# Analytic model for the equilibration in finite Bose systems

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# Content

# Derivation of the differential equation

- Hamiltonian consisting of free and interacting part

$$\hat{H} = \hat{H}_0 + \hat{H}_i$$

- von Neumann equation for weakly interacting bosons/fermions

$$\frac{\partial \langle \hat{n}_i \rangle}{\partial t} = -\left\langle \sum_{\gamma,\gamma'} |\gamma\rangle \langle\gamma| \, \Omega_{\gamma,\gamma'} \left( \langle\gamma|\hat{n}_i|\gamma\rangle - \langle\gamma'|\hat{n}_i|\gamma'\rangle \right) \right\rangle$$

- Transition amplitudes $\Omega_{\gamma,\gamma'}$

$$\Omega_{\gamma,\gamma'} = \left| \langle\gamma|\hat{H}_i|\gamma'\rangle \right|^2 G(\epsilon_\gamma, \epsilon_{\gamma'}) = \Omega_{\gamma',\gamma}$$

# Weak $2 \rightarrow 2$ interaction

- Non-vanishing matrix elements for $|\gamma\rangle \rightarrow |\gamma'\rangle$

$$n'_i = \begin{cases} n_i & \text{for} \quad i \neq i_1, i_2, i_3, i_4 \\ n_i - 1 & \text{for} \quad i = i_1, i_2 \\ n_i + 1 & \text{for} \quad i = i_3, i_4 \end{cases}$$

- Quantum Boltzmann equation for weakly interacting bosonic systems

$$\frac{\partial n_1}{\partial t} = \sum_{\epsilon_2, \epsilon_3, \epsilon_4}^{\infty} \left\langle V^2_{12,34} \right\rangle G(\epsilon_1 + \epsilon_2, \epsilon_3 + \epsilon_4) \Big[ (1 + n_1)(1 + n_2) n_3 n_4 - (1 + n_3)(1 + n_4) n_1 n_2 \Big]$$

# Nonlinear diffusion equation I

- Master equation

$$\frac{\partial n_1}{\partial t} = (1+n_1)\sum_{\epsilon_4} W_{4\to 1} n_4 - n_1 \sum_{\epsilon_4} W_{1\to 4}(1+n_4)$$

- $\sum \to \int g_4 \,\mathrm{d}\epsilon_4$ and Taylor expansion around $\epsilon_1$

$$\frac{\partial n}{\partial t} = -\frac{\partial}{\partial \epsilon}\left[vn\,(1+n) - n^2 \frac{\partial D}{\partial \epsilon}\right] + \frac{\partial^2}{\partial \epsilon^2}\,[Dn]$$

- Transport coefficients

$$D := \frac{1}{2} g_1 \int_0^\infty W(\epsilon_1, \epsilon_4)(\epsilon_1 - \epsilon_4)^2 \,\mathrm{d}\epsilon_4$$

$$v := \frac{1}{g_1}\frac{\partial}{\partial \epsilon_1}(g_1 D)$$

# Nonlinear diffusion equation II

- Assumption of constant transport coefficients

$$\frac{\partial n}{\partial t} = -v \frac{\partial}{\partial \epsilon} \left[ n (1+n) \right] + D \frac{\partial^2 n}{\partial \epsilon^2}$$

- Bose-Einstein distribution with temperature $T = -\frac{D}{v}$ as static solution

$$n_{eq} = \frac{1}{\exp\left(\frac{\epsilon - \mu}{T}\right) - 1}$$

# Analytic solution I

- Transformation to linear diffusion equation with partition function

$$n(\epsilon, t) = -\frac{D}{v}\frac{\partial}{\partial \epsilon}\ln(\mathcal{Z}) - \frac{1}{2} \quad \Rightarrow \quad \frac{\partial \mathcal{Z}}{\partial t} = D\frac{\partial^2 \mathcal{Z}}{\partial \epsilon^2}$$

- Partition function without boundary condition

$$\mathcal{Z}(\epsilon, t) = \int_{-\infty}^{+\infty} F(\epsilon, x, t) G(x)\,\mathrm{d}x$$

- Partition function with boundary condition $\lim_{\epsilon \downarrow \mu} n(\epsilon, t) = \infty$

$$\mathcal{Z}(\epsilon, t) = \int_{0}^{\infty} \underbrace{\left(F(\epsilon - \mu, x, t) - F(\epsilon - \mu, -x, t)\right)}_{\tilde{F}(\epsilon, x, t)} G(x + \mu)\,\mathrm{d}x$$

