Dynamic critical phenomena & η/s in Quark Gluon Plasma

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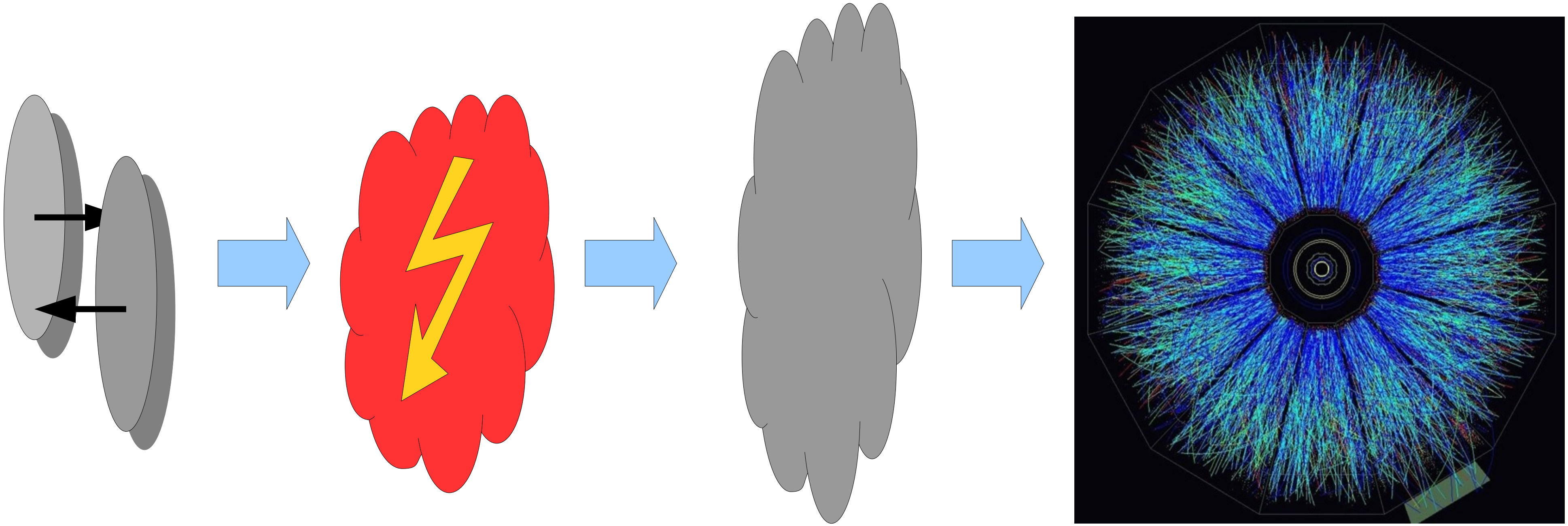
Introduction

a relativistic heavy ion collision



colliding heavy ions

Quark Gluon Plasma

hadronization, freeze-out

detection

Introduction

Experimental observations on relativistic heavy ion collisions can be well described by small transport coefficients or even ideal hydrodynamics.

⇒ Quark Gluon Plasma is a strongly coupled, **nearly perfect fluid** in the regime $1 \leq T/T_C \leq 2$!

Theoretical challenge: Calculate transport coefficients for QGP from an underlying field theory

Introduction

Small values of η/s in QGP are reached **near** the critical point of confinement/deconfinement:

fluid	P [Pa]	T [K]	η [Pa·s]	η/n [$\hbar$]	η/s [$\hbar/k_B$]
H_2O	$0.1{\cdot}10^6$	370	$2.9\cdot10^{-4}$	85	8.2
^{4}He	$0.1{\cdot}10^6$	2.0	$1.2\cdot10^{-6}$	0.5	1.9
H_2O	$22.6{\cdot}10^6$	650	$6.0\cdot10^{-5}$	32	2.0
^{4}He	$0.22{\cdot}10^6$	5.1	$1.7\cdot10^{-6}$	1.7	0.7
^{6}Li ($a=\infty$)	$12{\cdot}10^{-9}$	$23{\cdot}10^{-6}$	$\lesssim 1.7\cdot10^{-15}$	$\lesssim 1$	$\lesssim 0.5$
QGP	$88{\cdot}10^{33}$	$2{\cdot}10^{12}$	$\lesssim 5\cdot10^{11}$		$\lesssim 0.4$

$$\frac{1}{4\pi} \approx 0.08$$

Guess: The critical point corresponds to the minimal value.

This will turn out to be **wrong!**

Outline

- Dynamic properties of nearly perfect fluids
- Interlude: The QCD phase diagram
- Critical dynamics
- η/s in QGP from lattice simulations

Dynamic properties of nearly perfect fluids

Typical nonequilibrium situation

Consider a gas in a (sufficiently large) container
with no applied external field:

temperature, density,
average velocity of the
particles **not constant**
throughout the system

→

Equilibrium

transport of energy,
mass, momentum

Microscopic view

Equilibration through interactions of the particles:

- **mean free path λ,**
- most probable velocity $\bar{v}$
- **collision time $t_0 = \lambda/\bar{v}$**

λ , t_0 are the <u>microscopic scales of the system</u>

Non-uniformities in density or temperature of order λ will be washed out in the order of t_0.
Variations over long distances ($\gg \lambda$) may persist for a long time ($\gg t_0$).

Hydrodynamic regime

In the hydrodynamic regime λ is much less than

- size of the container (trap, fireball,...)
- wavelength of density fluctuations

} characteristic macroscopic scales of the system

⇒ **Separation of scales!**

Especially: temperature, entropy density are well-defined locally

Separation of scales

How to describe a system with different scales?

Hydrodynamic equations
(macroscopic point of view)

connection ?

Stochastic Langevin equations
(microscopic point of view)

Both are effective descriptions!

We will do it for a toy example: Brownian motion

Brownian motion: macroscopic

A particle of large mass m is suspended in a fluid of much lighter particles. It gets kicked from all sides and will perform a random walk.

t_0, λ: time / walked distance between two kicks

Let x be the position of the particle and n(x,t) its probability distribution.

Rate equations (Master equation) ⇒ n(x,t) satisfies the **Diffusion equation**

$$\frac{\partial n}{\partial t}(x,t) = D\Delta n(x,t)$$

Diffusion: a damped process

The diffusion equation describes a damped process:

$$\frac{\partial n}{\partial t}(x,t) = D\Delta n(x,t)$$

Fourier trf.

$$i\omega\tilde{n}(k,\omega) = -Dk^2\tilde{n}(k,\omega)$$

$$\omega = iDk^2$$

Consider mode of frequency ω:

$$n(x,t) \propto e^{i\omega t} \quad \Rightarrow \quad e^{-Dk^2 t}$$

Excitation gets damped:
Relaxation time is larger
for smaller k (longer wavelength)

Compare this to a propagating sound mode:

$$\omega = vk \quad \Rightarrow \quad n(x,t) \propto e^{-ik(x-vt)}$$

Diffusion in Hydrodynamics

Let n(x,t) be the density of a conserved quantity (e.g. particle number). Then the **continuity equation** is **exact**:

$$\frac{\partial n}{\partial t} + div(\vec{j}_n) = 0$$

Diffusion constant (transport coeffic.)

