Formelblatt

Vektor-Identitäten

$$
\begin{aligned}
\vec{A}\cdot(\vec{B}\times\vec{C}) &= \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}) \\
\vec{A}\times(\vec{B}\times\vec{C}) &= (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C} \\
(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) &= (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}) \\
\nabla\times\nabla\phi &= 0 \\
\nabla\cdot(\nabla\times\vec{A}) &= 0 \\
\nabla\times(\nabla\times\vec{A}) &= \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} \\
\nabla\cdot(\psi\vec{A}) &= \vec{A}\cdot\nabla\psi + \psi\nabla\cdot\vec{A} \\
\nabla\times(\psi\vec{A}) &= \nabla\psi\times\vec{A} + \psi\nabla\times\vec{A} \\
\nabla(\vec{A}\cdot\vec{B}) &= (\vec{A}\cdot\nabla)\vec{B} + (\vec{B}\cdot\nabla)\vec{A} + \vec{A}\times(\nabla\times\vec{B}) + \vec{B}\times(\nabla\times\vec{A}) \\
\nabla\cdot(\vec{A}\times\vec{B}) &= \vec{B}\cdot(\nabla\times A) - \vec{A}\cdot(\nabla\times B) \\
\nabla\times(\vec{A}\times\vec{B}) &= \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) + (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B}
\end{aligned}
$$

Differential-Operatoren

Kartesische Koordinaten (x_1, x_2, x_3):

$$
\begin{aligned}
\nabla\psi &= \vec{e}_1\frac{\partial\psi}{\partial x_1} + \vec{e}_2\frac{\partial\psi}{\partial x_2} + \vec{e}_3\frac{\partial\psi}{\partial x_3} \\
\nabla\cdot\vec{A} &= \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} \\
\nabla\times\vec{A} &= \vec{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \vec{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \vec{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right) \\
\nabla^2\psi &= \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}
\end{aligned}
$$

Zylinderkoordinaten (r, ϕ, z):

$$
\begin{aligned}
\nabla\psi &= \vec{e}_r\frac{\partial\psi}{\partial r} + \vec{e}_\phi\frac{1}{r}\frac{\partial\psi}{\partial\phi} + \vec{e}_z\frac{\partial\psi}{\partial z} \\
\nabla\cdot\vec{A} &= \frac{1}{r}\frac{\partial(r\,A_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \\
\nabla\times\vec{A} &= \vec{e}_r\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \vec{e}_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \vec{e}_z\frac{1}{r}\left(\frac{\partial\,(rA_\phi)}{\partial r} - \frac{\partial A_r}{\partial\phi}\right) \\
\nabla^2\psi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}
\end{aligned}
$$

Kugelkoordinaten (r, θ, ϕ)**:**

$$\begin{aligned}
\nabla\psi &= \vec{e}_r \frac{\partial\psi}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \vec{e}_\phi \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \\
\nabla\cdot\vec{A} &= \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \\
\nabla\times\vec{A} &= \vec{e}_r \frac{1}{r\sin\theta}\left[\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right] + \vec{e}_\theta\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial(r A_\phi)}{\partial r}\right] + \vec{e}_\phi \frac{1}{r}\left[\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right] \\
\nabla^2\psi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \\
&\equiv \frac{1}{r}\frac{\partial^2(r\psi)}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}
\end{aligned}$$

Kugelflächenfunktionen

Das Koordinatensystem:

$$\begin{aligned}
x &= r\,\sin\theta\,\cos\phi \\
y &= r\,\sin\theta\,\sin\phi \\
z &= r\,\cos\theta
\end{aligned}$$

Die $Y_{l,m}$ für $l = 0, 1, 2$ sind:

$$\begin{aligned}
Y_{0,0} &= \frac{1}{\sqrt{4\pi}} \\
Y_{1,0} &= \sqrt{\frac{3}{4\pi}}\cos\theta \\
Y_{1,1} &= -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} \\
Y_{1,-1} &= \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi} \\
Y_{2,0} &= \sqrt{\frac{5}{4\pi}}\frac{1}{2}(3\cos^2\theta - 1) \\
Y_{2,1} &= -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi} \\
Y_{2,-1} &= \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{-i\phi} \\
Y_{2,2} &= \sqrt{\frac{15}{2\pi}}\frac{1}{4}\sin^2\theta e^{2i\phi} \\
Y_{2,-2} &= \sqrt{\frac{15}{2\pi}}\frac{1}{4}\sin^2\theta e^{-2i\phi}
\end{aligned}$$

Die assozierten Legendre-Polynome für $l = 0, 1, 2$ sind:

$$\begin{aligned}
P_0^0(\cos\theta) &= 1 \\
P_1^0(\cos\theta) &= \cos\theta \\
P_1^1(\cos\theta) &= -\sqrt{1-\cos^2\theta} \\
P_1^{-1}(\cos\theta) &= \frac{1}{2}\sqrt{1-\cos^2\theta} \\
P_2^0(\cos\theta) &= \frac{1}{2}(3\cos^2\theta - 1) \\
P_2^1(\cos\theta) &= -3\sqrt{1-\cos^2\theta}\cos\theta \\
P_2^{-1}(\cos\theta) &= \frac{1}{2}\sqrt{1-\cos^2\theta}\cos\theta \\
P_2^2(\cos\theta) &= 3(1-\cos^2\theta) \\
P_2^{-2}(\cos\theta) &= \frac{1}{8}(1-\cos^2\theta)
\end{aligned}$$