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Introduction to Hydrodynamics

Used Literature:

- Lecture on „Numerical Fluid Dynamics"
  K. Dullemond
  http://www.mpia.de/homes/dullemon/lectures/fluiddynamics08/index.html

- „Fluid Mechanics", Landau & Lifshitz (Vol. 6)

- „Hydrodynamics", Greiner & Stock (Vol. 2a)

To describe the state of a moving fluid we need:

- conservation of mass, momentum & energy
  ⊕ equation of state
  => 6 equations for 6 unknowns:

$\vec{u}$ ... velocity
ρ ... density
e ... internal specific ($\hat{=}$ per mass) energy
P ... pressure

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There are two ways to describe the Fluid:

1. Euler form (used in grid-based simulations)

   - coordinate system is fixed in space
   - evolution of the medium / derivatives at ~~point $\vec{x}$~~ a fixed location $\vec{x}$ (time derivative $\partial/\partial t$)

2. Lagrange form (used in „Smooth particle hydrodynamics")

   - coordinate system is attached to moving mass element
   - derivatives refer to changes occuring within the element as it changes its state and location
   - time derivative $D_t = \partial/\partial t + \vec{u} \cdot \vec{\nabla}$

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1) Conservation of mass

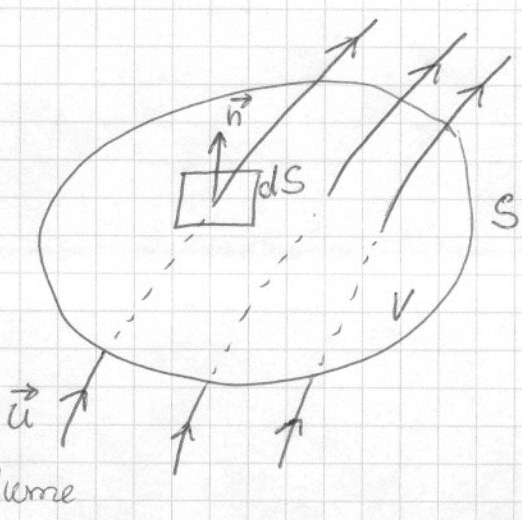
V ... Volume

$S = \partial V$ ... Surface

dS ... Surface element

$\vec{n}$ ... normal unit vector

$\vec{u}$ ... velocity



The variation of the mass in the volume must be entirely due to the in- or outflow of mass through S

$$\Rightarrow \frac{\partial}{\partial t} \int_V \rho \, dV = - \int_{\partial V} \underbrace{\rho \bar{u} \cdot \vec{n}}_{\text{mass flux}} \, dS \overset{\text{Gauss' Law}}{=} - \int_V \vec{\nabla} (\rho \bar{u}) \, dV$$

- since eq. must be true for any volume

$$\Rightarrow \partial_t \rho + \vec{\nabla} (\rho \bar{u}) = 0$$

($\hat{=}$ Continuity equation)

Lagrange form:

$$D_t \rho = - \rho \vec{\nabla} \vec{u} , \quad - \vec{\nabla} \vec{u} \hat{=} \text{compression}$$

=> a ~~gas~~ fluid element changes its density when the fluid motion converges

2) Conservation of momentum

$\rho \vec{u}$ .... momentum density ($\hat{=}$ mass flux)

total momentum in Volume $\hat{=}$ volume integral over $\rho \vec{u}$

=> same approach as in 1)

$$\Rightarrow \frac{\partial}{\partial t} \int \rho \vec{u} \, dV = - \int_{\partial V} \rho \vec{u} \, \vec{u} \cdot \vec{n} \, dS - \int_{\partial V} P \vec{n} \, dS$$

($- P \vec{n}$ ... force acting by gas outside the volume on gas inside the volume)

But: for Gauss' Law one needs inner product ~~a~~ write $\vec{n}$ at the surface => introduce unit tensor $\hat{I}$ (isotropic pressure)

$$\Rightarrow \partial/\partial t \int \rho \vec{u} \, dV = - \int_V \vec{\nabla} (\rho \vec{u} \vec{u} + P \hat{I}) \, dV$$

$$\Rightarrow \partial_t (\rho \vec{u}) + \vec{\nabla} (\rho \vec{u} \vec{u}) + \nabla (P \hat{I}) = 0$$