But what is j? Derivative expansion: $\vec{j}_n = -D\nabla n + ...$

$$\Rightarrow \quad \frac{\partial n}{\partial t} = -div(\vec{j}_n) = D div(\nabla n) = D\Delta n$$

Diffusion equation

The damping term Δn(x,t) is typical for hydrodynamic equations.

A nice calculation

Assume n(x,t) (satisfying the diffusion equation) to be the probability distribution of the heavy Brownian particle. (1-dim)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle |x|^2 \right\rangle$$

A nice calculation

Assume n(x,t) (satisfying the diffusion equation) to be the probability distribution of the heavy Brownian particle. (1-dim)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle |x|^2 \right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |x|^2\, n(x,t)\mathrm{d}x$$

A nice calculation

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$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle |x|^2 \right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |x|^2\, n(x,t)\mathrm{d}x = \int_\Omega |x|^2\, \frac{\partial n}{\partial t}(x,t)\mathrm{d}x$$

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$$= \int_\Omega |x|^2\, D\Delta n(x,t)\mathrm{d}x = \int_\Omega \underbrace{(\Delta |x|^2)}_{2} Dn(x,t)\mathrm{d}x$$

$$= 2D \underbrace{\int_\Omega n(x,t)\mathrm{d}x}_{1} = 2D$$

$$\left\langle |x|^2 \right\rangle = 2D\,|t|$$

We got this without solving the diffusion equation explicitly!

Brownian motion: microscopic

Let x be the position of the heavy particle, v its velocity.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{v}{\tau} + \xi(t)$$

damping term — stochastic force (noise)

„Stochastic Langevin equations“

The stochastic force describes the light particles. We do **not** know the explicit form of ξ(t). (Nor are we interested in.) We assume:

$$\langle \xi(t) \rangle = 0, \qquad \left\langle \xi(t)\xi(t') \right\rangle = g\delta(t - t')$$

Brownian motion: microscopic

Let x be the position of the heavy particle, v its velocity.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v$$

„Stochastic Langevin equations"

The ξ's are correlated on a time scale t_0 we do not see!

→ δ(t-t')

da

stochastic force (noise)

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Langevin equations: Motivation

We are only interested in the evolution of a subset of the degrees of freedom of the system. The remaining degrees of freedom enter the description via stochastic forces.

Examples:

- Brownian motion
- Dynamics of the order parameter near the critical point

Brownian motion: microscopic

Solve the Langevin equations for x(t) and v(t):

$$v(t) = v_0 e^{-t/\tau} + e^{-t/\tau} \int_0^t \mathrm{d}t' e^{t'/\tau} \xi(t')$$

This expression still contains g and ξ! Fit g to equilibrium values:

$$\frac{m}{2} \left\langle v(t)^2 \right\rangle = \frac{g\tau}{2m^2}(1 - e^{-2t/\tau}) + v_0^2 e^{-2t/\tau} \longrightarrow \frac{m}{2} \frac{g\tau}{2m^2} \stackrel{!}{=} \frac{1}{2} k_B T$$

$$\left\langle x(t)^2 \right\rangle \longrightarrow \frac{g\tau^2}{m^2} t \stackrel{!}{=} 2Dt$$

$$g = \frac{2k_B T m}{\tau}$$

Einstein relations

$$D = \frac{k_B T \tau}{m} = \mu k_B T$$

Conclusion

Fit g in order to get the correct equilibrium limit.

Once we know how to choose g properly, we can use the Langevin equations to describe non-equilibrium processes. (fluctuations, equilibration)

Dynamic vs. static properties

Dynamic properties (in the hydrodynamic regime):

- transport coefficients (D, η, ζ, κ_T)
- relaxation times
- multi-time correlation functions
- linear response to time-dependent perturbations

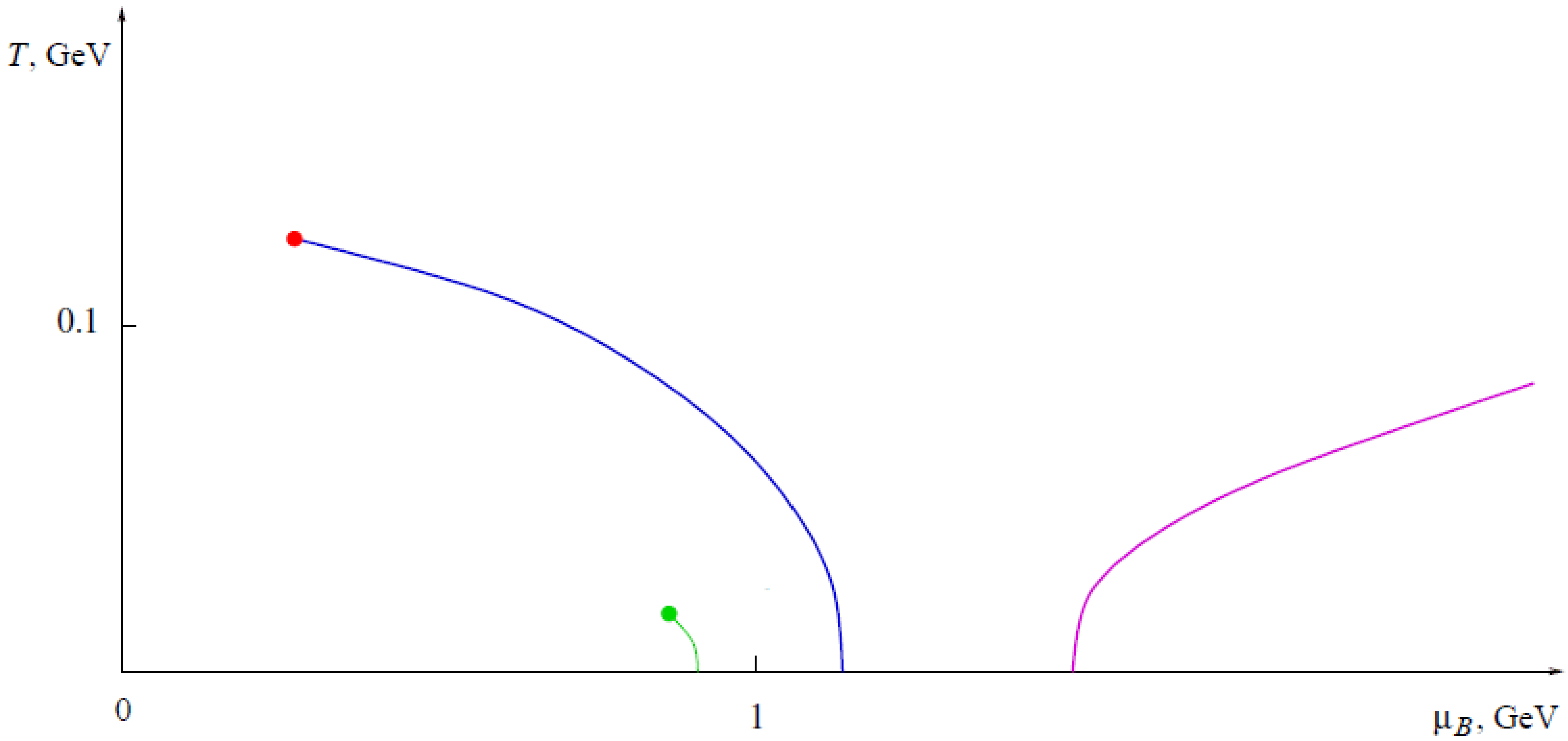
→ **not** simply described by single-time equilibirium distribution of the particles

