2. Homework sheet for PTP 4 (Quantum mechanics)

To be handed in on the 21.4 or 22.4 in the tutorial

Q 4: *Change of basis* **(6 points)**

$\{|a_1\rangle, |a_2\rangle\}$ form an orthonormal basis for a two-dimensional complex Hilbert space (the $\{a\}$ representation basis). In the Präsenzübungen you have already shown that the vectors

$$|b_1\rangle = \frac{1}{\sqrt{2}}\left(|a_1\rangle + i|a_2\rangle\right) \qquad |b_2\rangle = \frac{1}{\sqrt{2}}\left(|a_1\rangle - i|a_2\rangle\right)$$

also form an orthonormal basis (the $\{b\}$ representation basis).

a) Let $\hat{U}$ be the unitary change of basis operator, which changes from the $\{a\}$ representation to the $\{b\}$ representation

$$|b_1>=\hat{U}|a_1> \qquad \text{and} \qquad |b_2>=\hat{U}|a_2>$$

Which matrix corresponds to $\hat{U}$ in the $\{a\}$ representation? **(2 points)**

b) A vector $|\psi>$ is given in the $\{a\}$ representation by

$$|\psi>=\frac{1}{\sqrt{2}}\left(|a_1> + |a_2>\right) \qquad .$$

What are the components of $|\psi>$ in the $\{b\}$ representation, i.e. write $|\psi>$ as a linear combination of the basis vectors $|b_k>$. **(2 points)**

c) A linear operator $\hat{T}$ is given in the $\{a\}$ representation by the matrix

$$\mathbf{T^{(a)}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

What is the matrix $\mathbf{T^{(b)}}$, i.e. operator $\hat{T}$ in the b representation? **(2 points)**

Q 5: *Neutrino Oscillations* **(9 points)**

This question considers oscillations between electron neutrinos ν_e and muon neutrinos ν_μ. We assume that the neutrinos are so light that we can use the following relation between the energy E, momentum p and mass m:

$$E = \sqrt{p^2c^2 + m^2c^4} \approx pc + \frac{m^2c^4}{2pc}.$$

We further assume it is a good approximation to assume that neutrinos travel at the speed of light. Let $\hat{H}$ be the Hamiltonian operator of a free neutrino with momentum p and let $|\nu_1\rangle$ and $|\nu_2\rangle$ be the two Eigenvectors of $\hat{H}$:

$$\hat{H}|\nu_j\rangle = E_j|\nu_j\rangle \qquad E_j = pc + \frac{m_j^2c^4}{2pc}, \qquad j = 1,2$$

Here m_1 and m_2 are the masses of the Eigenstates $|\nu_1\rangle$ and $|\nu_2\rangle$, which we assume are not equal. Neutrino oscillations are due to a quantum mechanical effect whereby detected neutrinos are neither in the state $|\nu_1\rangle$ nor $|\nu_2\rangle$, but instead linear combinations of these two states:

$$|\nu_e\rangle = |\nu_1\rangle \cos(\theta) + |\nu_2\rangle \sin(\theta), \qquad |\nu_\mu\rangle = -|\nu_1\rangle \sin(\theta) + |\nu_2\rangle \cos(\theta)$$

Here θ is the so called mixing angle, which must be experimentally determined.

a) At time $t = 0$ a neutrino in state $|\nu_e\rangle$ with momentum p is created. Calculate the state $|\nu(t)\rangle$ at time t in the basis $|\nu_1\rangle, |\nu_2\rangle$, i.e. write this as a linear combination of $|\nu_1\rangle$ and $|\nu_2\rangle$. **(2 points)**

b) The likelihood, $P_e(t)$, that a neutrino at time t is found in state $|\nu_e\rangle$ is given by $P_e(t) = |\langle\nu_e|\,\nu(t)\rangle|^2$. Show that

$$P_e(t) = 1 - \sin^2(2\theta)\sin^2\left(\frac{\pi c t}{L}\right)$$

with the corresponding oscillation length scale $L = \frac{4\pi\hbar p}{|\Delta m^2|c^2}$ and $\Delta m^2 = m_1^2 - m_2^2$. **(4 points)**

c) Calculate the oscillation length L for an energy $E \approx pc = 4\text{MeV}$ (average energy of reactor neutrinos) and a mass difference $\Delta m^2 c^4 = 10^{-4}\,\text{eV}^2$.
(1 point)

d) The neutrino flux is measured with a detector at a distance l from the neutrino source. Calculate P_e as a function of l. **(1 point)**

e) From a large number of experiments we know that $|\Delta m^2|c^4 = 7.1(\pm 0.4)\cdot 10^{-5}\,\text{eV}^2$ and $\tan^2(\theta) = 0.45(\pm 0.02)$. Show that these values are consistent with the data from the KamLAND Experiment (see the figure, take $l = 180\,\text{km}$ and $E = pc \approx 4\,\text{MeV}$). **(1 point)**

For information about the KamLAND experiment see for example **http://kamland.lbl.gov/** and
http://www.pro-physik.de/Phy/leadArticle.do?laid=5437