Lagrange form:

$$\underbrace{\vec{u} \, \partial_t \rho + \rho \partial_t \vec{u} + \vec{u} \, \vec{\nabla} \cdot (\rho \vec{u})}_{= 0 \text{ (continuity equation}} + \rho \vec{u} \cdot \vec{\nabla} \vec{u} + \vec{\nabla} P = 0$$

$$\Rightarrow \rho \partial_t \vec{u} + \rho \vec{u} \cdot \vec{\nabla} \vec{u} + \vec{\nabla} P = 0$$

$$\rho D_t \vec{u} = - \vec{\nabla} P$$

=> fluid element will be accelerated due to a ~~force~~ pressure gradient

3) Conservation of energy

$e_{tot} = e + |\vec{u}|^2/2$ total specific energy (= internal plus kinetic)

- total energy $\hat{=}$ volume integral of $\rho(e + |\vec{u}|^2/2)$
- advection of energy through control volume $\hat{=}$ surface integral of $\rho(e + |\vec{u}|^2/2)\vec{u} \cdot \vec{n}$

$$\partial/\partial t \int_V \rho(e + |\vec{u}|^2/2)\, dV = -\int_{\partial V} \rho(e + |\vec{u}|^2/2)\vec{u} \cdot \vec{n}\, dS - \int_{\partial V} P\vec{u} \cdot \vec{n}\, dS$$

$\int_{\partial V} P\vec{u} \cdot \vec{n}\, dS$ $\hat{=}$ work that the exterior acts on the control volume

$$\Rightarrow \partial/\partial t \int \rho e_{tot}\, dV + \int \vec{\nabla}(\rho e_{tot} + P)\vec{u}\, dV = 0$$

$$\Rightarrow \partial_t(\rho e_{tot}) + \vec{\nabla}(\rho e_{tot} + P)\vec{u} = 0$$

Lagrange form:

$$D_t e_{tot} = -\frac{P}{\rho}\vec{\nabla}\vec{u} - \frac{1}{\rho}\vec{u}\vec{\nabla}P$$

$$D_t e_{tot} = -\frac{P}{\rho}\partial_i u_i - \frac{1}{\rho} u_i \partial_i P$$

$$D_t e_{tot} = D_t e + u_i \partial_t u_i + u_k u_i \partial_k u_i \quad \cancel{D_t e_{tot}}$$
$$= D_t e + u_i(\partial_t u_i + u_k \partial_k u_i) = D_t e + u_i D_t u_i$$

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Lagrange form:

$$D_t e_{tot} = -\frac{P}{\rho}\vec{\nabla}\vec{u} - \frac{1}{\rho}\vec{u}\,\vec{\nabla}P$$

$$D_t e_{tot} = -\frac{P}{\rho}\partial_i u_i - \frac{1}{\rho} u_i \partial_i P$$

$$D_t e_{tot} = D_t e + u_i \partial_t u_i + u_k u_i \partial_k u_i$$

$$= D_t e + u_i(\partial_t u_i + u_k \partial_k u_i) = D_t e + u_i D_t u_i$$

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use „Momentum conservation"

$$D_t u_i = -\frac{\partial_i P}{\varrho}$$

$$\Rightarrow D_t e + u_i D_t u_i = D_t e - u_i \frac{\partial_i P}{\varrho}$$

$$= -\frac{P}{\varrho}\partial_i u_i - \frac{1}{\varrho} u_i \partial_i P$$

$$\Rightarrow D_t e = -\frac{P}{\varrho}\partial_i u_i$$

$$\Rightarrow \boxed{D_t e = -\frac{P}{\varrho}\nabla\vec{u}} \quad (1)$$

$\Rightarrow$ internal energy increases due to compression

With 1st Law of Thermodyn:

$$de = T ds - P d(1/\varrho)$$

$$D_t e = T D_t s - P D_t(1/\varrho) = T D_t s - P\left(\frac{\partial}{\partial t} + \vec{u}\nabla\right)(1/\varrho)$$

$$= T D_t s - P\left(-\frac{1}{\varrho^2}\cdot\partial_t\varrho - \vec{u}\frac{1}{\varrho^2}\vec{\nabla}\varrho\right)$$

$$= T D_t s + \frac{P}{\varrho^2}(\partial_t\varrho + \vec{u}\vec{\nabla}\varrho) = T D_t s + \frac{P}{\varrho^2}(\partial_t\varrho$$

$$+ \vec{\nabla}(\vec{u}\varrho) - \varrho\nabla\vec{u})$$

$$= \boxed{T D_t s - \frac{P}{\varrho}\vec{\nabla}\vec{u}} \quad (2)$$

$\Rightarrow$ Comparison of (1) and (2)

$$\Rightarrow T D_t s = 0$$

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=> The entropy of a gas parcel does not change along its path of motion