Static properties:

- thermodynamic coefficients (C_V, compressibility,...)
- single-time correlation functions
- linear response to time-<u>in</u>dependent perturbations

The QCD phase diagram

The QCD phase diagram



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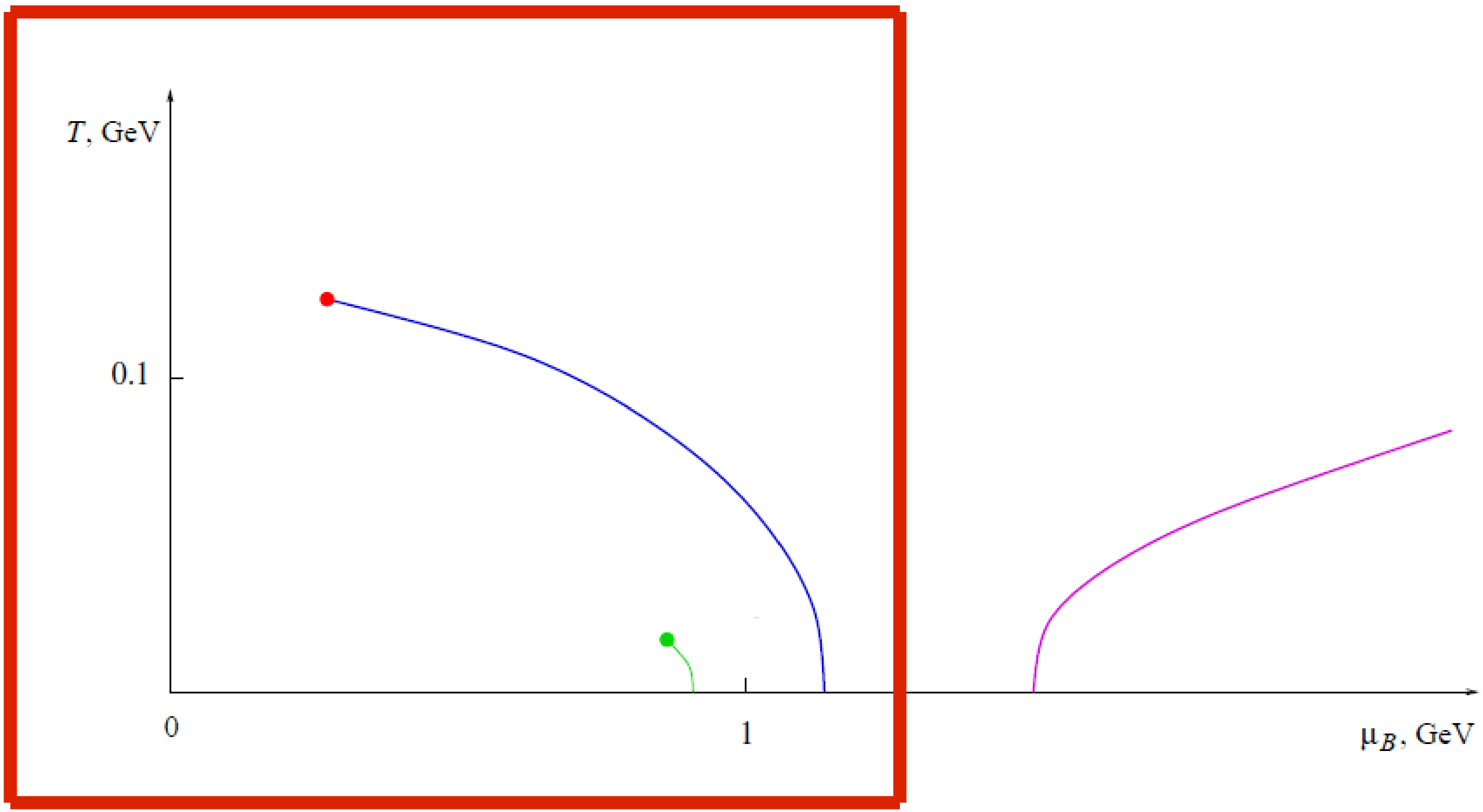

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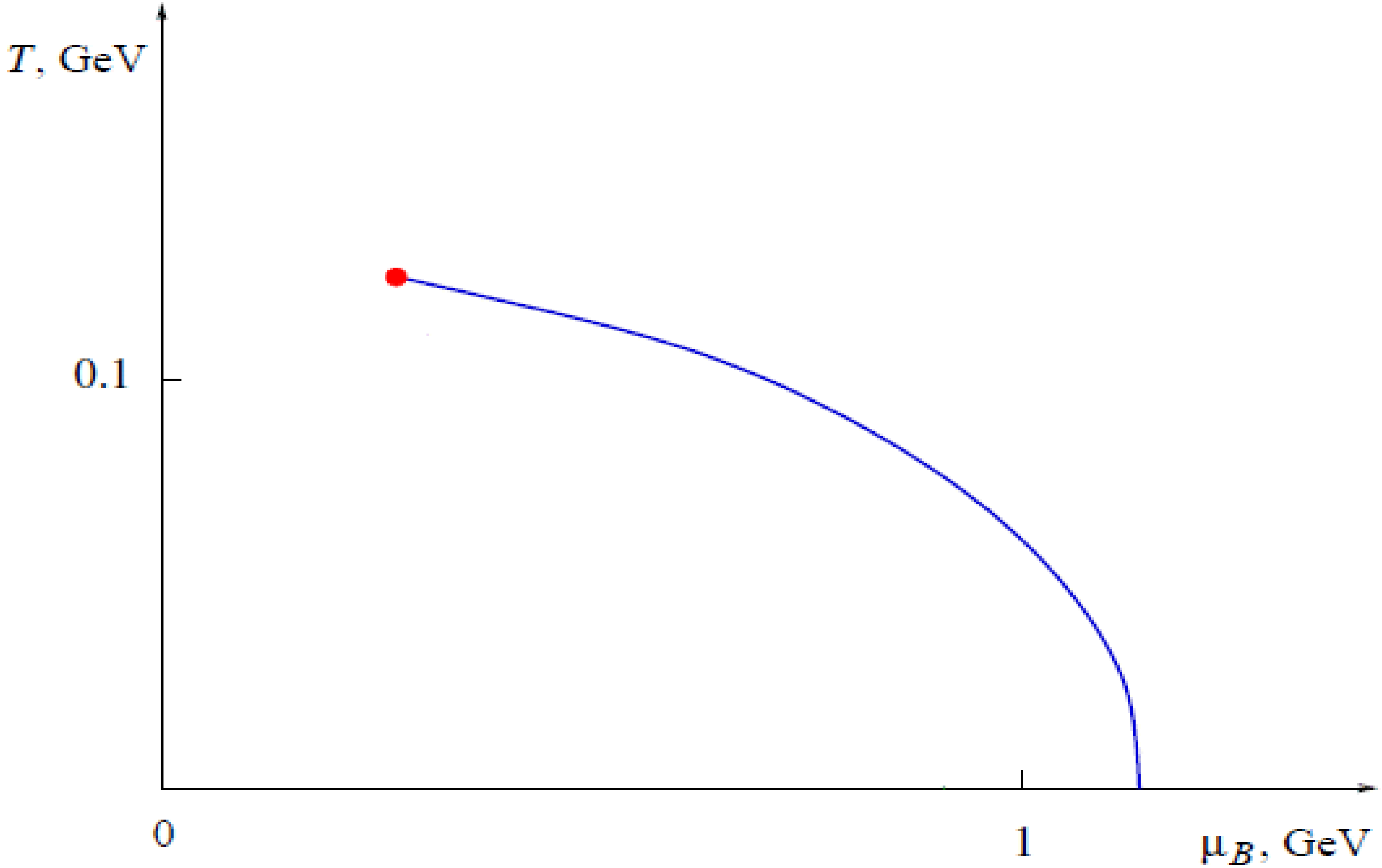

