6. Homework sheet for quantum mechanics

To be handed in on the 19.05.

Q 15: *δ-shaped potential* **(2 points)**

Consider a particle of mass m moving within a (one-dimensional) potential which is strongly repulsive or attractive within a small range around the point $x = x_0$. Such a scenario can be idealised by means of the potential

$$V(x) = -\frac{\hbar^2}{m} D\delta(x - x_0) + V_1(x)$$

where $D \in \mathbb{R}$ and $D > 0$ for an attractive and $D < 0$ for a repulsive potential. $V_1(x)$ is assumed to be continuous in the region of $x = x_0$. Assuming that the wavefunction is continuous in the region near $x = x_0$, show that the first derivative of the wavefunction has a step near $x = x_0$ of

$$\lim_{x \to x_0, x > x_0} (\psi'(x)) - \lim_{x \to x_0, x < x_0} (\psi'(x)) = -2D\,\psi(x_0).$$

Hint: Integrate the time-independent Schrodinger equation over the interval $x_0 - \epsilon < x < x_0 + \epsilon$ and consider the limit $\epsilon \searrow 0$.

Q 16: *Finite depth, rectangular potential well* **(5 points)**

Consider the following 1D square well potential

$$V(x) = \begin{cases} -V_0 & \text{for} \quad |x| < a \\ 0 & \text{for} \quad |x| > a \end{cases}$$

with $V_0 > 0$, $a > 0$. The wavefunctions $\psi_n(x)$ of bound energy Eigenstates of a particle in the reflection invariant potential are either odd or even functions of the position variable x.

Show the following:

$$\kappa a = K a \tan(Ka) \quad \text{for even Eigenfunctions} \tag{1}$$

$$\kappa a = -K a \cot(Ka) \quad \text{for odd Eigenfunctions} \tag{2}$$

$$(Ka)^2 + (\kappa a)^2 = \frac{1}{\hbar^2} 2\, m\, V_0\, a^2\,, \tag{3}$$

with $K = \frac{1}{\hbar}\sqrt{2m(V_0 - |E|)}$ and $\kappa = \frac{1}{\hbar}\sqrt{2m|E|}$

for the energy Eigenvalues $-V_0 < E < 0$ of bound states of a particle of mass m in this potential. Does the potential have bound states for an arbitrary V_0 and a? Justify your answer.

Comment and tip: Sketch the Eigenvalue conditions in a suitably chosen coordinate system ($\rightarrow$ graphical solution).

Q 17: *Tunnelling system* **(5 points)**
Since the seventies, double-well potentials and the influence of the tunnel effect on the energy spectrum of such configurations have been discussed, in order to explain the properties of glass at low temperatures. A simplified version of the double well is given by the following potential:

$$V(x) = \begin{cases} b\delta(x), & \text{for} \quad |x| \le a \\ \infty & \text{for} \quad |x| > a. \end{cases}$$

Determine the possible energy Eigenvalues and their corresponding Eigenfunctions.

Remark: The Eigenfunctions are either odd or even functions of the position variable x. In the case of an even function one gets for the energy Eigenvalues a transcendental conditional equation. A solution of this equation is not required. Instead sketch how it could be solved graphically.

Q 18: *Harmonic oscillator, occupation number representation* **(8 points)**

Consider again a harmonic oscillator (see Q 14).

a) Write down the position operator $\hat{Q}$ and the momentum operator $\hat{P}$ in terms of the annihilation and creation operators $\hat{a}$ and $\hat{a}^+$. **(2 points)**

b) Write down the operators $\hat{Q}^2$ and $\hat{P}^2$ in terms of the annihilation and creation operators $\hat{a}$ and $\hat{a}^+$. **(2 points)**

c) Calculate for the state $|n\rangle$, $n \in \mathbb{N}$, (occupation number representation!) the expectation values

$$\langle \hat{Q}^2 \rangle = \left\langle n \,\middle|\, \hat{Q}^2 \,\middle|\, n \right\rangle \qquad \text{and} \qquad \langle \hat{P}^2 \rangle = \left\langle n \,\middle|\, \hat{P}^2 \,\middle|\, n \right\rangle$$

Give the expectation values of the kinetic and potential energies. **(2 points)**

d) Calculate the position uncertainty $\Delta\hat{Q}$ and the momentum uncertainty $\Delta\hat{P}$ for the state $|n\rangle$.
What is given by $\Delta\hat{Q} \cdot \Delta\hat{P}$? **(2 points)**

"Ich habe hundertmal so viel über Quantenprobleme nachgedacht wie über die allgemeine Relativitätstheorie."

"I have thought a hundred times more over quantum mechanics than I have over GR."

A. Einstein zu Otto Stern, zitiert in Pais, "Einstein, Newton und der Erfolg"