Turbulence in the Early Universe

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Relaxation, Turbulence, and Non-Equilibrium Dynamics of Matter Fields in the Early Universe after Inflation

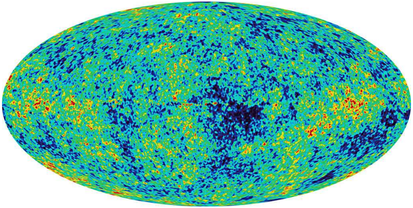
Puzzles of classical cosmology

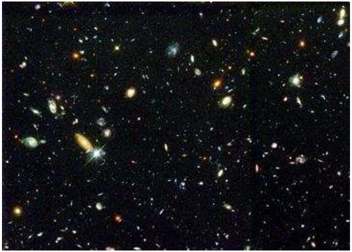
WHY THE UNIVERSE

- is so old, big and flat ?
  $t > 10^{10}$ years
- homogeneous and isotropic?
  $\delta T/T \sim 10^{-5}$
- contains so much entropy?
  $S > 10^{90}$
- does not contain unwanted relics?
  (e.g. magnetic monopoles)

can be solved with hypothesis of Inflation

Predictive power of Inflation





Fluctuations in inflaton field

$\Downarrow$

CMBR anisotropy
379,000 years after

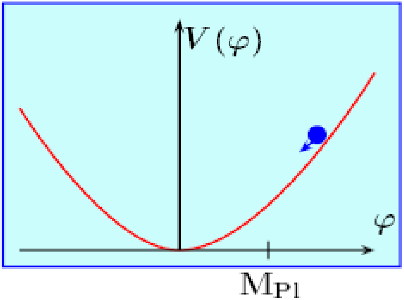
$\Downarrow$

Large-scale structure
13.7 billions years after

Equation of motion

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0$$

If $H \gg m$ the field rolls down slowly



$\varphi > M_{Pl}$ Inflation

$\varphi < M_{Pl}$ Reheating

During Inflation the Universe is empty, in a vacuum state.
How vacuum was turned into radiation ?

Particle physicist

Cosmologist

Initial linear stage

During Inflation the Universe is “empty”. But small fluctuations obey

$$\ddot{u}_k \; + \; [k^2 + m_{\text{eff}}^2(\tau)] \, u_k = 0$$

and it is not possible to keep fluctuations in vacuum
if m_{eff} is time dependent

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The source for $m_{\text{eff}} = m_{\text{eff}}(\tau)$ is time-dependence of classical backgrounds:

- Expansion of space-time, $a(\tau)$
- Evolution of the inflaton field, $\phi(\tau)$

Coupling to the inflaton

Scalar X	Fermion ψ
$m_{\text{eff}}^2 = m_X^2 + g^2\phi^2(t)$	$m_{\text{eff}} = m_\psi + g\phi(t)$

$$\ddot{u}_k + \left[k^2 + m_{\text{eff}}^2\right] u_k = 0$$

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Relevant parameter:

$$g^2 \;\rightarrow\; q \equiv \frac{g^2\phi^2}{4m_\phi^2}$$

Note: q can be very large since

$$\frac{\phi^2}{m_\phi^2} \approx 10^{12}$$

Kofman, Linde & Starobinsky (94)

Bose versus Fermi :

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Bose stimulation.
Occupation numbers grow,
$n = e^{\mu t}$

Explosive decay of the inflaton

Pauli blocking.
Occupation numbers
$n < 1$

Particles are massless at
$\phi(t) = -m_\psi/g$

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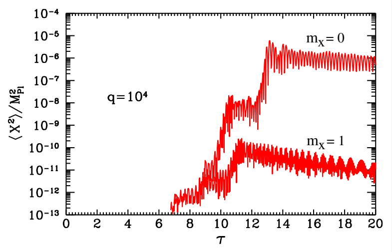

