3. Präsenzübung for quantum mechanics (PTP 4)

To be discussed in the tutorial on the 21st of April

Two points are given for active participation

Q P8: *Fourier transformation*

For a function $\phi(x)$, the Fourier transform is given by

$$\hat{\phi}(k) = \mathcal{F}[\phi(x);k] = \int_{-\infty}^{\infty} dx\, e^{-ikx}\phi(x)\,,$$

and the inverse transformation is

$$\phi(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx}\hat{\phi}(k)\,.$$

Show the following properties of the Fourier transform ($\alpha, \beta \in \mathbb{C}$, $a \in \mathbb{R}$):

$$\begin{array}{ll} (a) & \mathcal{F}[\alpha\phi(x) + \beta\psi(x);k] = \alpha\mathcal{F}[\phi(x);k] + \beta\mathcal{F}[\psi(x);k] \\ (b) & \mathcal{F}[\phi(x-a);k] = e^{-ika}\mathcal{F}[\phi(x);k] \\ (c) & \mathcal{F}[\phi(ax);k] = a^{-1}\mathcal{F}[\phi(x);k/a],\ a>0 \\ (d) & \mathcal{F}[\phi(-x);k] = \mathcal{F}[\phi(x);-k] \end{array}$$

Q P9: **δ-function**

The Θ-function (also known as the step or Heaviside function) is defined by:

$$\Theta(x-a) = \begin{cases} 0 & \text{for} \quad x<a \\ \frac{1}{2} & \text{for} \quad x=a \\ 1 & \text{for} \quad x>a \end{cases}$$

(a) Show that the derivative of the Θ-function has the property of the δ-function:

$$\delta(x-a) = 0 \qquad \text{for } x \neq a. \qquad \int_A f(x)\delta(x-a)dx = f(a) \quad \text{for } a \in A.$$

(b) Show the following relations hold:

$$\begin{aligned} &\delta(-x) = \delta(x) \\ &x\delta(x) = 0 \\ &x\delta'(x) = -\delta(x) \\ &\delta(bx) = b^{-1}\delta(x) \quad \text{for} \quad b>0 \end{aligned}$$

where the identities only have meaning inside the integral, i.e. that $g(x) = h(x)$ exactly when $\int f(x)g(x)dx = \int f(x)h(x)dx$.