3. Homework sheet for quantum mechanics

To be handed in on the 28th of April

Q 6: *Fourier transformation* **(6 points)**

The Fourier transform of a function $\phi(x)$ is given by

$$\hat{\phi}(k) = \mathcal{F}[\phi(x); k] = \int_{-\infty}^{\infty} dx\, e^{-ikx} \phi(x)\,.$$

The inverse transformation is then

$$\phi(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \hat{\phi}(k)\,.$$

a) Calculate the Fourier transformation of $\phi_a(x) = \Theta(x)\Theta(a-x)$ (defined in question P9). **(2 points)**

b) Calculate the Fourier transformation of $\phi(x) = \Theta(x+a)\Theta(a-x)$. Sketch the result and calculate the product $\Delta x \cdot \Delta k$ for suitably defined amplitudes Δx and Δk (provide the definitions as well). **(2 points)**

c) Calculate the Fourier transformation of the Gauss function $f(x) = Ne^{-\frac{1}{2}cx^2}$. Here N and c are real, positive constants. **(2 points)**

Q 7: *"Fourier" representation of the δ-function* **(4 points)**

The function $\delta_\epsilon(x-a)$ is given by:

$$\delta_\epsilon(x-a) = \frac{1}{2\epsilon}\Theta(a+\epsilon-x)\Theta(x-a+\epsilon).$$

The limit $\epsilon \to 0$ gives a representation of the δ function:

$$\delta(x-a) = \lim_{\epsilon \to 0} \delta_\epsilon(x-a).$$

Use this to learn a further, important representation of the δ function: Calculate the Fourier transformation of $\delta_\epsilon(x-a)$ and its limit $\epsilon \to 0$. The inverse transformation gives the "Fourier"' representation of $\delta(x-a)$:

$$\delta(x-a) = \frac{1}{2\pi} \int e^{ik(x-a)} dk.$$

Q 8: *Time independent Schrodinger equation, 1-dim. problem* **(4 points)**

Given a one-D potential $V(x)$, consider the corresponding 1-D time independent Schrodinger equation (see the lectures)

$$i\hbar \frac{\partial}{\partial t}\Psi(x,t) = \hat{H}\Psi(x,t) \qquad \text{with} \qquad \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

a) Make the separability ansatz

$$\Psi(x,t) \;=\; f(t)\,\psi(x).$$

Determine $f(t)$ and show that $\psi(x)$ is an Eigenfunction of the Hamiltonian operator $\hat{H}$, i.e. it satisfies

$$\hat{H}\psi(x) \;=\; E\psi(x) \qquad (*)$$

with a constant E. **(2 points)**
Note: $(*)$ denotes the time independent Schrodinger equation.

b) The potential $V(x)$ has at the position $x = a$ a discontinuity as shown in Fig. 1 . Let $\psi_I(x)$ be a solution of $(*)$ in the region $x \leq a$ and $\psi_{II}(x)$ be a solution of $(*)$ in region $x \geq a$. Explain why the solution of $(*)$ at the jump $x = a$ satisfies the matching conditions

$$\psi_I(a) = \psi_{II}(a) \qquad \text{and} \qquad \psi_I'(a) = \psi_{II}'(a).$$

(2 points)
Hint: Consider if $\psi(x)$ or $\psi'(x)$ had near $x = a$ a behaviour $\sim \Theta(x-a)$ and consider what consequences this has for $\psi''(x)$.