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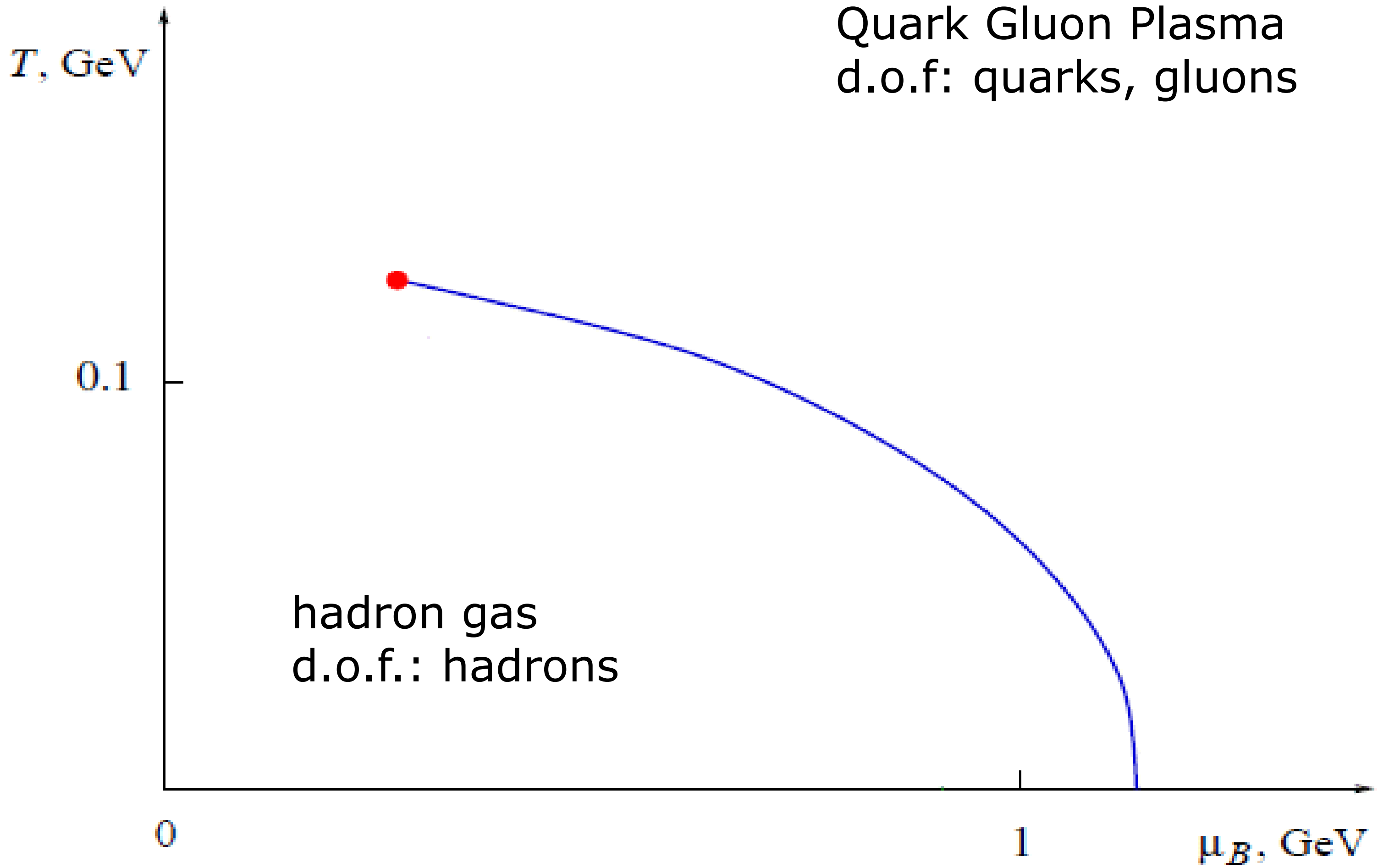