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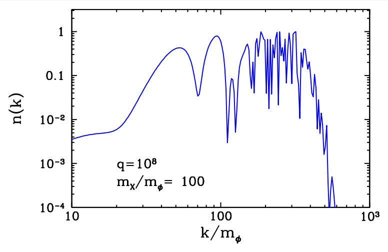
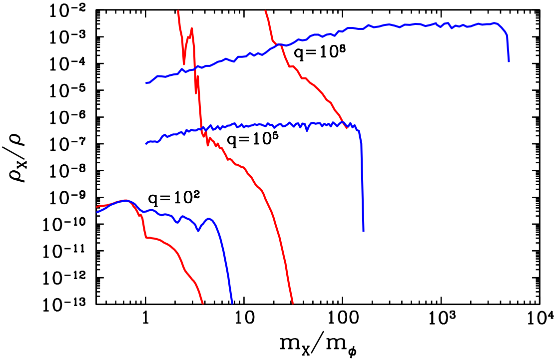


Matter creation: Bose versus Fermi



$\phi \propto \sin(mt)$

Particle production at the end of inflation in the model with inflaton potential

$$V(\phi) = m^2\phi^2$$

Red lines: fraction of produced Bosons
Blue lines: fraction of produced Fermions

Thermalization after Inflation

Questions:

- How system approaches equilibrium?
- When? What is thermalization temperature?

Are of general interest and important for practical applications.
It influences:

- Inflationary predictions
- Baryogenesis
- Abundance of gravitino and dark matter relics
- Primordial fluctuations

Thermalization after Inflation

To solve the problem one needs to understand the non-linear dynamics.

- At large occupation numbers it is possible to map quantum evolution into classical:
  quantum density matrix $\Rightarrow$ classical density matrix

  Khlebnikov, I.T. (1996)
- This allows numerical modelling on the lattice
  - starting from vacuum
  - through "parametric resonance", then through preheating
  - and down to physical effects in question

Approach:

- Lattice simulations
  (as a guidance)

Various quantities can be measured as functions of time:

- Zero mode, $\phi_0 = \langle \phi \rangle$
- Variance, $\langle \phi^2 \rangle$ - ϕ_0^2
- Particle number,
  $n_k = \langle a^\dagger(k) a(k) \rangle$
- Correlators,
  $\langle aa \rangle$, $\langle a^\dagger a^\dagger aa \rangle$, $\langle \pi^2 \rangle$, ...

- Kinetic theory
  (weak wave turbulence)

Falkovich & Shafarenko (1991)
Zakharov, L'vov, Falkovich (1992)

Compare to lattice results and extrapolate.

Micha & I.T. (2004)

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Weak wave turbulence

Theory. Steps and assumptions involved:

- Energy conservation in momentum space
  - Constant flux for stationary driven turbulence
- Self-similarity
  - Appears both in free and driven turbulence
- Bits of kinetic theory
  - Gives scaling exponents of collision integral

The model

Consider simplest $\lambda\varphi^4$ model.

In conformal frame, $\phi = \varphi/a$, and rescaled coordinates, $x^\mu \to \sqrt{\lambda}\varphi(0)\, x^\mu$, the equation of motion takes very simple form

$$\Box\phi + \phi^3 = 0$$

Initial conditions:
all fields are in vacuum + oscillating zero mode

- *Think about it as of relativistic generalization of Gross–Pitaevskii equation written for the real field.*
- *More complicated models will follow.*

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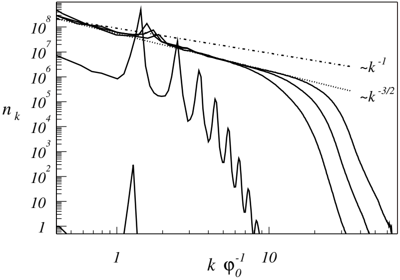
Turbulent spectra



Re-scale the field and coordinates by the current amplitude of the zero mode

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Here $x^\mu \to x^\mu\,\phi_0$ and therefore $k \to k\,/\phi_0$

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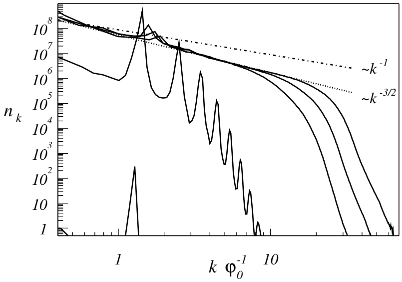