# Analytic solution II

- Green's function without boundary condition

$$F(\epsilon, x, t) = \exp\left(-\frac{(\epsilon - x)^2}{4Dt}\right)$$

- initial distribution (transformed)

$$G(x) = \exp\left(-\frac{1}{2D}\left(vx + 2v\int_0^x n_i(y)\,\mathrm{d}y\right)\right)$$

- equivalent expression to avoid infinities

$$G(x) = \exp\left(-\frac{1}{2D}\left(vx + 2vA_i(x)\right)\right)$$

# Initial condition

- describe stochastic cooling
- initial condition:

$$n_i(\epsilon) = \frac{1}{\exp\left(\frac{\epsilon-\mu}{T}\right) - 1}\Theta(\epsilon_i - \epsilon)$$

- $T = -\frac{D}{v} \quad \rightarrow \quad$ no description of cooling
- G - function

$$G(x) = \begin{cases} z^{-1}\exp\left(-\frac{vx}{2D}\right) - \exp\left(\frac{vx}{2D}\right) & \text{for} \quad x < \epsilon_i \\ z^{-1}\exp\left(-\frac{vx}{2D}\right) - \exp\left(-\frac{vx}{2D} + \frac{v\epsilon_i}{D}\right) & \text{for} \quad x > \epsilon_i \end{cases}$$

- fugacity: $z = \exp\left(\frac{\mu}{T}\right)$

# Partition function

- Partition function for non-fixed chemical potential

$$\mathcal{Z} = \sqrt{\pi D t}\exp\left(\frac{v^2 t}{4D}\right)\left[2z^{-1}\exp\left(-\frac{\epsilon v}{2D}\right) - \exp\left(\frac{\epsilon v}{2D}\right)\operatorname{erfc}\left(\frac{\epsilon - \epsilon_i + tv}{\sqrt{4Dt}}\right) - \exp\left(\frac{v\epsilon_i}{D} - \frac{\epsilon v}{2D}\right)\operatorname{erfc}\left(\frac{\epsilon_i - \epsilon + tv}{\sqrt{4Dt}}\right)\right]$$

- Complementary error function:

$$\operatorname{erfc}(x) := 1 - \operatorname{erf}(x) := \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,\mathrm{d}t$$

# Particle distribution for non-fixed chemical potential

- Derivative of the error functions:

$$\exp\left(\frac{\epsilon v}{2D}\right)\frac{\partial}{\partial\epsilon}\mathrm{erfc}\left(\frac{\epsilon-\epsilon_i+tv}{\sqrt{4Dt}}\right) + \exp\left(\frac{v\epsilon_i}{D}-\frac{\epsilon v}{2D}\right)\frac{\partial}{\partial\epsilon}\mathrm{erfc}\left(\frac{\epsilon_i-\epsilon+tv}{\sqrt{4Dt}}\right) = 0$$

- Particle distribution:

$$n(\epsilon,t) = \frac{1}{\exp\left(\frac{\epsilon-\mu}{T}\right)K(\epsilon,t)-1}$$

- Auxiliary function $K(\epsilon, t)$:

$$K(\epsilon,t) = \frac{2-\exp\left(\frac{\mu-\epsilon_i}{T}\right)\mathrm{erfc}\left(\frac{\epsilon_i-\epsilon+tv}{\sqrt{4Dt}}\right)}{\mathrm{erfc}\left(\frac{\epsilon-\epsilon_i+tv}{\sqrt{4Dt}}\right)}$$

# Non-fixed $\mu$ - Solution for small times

parameters: $D = 8000\,\mathrm{peV^2 s^{-1}}$, $v = -1000\,\mathrm{peVs^{-1}}$, $T = 8\,\mathrm{peV}$, $\epsilon_i = 7\,\mathrm{peV}$, $\mu = -0.68\,\mathrm{peV}$

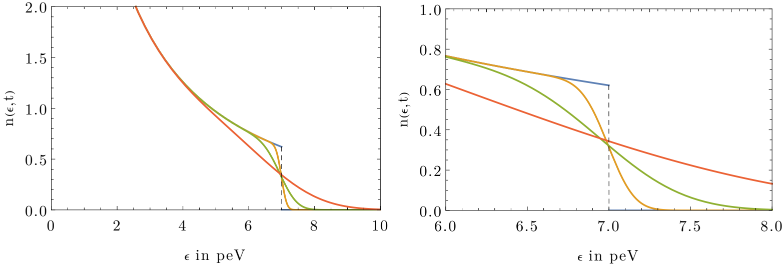


Figure: Solution for non-fixed $\mu$ at times 0 μs (blue), 1 μs (yellow), 10 μs (green) and 100 μs (red).