=> Equation of hydrodynamics conserve entropy

Summary: So far:

1) Conservation of mass:

$$\partial_t \rho + \vec{\nabla}(\rho \vec{u}) = 0$$

$$D_t \rho = -\rho \vec{\nabla} \vec{u}$$

2) Conservation of momentum

$$\partial_t (\rho \vec{u}) + \vec{\nabla}(\rho \vec{u} \vec{u} + P \hat{I}) = 0$$

$$\rho D_t \vec{u} = -\vec{\nabla} P$$

3) Conservation of energy:

$$\partial_t (\rho e_{tot}) + \vec{\nabla}(\rho e_{tot} \vec{u}) + \nabla(P \vec{u}) = 0$$

$$D_t e = -\frac{P}{\rho} \vec{\nabla} \vec{u} \quad \text{or} \quad D_t S = 0$$

Viscosity:

=> Momentum conservation (non-viscous)

$$\partial_t(\rho\vec{u}) + \nabla(\rho\vec{u}\vec{u} + P\hat{I}) = 0$$

$$\partial_t(\rho u_i) + \partial_k(\underbrace{\rho u_i u_k + \delta_{ik}P}) = 0$$

Π_{ik} ... momentum flux density tensor
flux of momentum i in direction k (or vice versa) due to mechanical ~~transfer~~ transport of different particles of fluid from place to place (first term) and due to pressure forces acting in the fluid

~~[illegible]~~

Viscosity:

- force involved in exchange of momentum between fluid particles
- internal friction (only) when fluid particles move with different velocities (rel. motion between various parts of the fluid) ⊛
- causes irreversible transfer of momentum from points where the velocity is large to those where it is small

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Equation of motion of viscous fluid => add to the ideal momentum flux a term $-\sigma_{ik}$ which gives irreversible transfer of momentum in the fluid

(equation of continuity is equally valid for any fluid, viscous or not)

Momentum flux density tensor in viscous fluids:

$$\Pi_{ik} = \underbrace{p\delta_{ik} - \sigma_{ik}}_{} + \rho u_i u_k$$

stress tensor: gives part of the momentum flux that is not due to the direct transfer of momentum with the mass of moving fluid

σ_{ik} ... viscous stress tensor

How does σ_{ik} look like?

- from ⊛ => σ_{ik} depends on space derivatives of velocity
- if velocity gradients are small => momentum transfer depends only on first derivatives of velocity (newtonian fluid)
- no term in σ_{ik} independent of $\partial u_i / \partial x_k$ since σ_{ik} must vanish for $\vec{u}$ = const.
- σ_{ik} must vanish when the whole fluid is in uniform rotation => in such motion no internal friction occurs

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$\Rightarrow$ for uniform rotation with angular velocity $\vec{\Omega}$

$$\Rightarrow \vec{u} = \vec{\Omega} \times \vec{r}$$

$$\Rightarrow \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \quad \text{vanishes for } \vec{u} = \vec{\Omega} \times \vec{r}$$

$\Rightarrow \sigma_{ik}$ must contain these symmetrical combinations of $\partial u_i / \partial x_k$

$\Rightarrow$ most general tensor of rank two, satisfying these two conditions:

$$\sigma_{ik} = a \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) + b \frac{\partial u_l}{\partial x_l} \delta_{ik}$$

usually is rewritten in slightly modified form:

$$\sigma_{ik} = \underbrace{\eta \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial u_l}{\partial x_l} \right)} + \underbrace{\xi \delta_{ik} \frac{\partial u_l}{\partial x_l}}$$

- traceless tensor
- vanishes when medium contracts uniformly with $\vec{u} \propto \vec{r}$
- acts when change in shape

- diagonal elements of equal size
  $\Rightarrow$ equal in all directions
- ~~acted~~ acts when change in volume at constant shape

η & ξ ... positive (proof see Landau & Lifshitz)
functions of pressure and temperature

η ... viscosity coefficient (shear)

ξ ... second viscosity coefficient (bulk)