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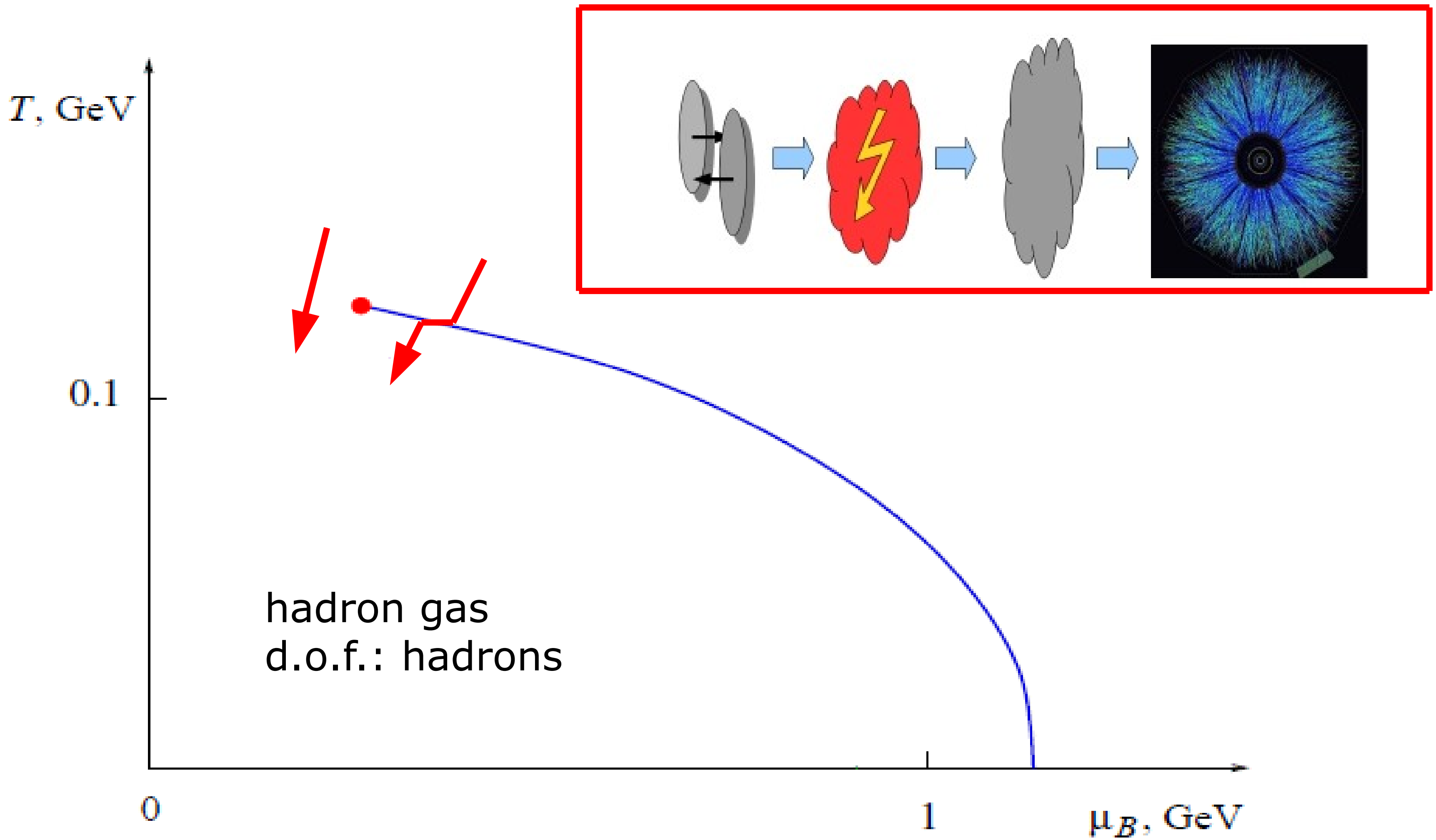
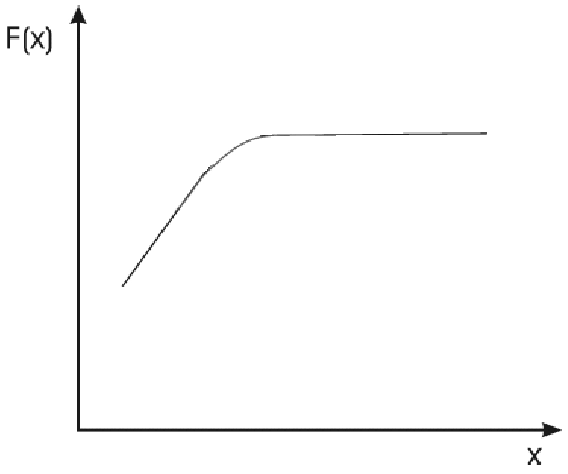
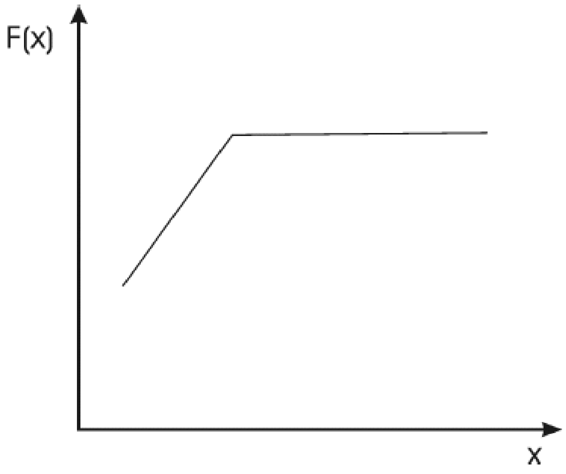
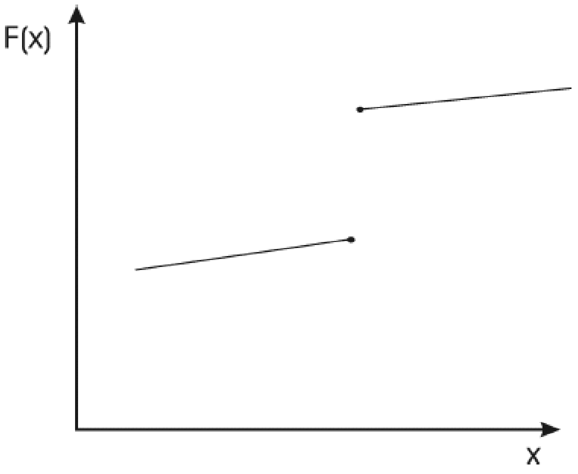


1st, 2nd order and crossover

Depending on the behavior of the free energy along a phase transition line, one distinguishes:







crossover

smooth

1st order phase transition

jump

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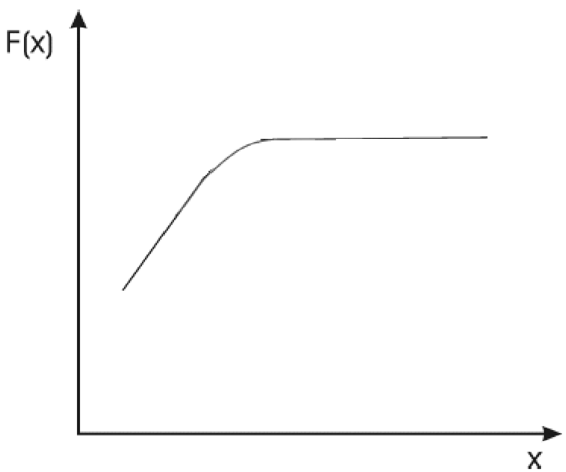
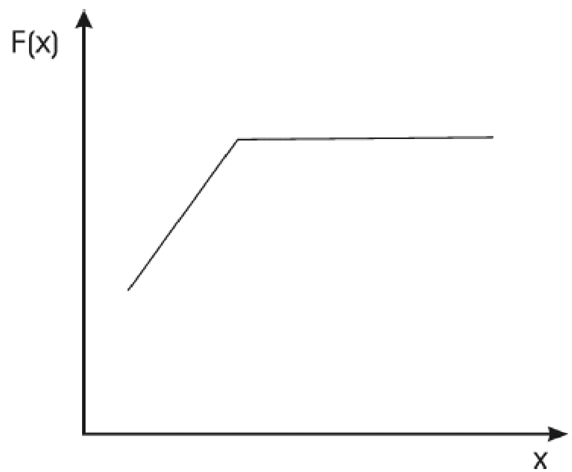