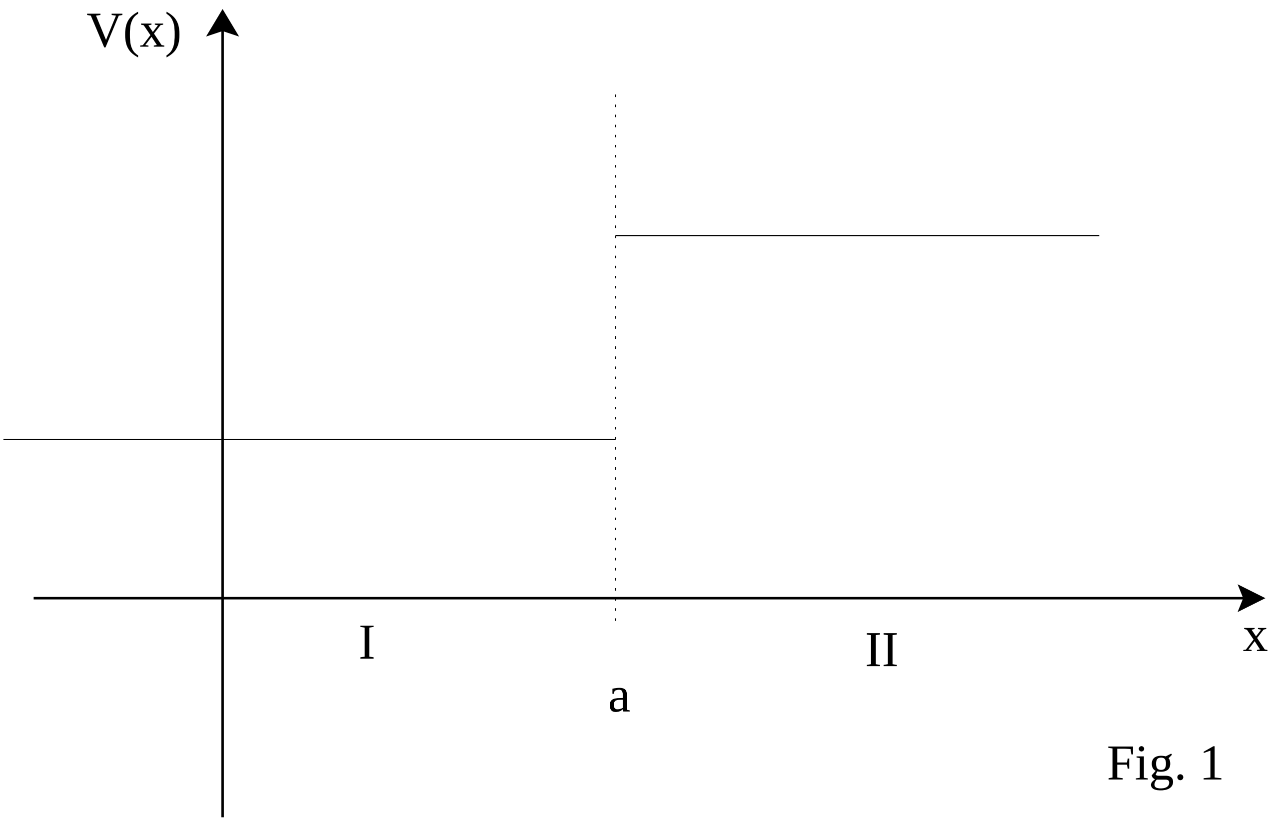

Fig. 1

Q 9: *1-D potential step* **(6 points)**

A wave of particles of mass m and energy $E < V_0$ travel from the left in a positive x direction to the potential step

$$V(x) = V_0\Theta(x) \qquad \text{with the constant} \qquad V_0 > 0 \quad .$$

a) Show that:

$$\psi(x) = \begin{cases} e^{ikx} + re^{-ikx} & \text{for} \quad x < 0 \quad \text{(region I)} \\ te^{-\kappa x} & \text{for} \quad x > 0 \quad \text{(region II)} \end{cases}$$

is a solution of the corresponding time independent Schrodinger equation. Determine k and κ in terms of E and V_0, and calculate r and t as functions of k and κ.

Instructions: Use the given matching conditions from Q 8b) for ψ. In region II consider only solutions which satisfy $\int_0^\infty dx|\psi(x)|^2 < \infty$. **(4 points)**

Remark: It can be taken without loss of generality that the amplitude of the incoming wave from the left is unity.

b) Calculate $|r|^2$ and interpret the result. What does $t \neq 0$ mean? **(1 point)**

c) Consider the limit of an infinitely high potential step $V_0 \to \infty$. Calculate for this limit r and t and show that one then requires $\psi(0) = 0$. **(1 point)**

Remark: This is the general condition for an infinitely high potential well.