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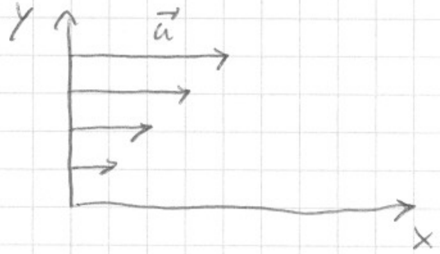
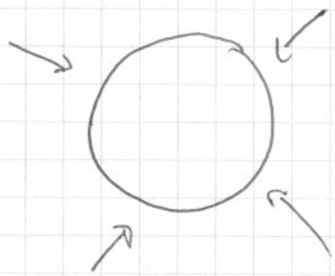
Examples

$u_i = (y, 0, 0)$



$$\sigma_{ik} = \begin{pmatrix} 0 & \eta & 0 \\ \eta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$u_i = (-x, -y, -z)$



compression

$$\sigma_{ik} = \begin{pmatrix} -3\zeta & & 0 \\ & -3\zeta & \\ 0 & & -3\zeta \end{pmatrix}$$

From momentum conservation

$$\Rightarrow \quad \partial_t (\rho u_i) = -\partial_k \Pi_{ik}$$

$$\Pi_{ik} = P\delta_{ik} + \rho u_i u_k - \sigma_{ik}$$

$$\partial_t (\rho u_i) + \partial_k (\rho u_i u_k) = u_i \big(\overbrace{\partial_t \rho + \partial_k (\rho u_k)}^{=0 \text{ (continuity eq.)}} \big) + \rho (\partial_t u_i + u_k \partial_k u_i)$$

$$= \rho (\partial_t u_i + u_k \partial_k u_i)$$

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$$\Rightarrow \rho(\partial_t u_i + u_k \partial_k u_i) \cancel{\frac{\partial P}{\partial x_i}}$$

$$= \partial_i P + \partial_k \left\{ \eta \left(\partial_k u_i + \partial_i u_k - \frac{2}{3} \delta_{ik} \partial_l u_l \right) \right\}$$

$$+ \partial_i (\xi \partial_l u_l)$$

$\Rightarrow$ most general form of equations of motion of a viscous fluid

- assume that η and ξ are constant in the fluid

$$\Rightarrow \frac{\partial \sigma_{ik}}{\partial x_k} = \eta \left(\frac{\partial^2 u_i}{\partial x_k^2} + \frac{\partial}{\partial x_i} \frac{\partial u_k}{\partial x_k} - \frac{2}{3} \frac{\partial}{\partial x_i} \frac{\partial u_l}{\partial x_l} \right) + \xi \frac{\partial}{\partial x_i} \frac{\partial u_l}{\partial x_l}$$

$$= \eta \frac{\partial^2 u_i}{\partial x_k^2} + \left(\xi + \frac{\eta}{3} \right) \frac{\partial}{\partial x_i} \frac{\partial u_l}{\partial x_l}$$

or in vector notation:

$$\rho \left\{ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} \right\} = - \vec{\nabla} P + \eta \Delta \vec{u} + (\xi + \eta/3) \vec{\nabla} \vec{\nabla} \vec{u}$$

(grad div $\vec{u}$)

for incompressible fluids: $\vec{\nabla} \vec{u} = 0$ (from continuity eq.)

$$\Rightarrow \boxed{\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} = - \frac{1}{\rho} \vec{\nabla} P + \eta/\rho \, \Delta \vec{u} \quad \Rightarrow \text{Navier-Stokes equation}}$$

$$\sigma_{ik} = \eta \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right)$$

$(\vec{u} \cdot \vec{\nabla}) \vec{u}$ ... convective acceleration, time independent acceleration of a fluid ~~~~ with respect to space

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$\eta/\varrho \; \Delta\vec{u}$ ... viscous forces

η/ϱ ... kinematic viscosity
$= \nu$

	η/ϱ [cm^2/s]	η [g/cm s]
Water	0.010	0.010
Air	0.150	0.00018
Alcohol	0.022	0.018
Glycerine	6.8	8.5

Reynolds number:

Tells how important viscosity is:

$$Re = \frac{\varrho u L}{\eta} = \frac{u L}{\nu} = \frac{\text{inertial force}}{\text{viscous force}}$$

u ... typical velocity
L ... typical spatial scale

$Re \gg 1$ => very inviscid flow; can often become turbulent
$Re \ll 1$ => very viscous flow, tend to be laminar