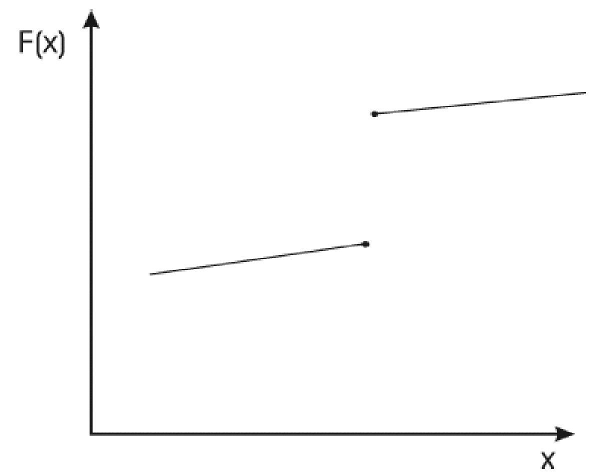
crossover

smooth



in between:
2nd order phase transition

kink



1st order phase transition

jump

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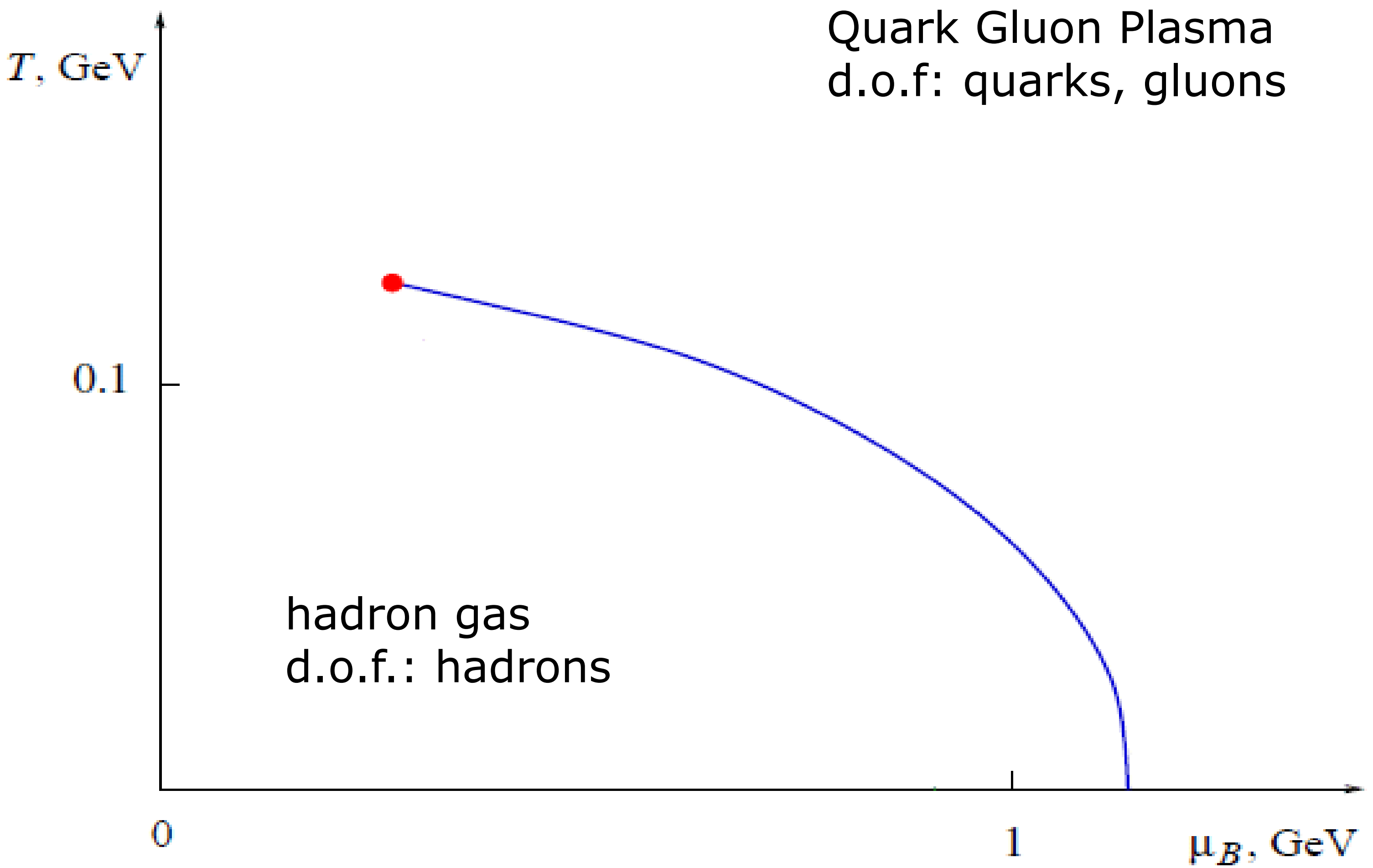

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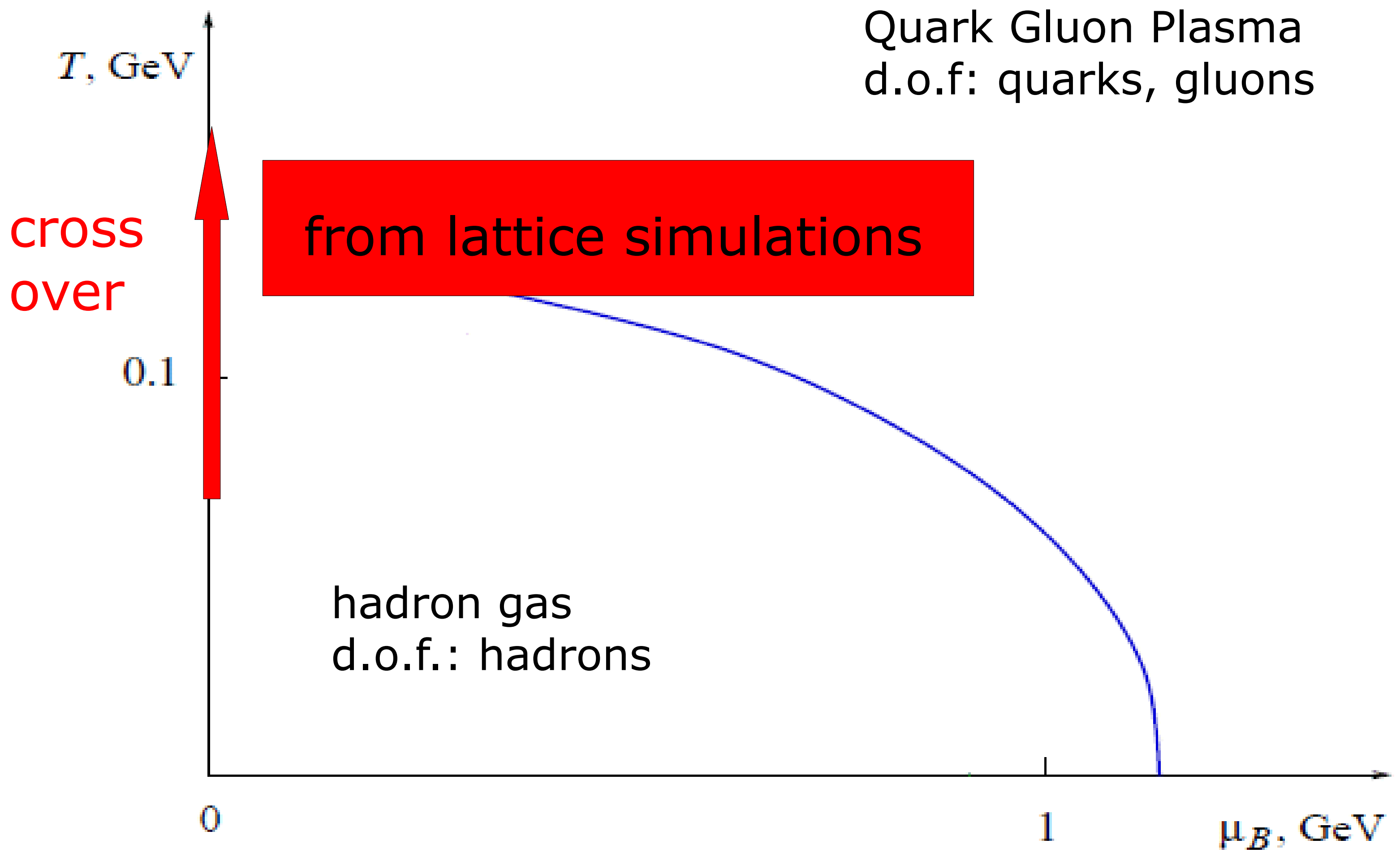