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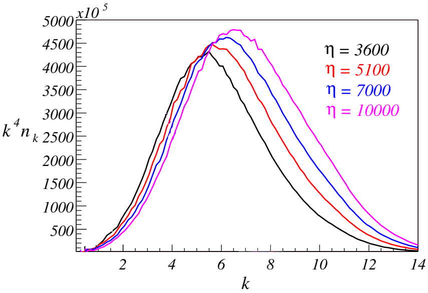
Let $n \sim k^{-s}$. Theory of a stationary wave turbulence predicts

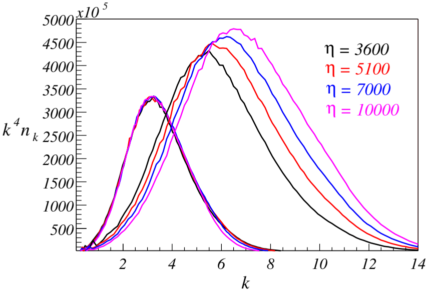
- $s = \frac{5}{3}$ for 4-particle interaction
- $s = \frac{3}{2}$ for 3-particle interaction

Self-similar evolution



Particle numbers on the lattice
in the regime of free turbulence

Self-similar evolution

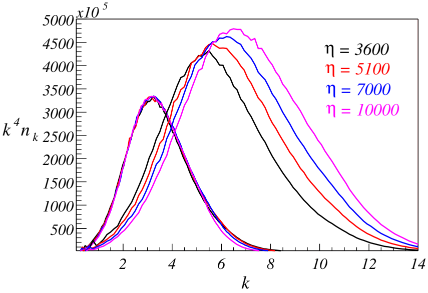


Particle numbers on the lattice in the regime of free turbulence evolve self-similarly

$$n(k,\eta) = \tau^{-q}\, n_0(k\tau^{-p})$$

with $p = \frac{1}{5}$

Self-similar evolution



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Theory:

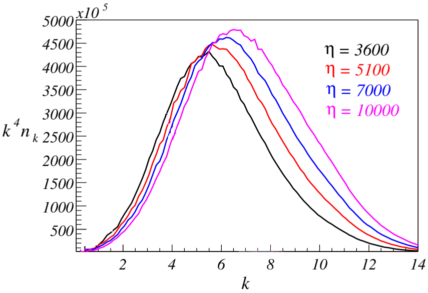
Free turbulence

- $p = \frac{1}{7}$ for 4-particle interactions
- $p = \frac{1}{5}$ for 3-particle interactions

Driven turbulence

- $p = \frac{3}{7}$ for 4-particle interactions
- $p = \frac{2}{5}$ for 3-particle interactions

Self-similar evolution



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Free turbulence

- $p = \frac{1}{7}$ for 4-particle interactions
- $p = \frac{1}{5}$ for 3-particle interactions

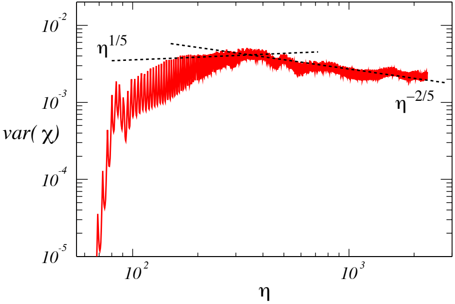
Thermalization:

Position of the peak moves as

$$k(\tau) = k_0\, \tau^p$$

Thermalization will occur when $k^4_{\max} \sim T^4 \sim$ (initial energy).

