta P}{\delta t} \right\rangle_{\mathrm{b}} \rightarrow \text{fixes } 2^{\text{nd}} \text{ coefficient}$$

[J. Dunkel and P. Hänggi, Phys. Rep. **471**, 1 (2009)]

Mean relaxation behavior

More mathematically precise:

$$\left\langle \frac{\mathrm{d}P(t)}{\mathrm{d}t} \,\middle|\, P(t) = p \right\rangle \overset{!}{=} \left\langle \frac{\delta P(t)}{\delta t} \,\middle|\, P(t) = p \right\rangle_{\mathrm{b}}$$

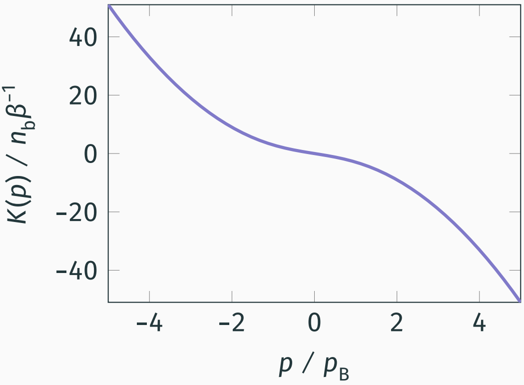
LHS: stochastic differential calculus

$$\left\langle \frac{\mathrm{d}P(t)}{\mathrm{d}t} \,\middle|\, P(t) = p \right\rangle = -\alpha_{\circ}(p)\, p + \partial_p \Delta(p)$$

RHS: microscopic models

$$\left\langle \frac{\delta P(t)}{\delta t} \,\middle|\, P(t) = p \right\rangle_{\mathrm{b}} =: K(p) \quad \text{(mean drift force)}$$

Example: Non-relativistic gas of hard spheres



bath number density

$$n_{\mathrm{b}} = N_{\mathrm{b}}/L$$

bath velocity PDF

$$\psi_{\mathrm{b}}(u) \propto \exp\left(-\tfrac{1}{2}\beta m u^2\right)$$

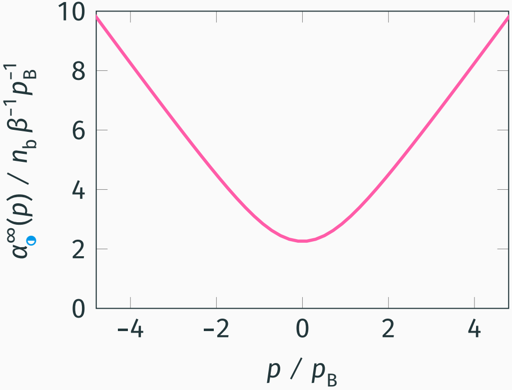
momentum scale

$$p_{\mathrm{B}} = M\sqrt{2\langle u^2\rangle_{\mathrm{b}}}$$

$$K \approx -2 n_{\mathrm{b}} \beta^{-1}\left[\pi^{-1/2}\xi \exp\left(-\xi^2\right) + \left(\xi^2 + \tfrac{1}{2}\right)\mathrm{erf}(\xi)\right] \quad \text{with} \quad \xi = \frac{p}{p_{\mathrm{B}}}$$

[J. Dunkel and P. Hänggi, Phys. Rep. **471**, 1 (2009), Fig. 1]

Example: Non-relativistic gas of hard spheres



mean relaxation

$$K(p) \overset{!}{=} -\alpha_{◑}(p)\, p + \partial_p \Delta(p)$$

fluctuation–dissipation

$$\Delta(p) = \beta^{-1} M\, \alpha_{◑}(p)$$

$$\alpha_{◑}(p) = -\frac{K(p)}{p} - \frac{M}{\beta p}\partial_p \alpha_{◑}(p) \approx -\frac{K(p)}{p} =: \alpha^{\infty}_{◑}(p) \quad \text{for} \begin{cases} \text{cold system} \\ \text{fast particle} \end{cases}$$

[J. Dunkel and P. Hänggi, Phys. Rep. **471**, 1 (2009), Fig. 2a]

Outlook

Status quo

Take-home message

- systematic bottom-up construction of diffusion-FPE
- asymptotic momentum state ⇒ FDR
- microscopic forces ⇒ mean relaxation

} friction & noise

Open issues

- form of asymptotic momentum state?
- choice of coordinates/frame?
- microscopic model for relaxation behavior?
- inclusion of transverse d.o.f.?

Thank you!

Further reading

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Backup

Langevin dynamics in rapidity space

- Rapidity

$$y = \operatorname{arsinh}(p/M) =: G(p) \quad \Rightarrow \quad Y(t) := G[P(t)]$$

- Marginal rapidity PDF

$$\psi(t;y)\,\mathrm{d}y := \mathbb{P}\{Y(t) \in [y, y+\mathrm{d}y]\}, \quad \int \mathrm{d}y\, \psi(t;y) = 1$$

$$\psi(t;y) = \phi(t;p(y))\,\underbrace{\left|G^{-1}(y)\right|}_{M\cosh(y)}$$

- Transformation of e.o.m.

$$\mathrm{d}Y = \underbrace{G'(P)}_{1/E(P)} \circledcirc \mathrm{d}P - \lambda_{\circledcirc}\,\Delta(P)\,\underbrace{G''(P)}_{-P/E(P)^3}\,\mathrm{d}t$$

$$= -\left[\frac{\alpha_{\circledcirc}(P)}{E(P)} - \lambda_{\circledcirc}\frac{\Delta(P)}{E(P)^3}\right] P\,\mathrm{d}t + \frac{[2\Delta(P)]^{1/2}}{E(P)} \circledcirc \mathrm{d}W$$