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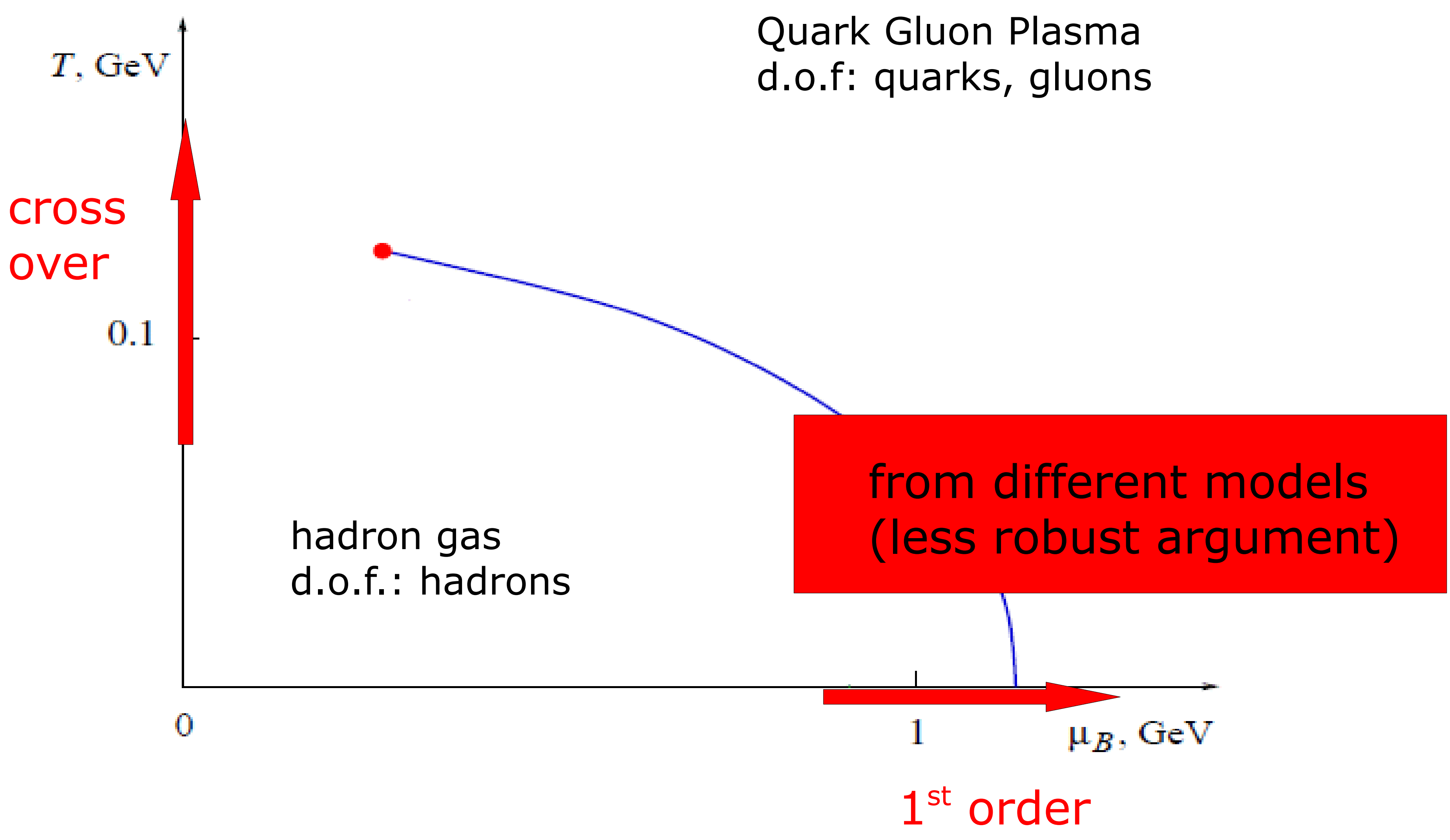
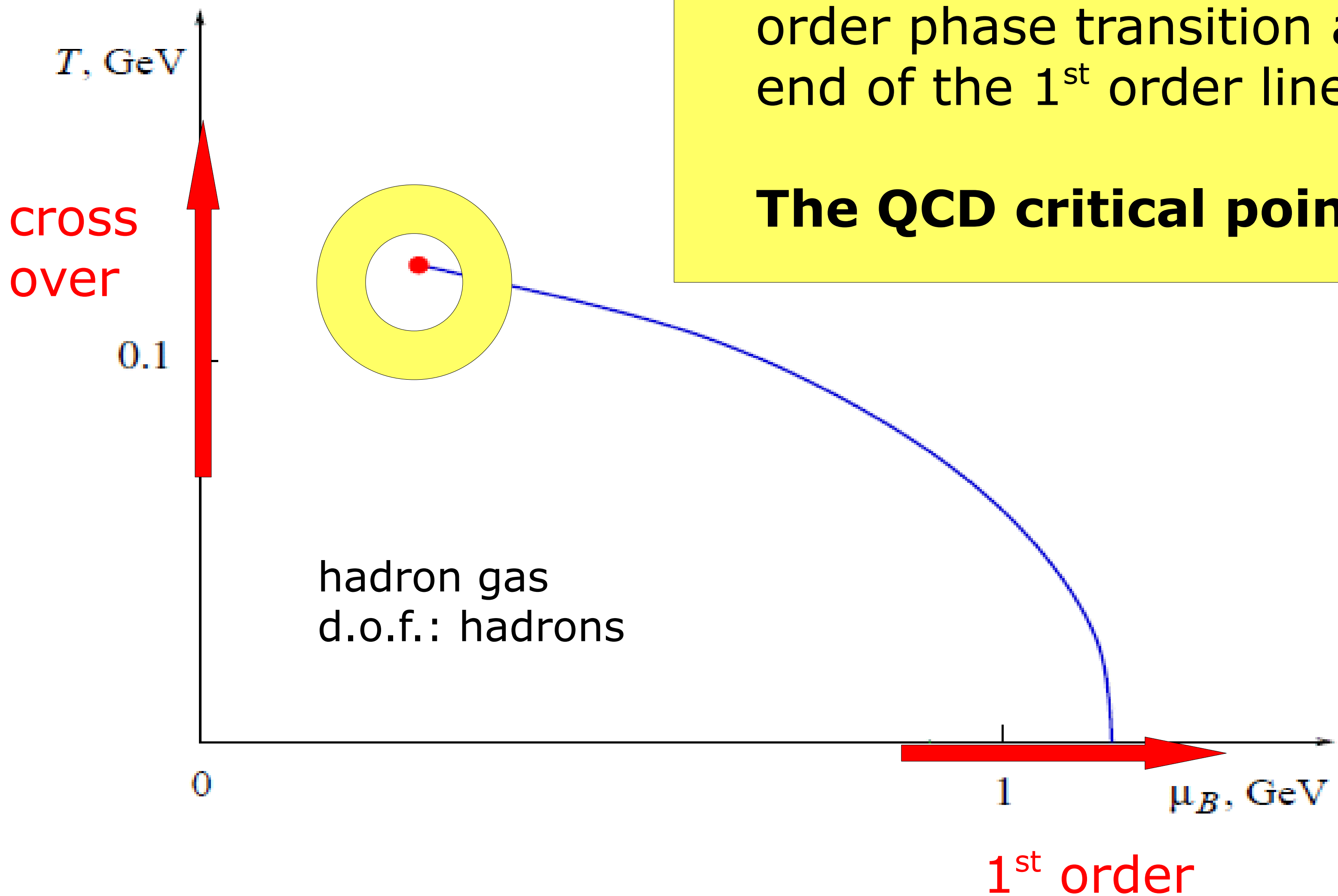


The QCD phase diagram

There has to be a point of 2nd order phase transition at the end of the 1st order line:

The QCD critical point!



