5. Homework sheet for quantum mechanics

To be handed in on the 12.5

Q 13: *Potential barrier, tunnel effect* **(10 points)**
A current of particles of mass m and energy E travel from the left in the positive x-direction to the potential barrier

$$V(x) = V_0\Theta(x)\Theta(L-x)$$

with height $V_0 > 0$ and width $L > 0$.

a) As in Q9 one can make the ansatz (NB ansatz is a relatively rare example of a German word used unaltered in English)

$$\psi(x) = \begin{cases} e^{ik_0x} + re^{-ik_0x} & \text{for} \quad x < 0 \quad \text{(region I)} \\ Ae^{ikx} + Be^{-ikx} & \text{for} \quad 0 < x < L \quad \text{(region II)} \\ te^{ik_0x} & \text{for} \quad x > L \quad \text{(region III)} \end{cases}$$

for the solution of the time independent Schrodinger equation with energy $E > V_0$. We can assume without loss of generality that the amplitude of the incoming wave from the left is unity.

How do k_0 and k depend on E and V_0? **(1 point)**

b) Calculate the probability density current j_I and j_{III} in the regions I and III as functions of r and t. Show that the probability of transmission $T := j_{\text{trans}}/j_{\text{in}}$ is equal to $|t|^2$.
Remark: One has $j_I = j_{\text{in}} - j_{\text{refl}}$ with j_{in} = the probability density current of the incoming particles, j_{refl} = the probability density current of the reflected partices and $j_{III} = j_{\text{trans}}$ = the probability density current of the transmitted particles. **(1 point)**

c) Calculate the transmission probability T as a function of $\hbar, E, V_0, L$ and m for the case $E > V_0$. Discuss the result:

$$T = \frac{1}{1 + \frac{V_0^2}{4E(E-V_0)} \sin^2\left(\frac{L}{\hbar}\sqrt{2m(E-V_0)}\right)}$$

Remark: Consider in your discussion the special case $V_0 \to 0$, $L \to 0$, $m \to 0$, $E \to \infty$ and $E \to V_0$. When is $T = 1$? Sketch T as a function of $\kappa := \sqrt{\frac{2m}{\hbar^2}(E - V_0)}$. What is given in the limit of $\kappa \to 0$? **(5 points)**

d) What is T in the case $E < V_0$ (you can solve this without lengthy calculations) and discuss the result. Consider especially the limit $\kappa L \gg 1$. What behaviour does $T(\kappa)$ have in this case ? **(2 points)**

e) Calculate T for electrons with $E = 1\text{eV}$, for $V_0 = 2\text{eV}$ and $L = 10^{-10}\text{m}$ as well as protons with the same kintic energy for the same barrier.
(1 point)

Q 14: *Harmonic oscillator* **(10 points)**

In the lectures you have considered the stationary solution of the Schrodinger equation for a harmonic oscillator in 1D using the algebraic method of the creation and annihilation operators

$$\hat{a} = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\,\hat{Q} + \frac{i}{\sqrt{m\omega}}\,\hat{P}\right), \qquad \hat{a}^{+} = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\,\hat{Q} - \frac{i}{\sqrt{m\omega}}\,\hat{P}\right)$$

with $\hat{Q} = x$ and $\hat{P} = -i\hbar\frac{d}{dx}$.

a) The dimensionless length is defined by $\xi := \frac{x}{x_0}$ with $x_0 = \sqrt{\frac{\hbar}{m\omega}}$
Show that $\hat{a} = \frac{1}{\sqrt{2}}\left(\xi + \frac{d}{d\xi}\right)$ and $\hat{a}^{+} = \frac{1}{\sqrt{2}}\left(\xi - \frac{d}{d\xi}\right)$. **(1 point)**

From $\hat{a}|0\rangle = 0$ one gets for the ground state wave function $\psi_0(\xi)$ (see the lectures)

$$\psi_0(\xi) = (\pi)^{-\frac{1}{4}}\, e^{-\frac{1}{2}\xi^2}$$

b) What meaning does the length x_0 have for the classical oscillator ? **(1 point)**
Tip: Consider a classical harmonic oscillator with the total energy $E_0 = \frac{1}{2}\hbar\omega$.

c) $|\psi_0(\xi)|^2 d\xi$ is the probability to find a particle in the interval $[\xi, \xi + d\xi]$. Give the probability of finding a particle in the interval $[x, x + dx]$! **(1 point)**

d) Show that the wave function of the excited state is given by
(4 points)

$$\psi_n(\xi) = \frac{1}{\sqrt{2^n n!}}\left(\frac{1}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\xi^2} H_n(\xi)$$

Tip: $\psi_n(\xi)$ is given by repeated application of a^{+} on the ground state wave function $\psi_0(\xi)$. Consider and use the operator equation

$$e^{-\frac{\xi^2}{2}}\left(\xi - \frac{d}{d\xi}\right)^n e^{\frac{\xi^2}{2}} = (-1)^n \frac{d^n}{d\xi^n}$$

Finally use the demonstrated representation for $H_n(\xi)$ in the Präsenzaufgabe P11 c) .

e) Sketch and discuss the progression of the first three energy states $\psi_0(\xi), ..., \psi_2(\xi)$. **(2 points)**

f) What is the probability that the particle in the groundstate ψ_0 is outside the classically allowed intervals (= interval between the turning points of a classical harmonic oscillator with total energy $E_0 = \frac{1}{2}\hbar\omega$)? **(1 point)**
(Remark: you may use: $\text{erf}(1) = \frac{2}{\sqrt{\pi}}\int_0^1 e^{-\xi^2} d\xi = 0.8427$).