INSTITUT FÜR THEORETISCHE PHYSIK HEIDELBERG
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COMPUTATIONAL STATISTICS – SUMMER SEMESTER 2022 –

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Please, send (in pdf) or hand in the solution to this exercise sheet to your tutor, following their instructions, and carefully respecting the delivery date shown below. All exercise sheets will be graded. You can solve them individually or in pairs (with another student of the same tutorial group). In the latter case deliver please only one document with the solutions. Write the name of the file in the following format: X.Name1.Surname1 or X.Name1.Surname1-Name2.Surname2.pdf, with X being the number of the corresponding exercise sheet.

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# Exercise sheet 3

RECEIVED: May. 09 - DELIVERY: May. 16

## 1 Probability interval of a Gaussian Distribution (2pt)

Show that for a Gaussian distribution

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{1}$$

the $n\sigma$-intervals are given in terms of the error-function ,

$$\int_{-n\sigma}^{+n\sigma} p(x)dx = \mathrm{erf}\left(\frac{n}{\sqrt{2}}\right). \tag{2}$$

where the error-function is defined as:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt\, e^{-t^2}. \tag{3}$$

## 2 Random Walk (6pt)

Random walks are quite common in physics. The basic example is the thermal motion of a particle in a liquid. Imagine a particle in in that liquid, hit by other particles due to their random thermal motion. For simplicity we can consider only one dimension where the particle will move to the left / right with a certain probability in a given time interval depending on the collision rates. The path of such an particle will be completely unordered: this is called Brownian motion. Since this can be described by a random experiment with two possible outcomes (left / right movement) the binomial distribution can be applied here. Consider a binomial random walk with $n$ steps.

1. What is the average distance $\langle x \rangle$ travelled after $N$ steps, each with probability $p$ to the right and probability $q = 1 - p$ to the left? What is the condition that $\langle x \rangle = 0$? (2pt)

2. What is the probability that after $N = 5$ steps the walk reached a distance $x = 3$ to the right? and $x = 4$? and $x \geq 3$? (2pt)

3. Show that the higher order moments $\langle k^m \rangle$, $m \geq 2$ for our binomial-walk can be written as

$$\langle k^m \rangle = \sum_k k^m g(p,q) = (p\frac{\partial}{\partial p})^m (p+q)^n \tag{4}$$

   where $k$ is the number of steps to the right. (2pt)

## 3 Further Properties of PDF's (8pt)

We would like now to calculate two further properties of probability density functions. When talking about PDF's we can also consider the shape of a function. For example how peaked or flat is a function, is it symmetric or is it tilted to one side? These properties are described by the so called kurtosis and skewness. The kurtosis is a measure for the flatness of a PDF and is definded by

$$\kappa = \frac{\langle x^4_{central} \rangle}{\langle x^2_{central} \rangle^2}. \tag{5}$$

The skewness is a measure of asymmetry and is defined by

$$s = \frac{\langle x^3_{central} \rangle}{\langle x^2_{central} \rangle^{\frac{3}{2}}}. \tag{6}$$

where $\langle x^n \rangle_{central}$ is the central moment, i.e. moment around the mean, given by $= \langle (x - \langle x \rangle)^n \rangle$

Calculate the following:

1. For a Gaussian distribution with zero mean $\mu = 0$ and unit variance $\sigma$ calculate the kurtosis and skewness (2pt).

2. Show that the general formula for the skewness and kurtosis for the binomial distribution is:

$$\kappa_{bin} = \frac{1-6pq}{npq} + 3 \tag{7}$$

$$s_{bin} = \frac{q-p}{\sqrt{npq}} \tag{8}$$

and for the Poisson distribution is given by:

$$\kappa_{poisson} = \frac{1}{\nu} + 3 \tag{9}$$

$$s_{poisson} = \frac{1}{\sqrt{\nu}} \tag{10}$$

(2pt)

3. For a binomial distribution with $p = 0.1$, as well as for $p = 0.5$, calculate the skewness and kurtosis for $n = 5,\ n = 10,\ n = 50$ (2pt).

4. Plot and compare the behavior of the skewness and kurtosis for the binomial and the Poisson distribution for high values of $n$ and $\nu$ with respect to the skewness and kurtosis of a Gaussian. What do we observe and what can we conclude ? (2pt)