# Non-fixed $\mu$ - Large times

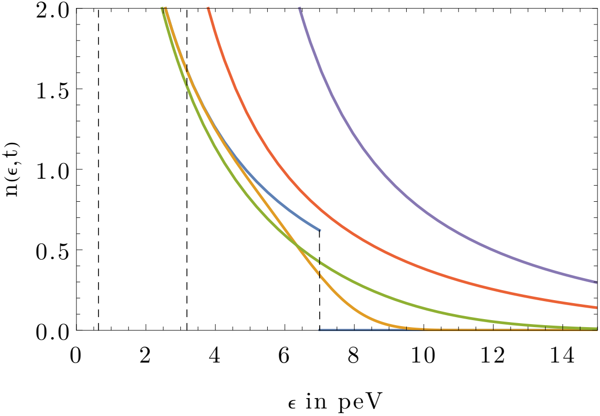


Figure: Solution for non-fixed $\mu$ at times 0 µs (blue), 100 µs (yellow), 1 ms (green), 10 ms (red) and at $t \to \infty$ (purple).

# Partition function

- Partition function for fixed chemical potential

$$\mathcal{Z}(\epsilon, t) = \sqrt{4Dt}\exp\left(\frac{v^2 t}{4D} - \frac{\mu}{2T}\right)\left(\exp\left(\frac{\epsilon-\mu}{2T}\right)\Sigma_1(\epsilon, t) - \exp\left(\frac{\mu-\epsilon}{2T}\right)\Sigma_2(\epsilon, t)\right)$$

- Auxiliary functions

$$\Sigma_1(\epsilon, t) = \mathrm{erfc}\left(\frac{2\mu - \epsilon_i - \epsilon + tv}{\sqrt{4Dt}}\right) - \exp\left(\frac{\mu - \epsilon_i}{T}\right)\mathrm{erfc}\left(\frac{\epsilon_i - \epsilon + tv}{\sqrt{4Dt}}\right)$$

$$\Sigma_2(\epsilon, t) = \mathrm{erfc}\left(\frac{\epsilon - \epsilon_i + tv}{\sqrt{4\,D\,t}}\right) - \exp\left(\frac{\mu - \epsilon_i}{T}\right)\mathrm{erfc}\left(\frac{\epsilon - 2\mu + \epsilon_i + tv}{\sqrt{4Dt}}\right)$$

# Fixed $\mu$ - Solution

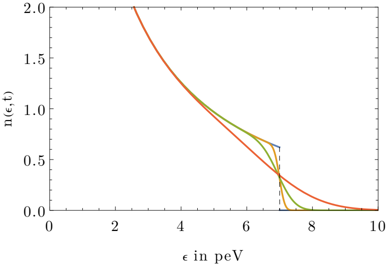


Figure: Solution for fixed $\mu$ at times 0 μs (blue), 1 μs (yellow), 10 μs (green) and 100 μs (red).

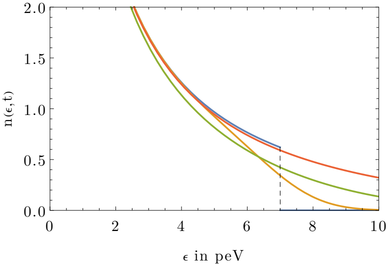


Figure: Solution for fixed $\mu$ at times 0 μs (blue), 100 μs (yellow), 1 ms (green) and 10 ms (red).