Critical dynamics

Langevin eq. for order parameter

Hydrodynamic regime: time scales for the conserved quantities:

$$\tau \sim 1/(Dk^2)$$

➡ All other time scales are small compared to the long wavelength excitations of the conserved quantities.

At the critical point: Fluctuations of the order parameter become macroscopic ($\xi \to \infty$).

➡ Order parameter has to be included into the hydrodynamic description.

➡ Langevin equations for conserved quantities & order parameter. Everything else: stochastic forces

Generic example

No conserved quantities, order parameter m(x,t) (e.g. local magnetization):

$$\frac{\partial m}{\partial t}(\vec{x},t) = -\lambda \frac{\delta F}{\delta m(\vec{x},t)} + \xi(\vec{x},t)$$

F[m] is given by the **static** Ginzburg-Landau functional:

$$\underbrace{F[m] = \int \mathrm{d}^3x \left\{ \frac{a}{2}(\nabla m)^2 + \frac{b}{2}(T - T_C)m^2 + \frac{c}{4}m^4 - mh \right\}}$$

underlying field theory!

Generic example

Without quartic term:

$$\left\langle \widetilde{m}(\vec{k},t)\widetilde{m}(-\vec{k},0) \right\rangle \sim e^{-t/\tau_k}$$

τ_k is the **momentum-dependent relaxation time** of the order parameter. For small k (long wavelength) it diverges like

$$\tau_0 \sim \xi^2$$

Relaxation time goes to ∞ at T_C: „**Critical slowing down**“: The order parameter just can't calm down!

Generic example

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In general: $$\tau \sim \xi^z$$ z: dynamic critical exponent

How to explain critical slowing down?

Relaxation times are usually given by

$$\frac{1}{\tau} = \frac{\text{transport coefficient}}{\text{susceptibility}} \rightarrow 0 \qquad (T \rightarrow T_C)$$

„**Conventional theory**" (1950's)

Transport coefficients remain finite ≠0 at the critical point.

Now we know:

Transport coefficients can go to 0 or ∞ at T_C, but critical slowing down still holds.

How does η behave for $T \rightarrow T_C$?

Dynamic universality classes

Hypothesis: There exist universality classes for the dynamic behavior of physical systems near the critical point.

These depend on

- conservation laws,
- Poisson-bracket relations (commutators) between the order parameter and conserved quantities

} dynamics

- and the static universality class properties. (dimensionality, symmetry of the order parameter,...)

(Classification due to Hohenberg and Halperin)

Dynamic universality classes

Model H: describes binary fluids at the consolute point, the gas-liquid critical point and the QCD critical point

Model H corresponds to z≈3.

Transport coefficients of model H **diverge** at the critical point:

$$\eta \sim \xi^{0.06} \qquad \zeta \sim \xi^{2.8} \qquad D_{heat} \sim \xi^{0.9}$$

We conclude: The minimal value of η/s in QGP cannot be located at the critical point.

η/s in QGP from lattice simulations

Why computers?

Example: How to find the QCD critical point?

The QCD-Lagrangian is known, the partition function Z of QCD is given by the **path integral** over the Lagrangian.

→ calculate Z

→ look for singularities, jumps, kinks, …

→ done.

But: Z is incredibly hard to get, since the path integral sums over an **infinite number of degrees of freedom** and is therefore infinite dimensional.

Discretization of spacetime

One possible way out: Discretize spacetime

Continuum

(infinite number of points)

→

Lattice

(finite number of points)

A typical number of lattice points could be $8 \times (20)^3$.

The quantities on the lattice are still difficult to handle, but can be calculated e.g. by using Monte Carlo methods.

Perturbative / Non-perturbative

Note: Perturbation theory (= expansion in coupling g) cannot be applied near the critical point because of strong coupling.

Lattice calculations can be performed with every g and are therefore **non-perturbative**.

Entropy density s

s in η/s is not a big deal!

1st law of thermodynamics:

$$T\mathrm{d}S = \mathrm{d}E + P\mathrm{d}V \qquad \Rightarrow \qquad T\frac{\mathrm{d}S}{\mathrm{d}V} = \frac{\mathrm{d}E}{\mathrm{d}V} + P$$

But E, S are extensive:

$$S \propto V \Rightarrow \frac{\mathrm{d}S}{\mathrm{d}V} = \frac{S}{V} = s$$

$$s = \frac{\varepsilon + P}{T}$$

There is a standard method to calculate ε+P on the lattice.

Entropy density s is known with an accuracy of 1%.

Shear viscosity η

In order to get η, we use the Kubo relations. They appear in

Linear Response Theory.

Main idea of linear response theory:

Apply a **small** field to a system. The answer to this perturbation will be given in terms of the **equilibrium properties** of the system.

→ Many beautiful results (Fluctuation-Dissipation-Theorem, Onsager reciprocity, Kubo relations,...) and applications!

Fluctuation-Dissipation-Theorem

Fluctuation-Dissipation-Theorem:

Fluctuations in thermal equilibrium are related to linear response to small perturbations:

Correlation fcts. of fluctuations without external field	⟺	Response to fields, energy dissipation

Usually, one can simulate/measure one of them and gets information about the other one.

η in QGP

Get retarded correlator G_{ret} on the lattice
and calculate η by a special case of the FDT:

$$\eta = \lim_{\omega \to 0} \frac{\rho(\omega)}{2\omega}$$

a „**Kubo relation**"

$$G_{ret}(\vec{x},t) = -i\theta(t)\left\langle [T^{xy}(\vec{x},t), T^{xy}(\vec{0},0)] \right\rangle$$

$$\rho(\omega) = -2\mathrm{Im}\widetilde{G_{ret}}(\vec{k}=\vec{0},\omega)$$

energy momentum
tensor of gluon-field

ρ(ω) is called „spectral funtion".

An ill-posed problem

Problem: Lattice calculations are done in euclidean time. We obtain G_E (instead of G_{ret}) and $\rho(\omega)$ is given by

$$G_E(\tau) \stackrel{!}{=} \int \frac{\mathrm{d}\omega}{2\pi} \rho(\omega) \frac{\cosh[\omega(\tau - 1/(2T))]}{\sinh[\omega/(2T)]}$$

Inverting this integral transform is an **ill-posed problem**.

Results on η/s

Recent developments to overcome these difficulties:

- Improvements of maximum entropy method
- Different **parametrizations** of ρ(ω)
- Multi-level algorithms
- Non-zero spatial momentum: ρ(ω,k) instead of ρ(ω)
- Smoothness assumptions (also suggested by gauge/gravity duality for N=4 SUSY QCD)

H.B. Meyer, Phys. Rev. D 76, 101701 (2007):

$$\frac{1}{4\pi} \approx 0.08$$

$$\eta/s = \begin{cases} 0.134(33)\ (T=1.65T_C) \\ 0.102(56)\ (T=1.24T_C) \end{cases}$$