INSTITUT FÜR THEORETISCHE PHYSIK HEIDELBERG
PROF. DR. LUCA AMENDOLA
COMPUTATIONAL STATISTICS – SUMMER SEMESTER 2022 –

Tutors:

- Group 1: Ziyang Zheng
- Group 2: Nico Lorenz
- Group 3: Ziad Sakr
- Group 4: Adrian Schirra

Please, send (in pdf) or hand in the solution to this exercise sheet to your tutor, following their instructions, and carefully respecting the delivery date shown below. All exercise sheets will be graded. You can solve them individually or in pairs (with another student of the same tutorial group). In the latter case deliver please only one document with the solutions. Write the name of the file in the following format: X.Name1.Surname1 or X.Name1.Surname1-Name2.Surname2.pdf, with X being the number of the corresponding exercise sheet.

Exercise sheet 8

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Exercise 8.1: Marginalization over a multiplicative factor (5pt)

In the lecture, you have seen how to marginalize over a constant offset, now we see the marginalization over a multiplicative constant. Consider the measurement of astrophysical distances d which are however known only up to a factor, $d = \alpha \hat{d}$ (this is actually quite common since we often know only how relative distances). We collect several values d_i, we assume them uncorrelated Gaussians, and build the likelihood

$$L(\alpha) = N \exp -\frac{1}{2} \sum_i \frac{(\alpha \hat{d}_i - d_{th,i})^2}{\sigma_i^2} \tag{1}$$

The theoretical distances $d_{th,i}$ depend on a number of parameters that we do not need to specify. Here N is an irrelevant normalization constant.

a) Marginalize $L(\alpha)$ over α. (3pt) (Hint: what is the correct domain of integration for α?)

b) If all the theoretical parameters are known, which would be the Maximum Likelihood Estimator of α (2pt)?

Exercise 8.2 Likelihood of Cosmological Parameters (6pt)

The goal is to perform a maximum likelihood analysis on a set of supernova data (see in Dropbox the file data/supernovae.csv). We assume uniform priors, so the likelihood coincides with the posterior. At the end, we want the position of the maximum of the likelihood. This gives us the most likely values of the matter density of the Universe, Ω_M, and the dark energy density, Ω_L.

First, define the following functions in your code:

1. The comoving distance (in units of $1/H_0$):

$$d_C(z, \Omega_M, \Omega_L) = \int_0^z \frac{dz'}{\sqrt{\Omega_M(1+z')^3 + \Omega_L}}$$

2. The luminosity distance:
If $\Omega_k = 0$: $d_L(z, \Omega_M, \Omega_L) = (1+z) d_C(z, \Omega_M, \Omega_L)$
If $\Omega_k > 0$: $d_L(z, \Omega_M, \Omega_L) = (1+z)\sinh(\sqrt{\Omega_k} d_C(z, \Omega_M, \Omega_L))/\sqrt{\Omega_k}$
If $\Omega_k < 0$: $d_L(z, \Omega_M, \Omega_L) = (1+z)\sin(\sqrt{-\Omega_k} d_C(z, \Omega_M, \Omega_L))/\sqrt{-\Omega_k}$
Here, $\Omega_k = 1 - \Omega_M - \Omega_L$.

3. The distance (called "distance modulus" in the supernovae file) as predicted by the theory is $\alpha + \mu(z, \Omega_M, \Omega_L)$ where:

$$\mu(z, \Omega_M, \Omega_L) = 5 \log_{10} d_L(z, \Omega_M, \Omega_L)$$

We marginalize over the unknown parameter α.

4. The auxiliary function S_n:

$$S_n(\Omega_M, \Omega_L) = \sum_i \frac{(m_i - \mu(z_i))^n}{\sigma_i^2}$$

Here, m_i is the distance modules of the i-th supernova at redshift z_i.

5. The log-likelihood already marginalized over α is

$$L(\Omega_M, \Omega_L) = -\frac{1}{2}(S_2 - \frac{S_1^2}{S_0})$$

Next, find the position of the maximum for L in two cases: The flat case, where the curvature of the Universe vanishes, i.e. $\Omega_k = 0$ (and therefore $\Omega_L = 1 - \Omega_M$ so the problem is uni-dimensional), and the general case. In either case, both parameters Ω_M, Ω_L should be between 0 and 1. Create a 2D grid (not necessarily a regular one) of $N \times N$ points on the domain $[0,1] \times [0,1]$ and evaluate L on each point. Then find where the maximum is. Do the same for the flat case on a 1D grid in the interval $[0,1]$. Attention: In the general case, the computation time goes as N^2 and can be quite long. Choose small N's or use a small subset of the entire supernova catalog to test your code. Your result should be close to $(\Omega_M, \Omega_L) = (0.3, 0.7)$. Then you can do a more precise run by increasing the grid points near the maximum.

Bonus: you can also normalize the likelihood (not the log-likelihood!) in the considered domain and draw the confidence regions on the Ω_M, Ω_L plane at 68%.