# Partition function I - $G(x+\mu)$

- G - function for arbitrary initial temperature $T_i$

$$G(x+\mu) = \begin{cases} \exp\left(\frac{x-\mu}{2T_f}\right)\left(1-\exp\left(-\frac{x}{T_i}\right)\right)^{\frac{T_i}{T_f}} & \text{for} \quad x < \epsilon_i - \mu \\ \exp\left(\frac{x-\mu}{2T_f}\right)\left(1-\exp\left(\frac{\mu-\epsilon_i}{T_i}\right)\right)^{\frac{T_i}{T_f}} & \text{for} \quad x > \epsilon_i - \mu \end{cases}$$

- Separation into two parts:

$$\mathcal{Z}(\epsilon,t) = \underbrace{\int_0^{\epsilon_i-\mu} \tilde{F}(\epsilon,x,t)\, G(x+\mu)\,\mathrm{d}x}_{\mathcal{Z}_1(\epsilon,t)} + \underbrace{\int_{\epsilon_i-\mu}^{\infty} \tilde{F}(\epsilon,x,t)\, G(x+\mu)\,\mathrm{d}x}_{\mathcal{Z}_2(\epsilon,t)}$$

# Partition function II - $\mathcal{Z}_1(\epsilon, t)$

- Result for $\mathcal{Z}_1(\epsilon, t)$

$$\mathcal{Z}_1(\epsilon, t) = \sum_{k=0}^{\infty} \binom{\frac{T_i}{T_f}}{k} (-1)^k \exp\left(-\frac{\mu}{2T_f}\right) \underbrace{\int_0^{\epsilon_i - \mu} \tilde{F}(\epsilon, x, t)\, \mathrm{e}^{\alpha x}\, \mathrm{d}x}_{\mathcal{Z}_1^k(\epsilon, t)}$$

- Generalized binomial theorem and generalized binomial coefficients

$$(1+x)^s = \sum_{k=0}^{\infty} \binom{s}{k} x^k \quad , \quad \binom{s}{k} := \frac{s\,(s-1)\ldots(s-k+1)}{k!}$$

- Auxiliary function $\alpha := \frac{1}{2T_f} - \frac{k}{T_i}$

# Partition function III - $\mathcal{Z}_1^k(\epsilon, t)$

- Result for $\mathcal{Z}_1^k(\epsilon, t)$

$$\mathcal{Z}_1^k = \sqrt{\pi D t}\, \mathrm{e}^{\alpha^2 D t} \left[ \mathrm{e}^{\alpha(\epsilon-\mu)} \Lambda_1^k(\epsilon, t) - \mathrm{e}^{\alpha(\mu-\epsilon)} \Lambda_2^k(\epsilon, t) \right]$$

- Auxiliary functions

$$\Lambda_1^k(\epsilon, t) = \mathrm{erf}\left( \frac{\epsilon - \mu + 2Dt\alpha}{\sqrt{4Dt}} \right) - \mathrm{erf}\left( \frac{\epsilon - \epsilon_i + 2Dt\alpha}{\sqrt{4Dt}} \right)$$
$$\Lambda_2^k(\epsilon, t) = \mathrm{erf}\left( \frac{\mu - \epsilon + 2Dt\alpha}{\sqrt{4Dt}} \right) - \mathrm{erf}\left( \frac{2\mu - \epsilon - \epsilon_i + 2Dt\alpha}{\sqrt{4Dt}} \right)$$

# Partition function IV - $\mathcal{Z}_2(\epsilon, t)$

- Result for $\mathcal{Z}_2(\epsilon, t)$

$$\mathcal{Z}_2 = \left(1 - \exp\left(\frac{\mu - \epsilon_i}{T_i}\right)\right)^{\frac{T_i}{T_f}} \exp\left(-\frac{\mu}{2T_f}\right)\sqrt{\pi D t}\exp\left(\frac{Dt}{4T_f^2}\right) \times \left[\exp\left(\frac{\epsilon - \mu}{2T_f}\right)\Lambda_3(\epsilon, t) - \exp\left(\frac{\mu - \epsilon}{2T_f}\right)\Lambda_4(\epsilon, t)\right]$$

- Auxiliary functions

$$\Lambda_3(\epsilon, t) = \mathrm{erfc}\left(\frac{\epsilon_i - \epsilon + tv}{\sqrt{4Dt}}\right)$$

$$\Lambda_4(\epsilon, t) = \mathrm{erfc}\left(\frac{\epsilon - 2\mu + \epsilon_i + tv}{\sqrt{4Dt}}\right)$$

