Twelfth homework sheet for quantum mechanics

To be handed in on 30.06 in the tutorial.

Note: As already announced in the lectures, the points from Q27 will be awarded as bonus points.
Beware: The second exam is on Saturday, the 04.07.09 from 09.30 until 11:30 in HS 1 and HS 2, INF 227 (KIP).

Q 31: *Two state system (again)* **(7 points)**
Consider a NH_3-molecule: By a measurement one can find the the N-atom above or below the three H-atoms spanning levels. We describe the measurement "position of the N-atoms" through the operator

$$\hat{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The measurement can take the value 1 (N-atom above) or -1 (N-atom below). The analogous Eigenstates of $\hat{\Sigma}$ are

$$|\Psi_u\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\Psi_d\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The Hamiltonian operator of the system has in the basis of both states the form

$$\hat{H} = \begin{pmatrix} E & W \\ W & E \end{pmatrix}$$

a) Determine the normalised energy Eigenstates and their corresponding energies. **(3 points)**

b) The molecules are at time $t = 0$ in state $|\psi_u\rangle$. Determine the expectation value of $\hat{H}$ for arbitrary time t. **(2 points)**

c) Determine the probability that the N-atom with a measurement at time is found at time t above or below. Give the time evolution of the expectation value of $\hat{\Sigma}$.**(2 points)**

Q 32: *Position and momentum space representation* **(9 points)**
Consider the time independent Schrodinger equation for a particle of mass m in a potential $V(x)$ (with 1D movement)

$$\left(\frac{\hat{P}^2}{2m} + V(\hat{Q})\right)|u\rangle = E|u\rangle$$

in the position space representation

$$\frac{p^2}{2m}\langle p|u\rangle + \langle p|V(\hat{Q})|u\rangle = E\langle p|u\rangle$$

a) Show that, for the wavefunction $v(p) = \langle p|u\rangle$ of the momentum representation satisfies the formal integral equation

$$\frac{p^2}{2m}v(p) + \int_{\mathbf{R}} dp' K(p-p')v(p') = Ev(p) \qquad (*)$$

with the kernel function

$$K(p-p') := \frac{1}{2\pi\hbar}\int_{\mathbf{R}} dx\, V(x) \exp\left[-\frac{i}{\hbar}(p-p')x\right]$$

(assuming that the integral defining the kernel exists).
(2 points)
Hint: Use the completeness relation of the momentum Eigenstates and the spectral representation of $V(\hat{Q})$.

b) Consider now a particle of mass m in the attractive δ-shaped potential

$$V(x) = -\frac{\hbar^2}{m} D\delta(x) \quad , \quad D > 0.$$

Solve for this potential the equation (*) for $E < 0$, and show that there is only one non-trivial solution, if $E = -\frac{\hbar^2 D^2}{2m}$. What is the normalised wavefunction $v(p) = \langle p|u\rangle$ of the momentum representation for this bound state?
(4 points)

c) Calculate from $v(p)$ the corresponding wavefunction $u(x) = \langle x|u\rangle$ of the position space representation. **(3 points)**

Hint:

$$\int_R \frac{dx}{(x^2+\alpha^2)^2} = \frac{\pi}{2\alpha^3} \; , \qquad \int_R dx\, \frac{e^{i\beta x}}{x^2+\alpha^2} = \frac{\pi}{\alpha} e^{-\alpha|\beta|} \; , \quad \alpha \in \mathbb{R}^+, \, \beta \in R$$

Q 33: *A time dependent perturbed system* **(6 points)**
A charged Harmonic oscillator (charge q, mass m, angular frequency ω) is at time $t_0 = -\infty$ in its groundstate. In the time interval $(-\infty, +\infty)$ it feels the force of one of the time dependent homogenous electric fields:

(a)

$$E(t) = \frac{A}{\tau_0} e^{-\frac{t^2}{\tau_0^2}}$$

(b)

$$E(t) = \frac{A}{\tau_0} e^{-i\Omega t - \frac{|t|}{\tau_0}}$$

Calculate for both cases at first order in time dependent perturbation theory the probability that the oscillator at time $t = +\infty$ is found in the n-th energy Eigenstate ($n \neq 0$). Discuss the results.