Field variance



$$var(\chi,\eta) = \tau^{v}\, var(\chi,0)$$

Time dependence of the variance of χ field in the model $h = 10g$

Theory:

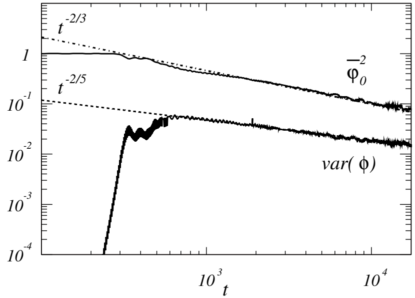
Driven turbulence

- $v = +\frac{1}{7}$ for 4-particle interactions
- $v = +\frac{1}{5}$ for 3-particle interactions

Free turbulence

- $v = -\frac{2}{7}$ for 4-particle interactions
- $v = -\frac{2}{5}$ for 3-particle interactions

Amplitude of the zero mode.



$$\phi_0^2(\eta) = \tau^{-z}$$

Theory:

Free turbulence

- $z = \frac{2}{5}$ for 4-particle interactions
- $z = \frac{2}{3}$ for 3-particle interactions

Summary

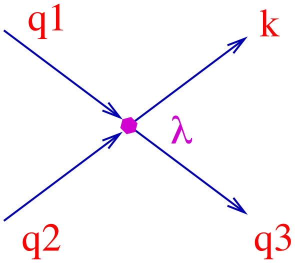
All scaling exponents agree with predictions for 3-particle interactions, which for k-independent matrix elements are

$$
\begin{aligned}
p &= 1/(2m-1) \\
s &= d - m/(m-1) \\
v &= 2/(2m-1) \\
z &= 2/(d(m-1)-m)
\end{aligned}
$$

with $d = 3$ and $m = 3$

Bose-condensate dominates

How 3-particle interactions can appear in $\lambda\phi^4$-theory?



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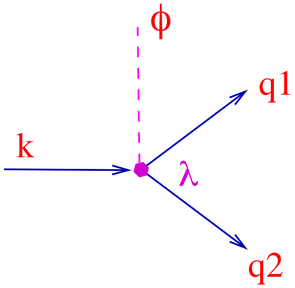


3-particle collision integral can be obtained from the 4-particle one with the substitution

$$\frac{n_p}{\omega_p} \to \frac{n_p}{\omega_p} + (2\pi)^3\delta^{(3)}(\vec{p})\bar{\phi}_0^2$$

Bose-condensation on a lattice

- The condensate quickly recovers to the original value after being set to zero "by hands".
- This prohibited direct check of scaling laws for $m = 4$.
- For a dedicated lattice studies of Bose-condensation see

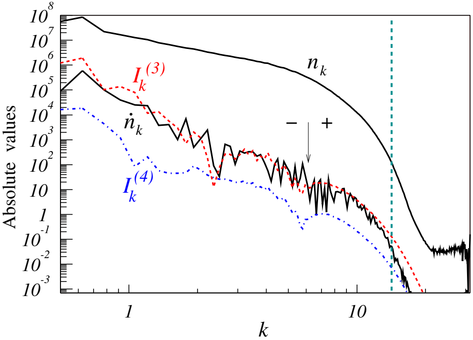
Damle, Najumdar, and Sachdev (1996)

Khlebnikov & I.T. (1999)

Berges & Sexty (2012)

Nowak & Gasenzer (2012)

Test of kinetic description



Collision integrals and $\dot{n}(k)$ at $\eta = 5000$.

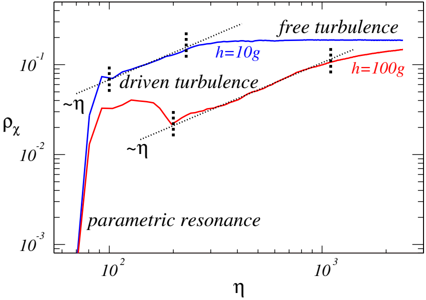
$I_k^{(3)}$ agrees with $\dot{n}(k)$ to the left of the vertical dashed line

Red line: 3-particle collision integral, $I_k^{(3)}$

Blue line: 4-particle collision integral, $I_k^{(4)}$

Three major epochs of reheating

$$V(\phi,\chi) = \frac{1}{4}\phi^4 + \frac{g}{2}\phi^2\chi^2 + \frac{h}{4}\chi^4$$

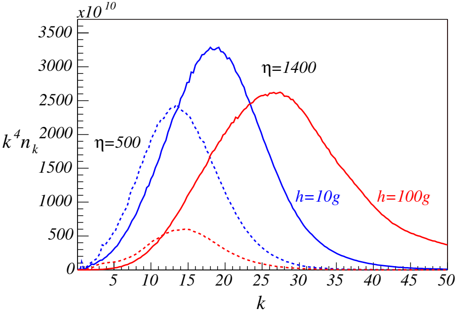


- Parametric resonance
- Driven turbulence
- Free turbulence

At large h and/or g the parametric resonance stops early.