# Partition function V - Final result

$$\mathcal{Z} = \sqrt{4Dt}\exp\left(-\frac{\mu}{2T_f}\right)\sum_{k=0}^{\infty}\binom{\frac{T_i}{T_f}}{k}(-1)^k$$

$$\times\left(\mathrm{e}^{\alpha^2 Dt}\left[\mathrm{e}^{\alpha(\epsilon-\mu)}\Lambda_1^k(\epsilon,t) - \mathrm{e}^{\alpha(\mu-\epsilon)}\Lambda_2^k(\epsilon,t)\right]\right.$$

$$+\exp\left(\frac{(\mu-\epsilon_i)k}{T_i}\right)\exp\left(\frac{Dt}{4T_f^2}\right)$$

$$\left.\times\left[\exp\left(\frac{\epsilon-\mu}{2T_f}\right)\Lambda_3(\epsilon,t) - \exp\left(\frac{\mu-\epsilon}{2T_f}\right)\Lambda_4(\epsilon,t)\right]\right)$$

# Equilibration for $T_i = 20\,\text{peV}$ - Small times

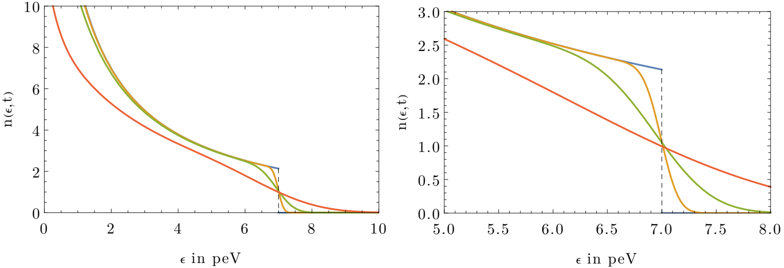


Figure: Equilibration starting with $T = 20\,\text{peV}$ at $0\,\mu\text{s}$ (blue), $1\,\mu\text{s}$ (yellow), $10\,\mu\text{s}$ (green) and $100\,\mu\text{s}$ (red).

# Equilibration for $T_i = 20\,\text{peV}$ - Large Times

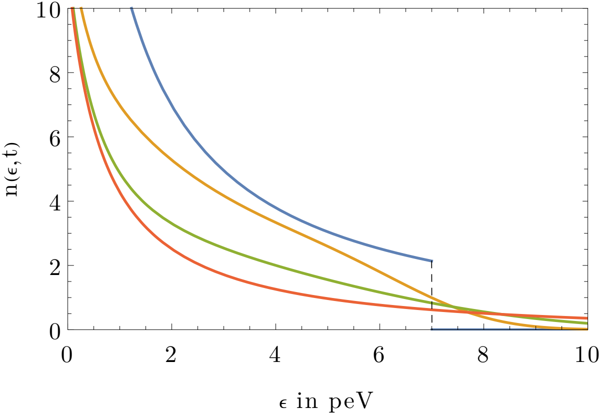


Figure: Equilibration starting with $T = 20\,\text{peV}$ at $0\,\mu\text{s}$ (blue), $100\,\mu\text{s}$ (yellow), $500\,\mu\text{s}$ (green) and $t \to \infty$ (red).

# Number of particles

- Total particle number seperated into thermal and condensate part

$$N = N_c(t) + N_t(t) = n(0,t) + \int_0^\infty g(\epsilon)\, n(\epsilon, t)\, \mathrm{d}\epsilon$$

- Density of states for ideal uniform Bose gas

$$g(\epsilon) = g_0\, \epsilon^k \quad , \quad g_0 = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}$$

- Particle number at $t = 0$

$$N = \frac{1}{z^{-1} - 1} + g_0 \int_0^{\epsilon_i} \frac{\sqrt{\epsilon}}{\exp\left(\frac{\epsilon - \mu}{T_i}\right) - 1}\, \mathrm{d}\epsilon$$

# Non-conserved particle number for constant $\mu$

Parameters: $g_0 = 100\,\mathrm{peV}^{-3/2}$, $k = \frac{1}{2}$

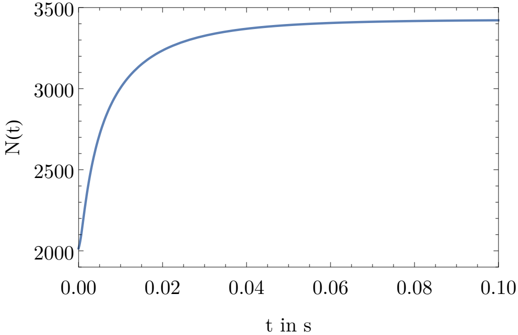


Figure: Particle number for fixed chemical potential $\mu = -0.68\,\mathrm{peV}$.

# Time dependant chemical potential

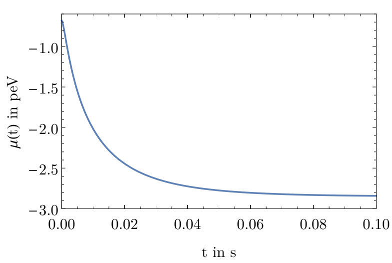


Figure: Time dependant chemical potential on linear time scale (left) and logarithmic time scale (right).

# Particle number conserving equilibration I

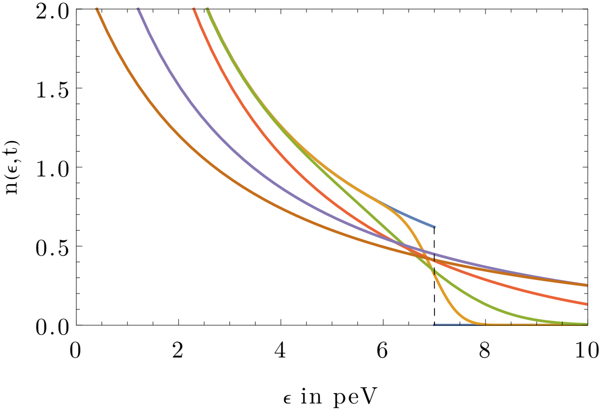


Figure: Equilibration for $T_i = T_f$ with time dependant, particle number conserving chemical potential at 0 µs (blue), 10 µs (yellow), 100 µs (green), 1 ms (red), 10 ms (purple) and 100 ms.

# Arbitrary temperatures

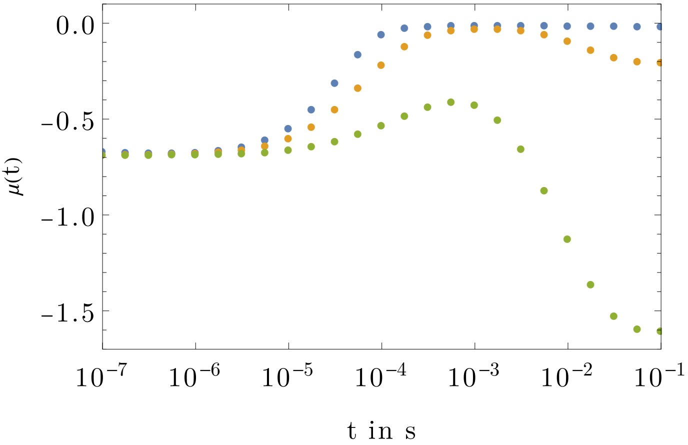


Figure: Time dependant chemical potential for initial temperature at 20 peV (blue), 15 peV (yellow) and 10 peV (green).

# Particle number conserving equilibration II

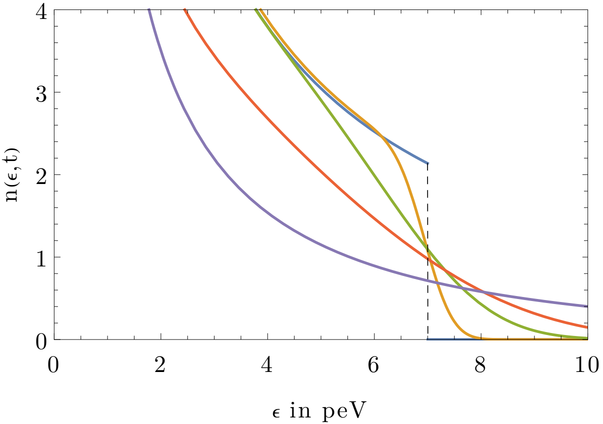


Figure: Equilibration for $T_i = 20\,\text{peV}$ with time dependant, particle number conserving chemical potential at 0 µs (blue), 10 µs (yellow), 100 µs (green), 316 µs (red) and $t \to \infty$ (purple).

# Conclusion

- Converging solutions for the nonlinear diffusion equation
  - Analytic particle distributions
  - Two approaches, both not particle conserving
- Description of cooling, transition to different temperature
- Implementation of particle number conservation
  - Rise in condensed particles
  - No solutions anymore

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