Three major epochs of reheating

$$V(\phi,\chi) = \frac{1}{4}\phi^4 + \frac{g}{2}\phi^2\chi^2 + \frac{h}{4}\chi^4$$



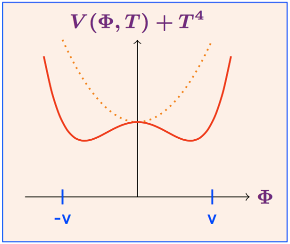
- Parametric resonance
- Driven turbulence
- Free turbulence

But distributions evolve faster at late times.

Non-thermal Phase Transitions

The effective mass of the Higgs field

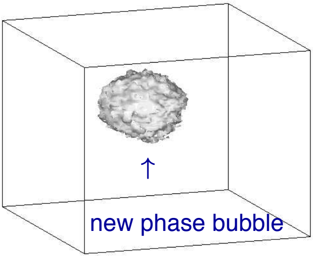
$$m_\Phi^2 = -\mu^2 + g^2 \langle X^2 \rangle$$



- Symmetry is restored when $\langle X^2 \rangle > \frac{\mu^2}{g^2}$
- In thermal equilibrium $\langle X^2 \rangle = \frac{T^2}{12}$
- In turbulent state before thermalization $\langle X^2 \rangle$ is much larger.

This may result in GUT phase transitions even if reheating temperature is low.

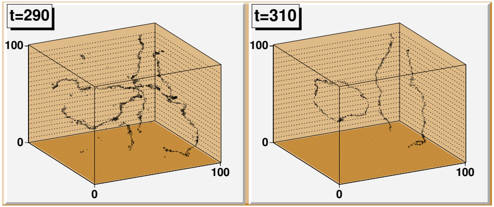
Non-thermal Phase Transitions



First order phase transition

$$V(\phi, X) = \lambda(\phi^2 - v^2)^2 + g^2\phi^2 X^2$$

$g^2/\lambda = 200$



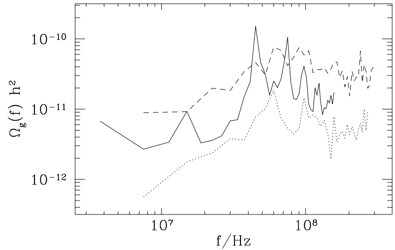
String formation

$$V(\phi) = \lambda(\phi_1^2 + \phi_2^2 - v^2)^2$$

$v \sim 10^{16}$ GeV

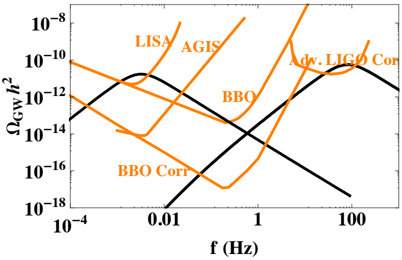
GW spectrum generated by:

Preheating



Khlebnikov & I.T. (1997)

Turbulence from phase transitions with $T_c = 10^2$ GeV and $T_c = 10^7$ GeV



Caprini, Durrer, Servant (2009)

Conclusions

- We identify three different stages of the Universe reheating
  - ``Parametric resonance.'' Fast exponential growth of energy in fluctuations, but only a small fraction of energy is transferred during this stage.
  - Driven turbulence. Linear growth. Major mechanism of energy transfer.
  - Free turbulence. Long stage of thermalization.
- Turbulent evolution is self-similar.
- Bose-condensate of zero mode governs evolution.
- Explicit expressions for particle occupation numbers.
- Estimates for reheating time and temperature.