7th homework sheet for quantum mechanics

To be handed in on the 26.05.

Q 19: *Angular momentum, matrix representation* **(8 points)**

In the lectures you have been introduced to the operators $L_+ = L_x + iL_y$ and $L_- = L_x - iL_y$.

a) Show that

$$L_+L_- = L_x^2 + L_y^2 + \hbar L_z$$
$$L_-L_+ = L_x^2 + L_y^2 - \hbar L_z$$

(1 point)

b) Let $|l,m\rangle$ be the Eigenstate of both L_z and $\vec{L}^2$ with $L_z\,|l,m\rangle = \hbar|l,m\rangle$ and $\vec{L}^2\,|l,m\rangle = \hbar^2 l(l+1)\,|l,m\rangle$ (see the lectures). Show that

$$L_\pm\,|l,m\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}\,|l,m\pm 1\rangle$$

(2 points)

c) Determine the matrix representation of the angular momentum operators L_x, L_y as well as L_+ and L_- in the basis of the angular momentum Eigenstates $|l,m\rangle$ for $l = 1$.
Tip: Calculate L_x and L_y in terms of L_+ and L_- and use the result in section b).
(5 points)

Q 20: *Gaussian wave packet* **(12 points)**

Remark: This exercise requires a lot of calculation but is a typical and informative problem. The following integrals can often be solved in terms of the standard integral

$$\int_{-\infty}^{\infty} dx\, e^{-c(x-d)^2} = \sqrt{\frac{\pi}{c}}$$

with the in general complex constants $c, d \in \mathbb{C}$, $Re(c) > 0$ or sometimes through simple symmetry arguments

Consider a 1D problem of a force free particle of mass m. The corresponding time dependent Schrodinger equation is

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t). \quad (1)$$

a) Show that the 1D plane wave

$$\psi(x,t) = \alpha \cdot \exp\left\{\frac{i}{\hbar}\left(px - \frac{p^2}{2m}t\right)\right\}$$

(with a constant α) is a solution of the time dependent Schroedinger equation (1). **(1 point)**

The general solution of the Schrodinger equation (1) is then given by a superposition of such plane waves

$$\psi(x,t) \;=\; \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar}\,\phi(p)\exp\left\{\frac{i}{\hbar}\left(px - \frac{p^2}{2m}t\right)\right\} \tag{2}$$

One such superposition of planes waves is described as a *wavepacket*.

b) The amplitude function $\phi(p)$ allows one to determine the initial conditions $\psi(x, t = 0)$. Explain how! **(2 points)**

Now consider the amplitude function of a so called 1D Gaussian wave packet

$$\phi(p) \;=\; A\exp\left\{-\frac{(p-p_0)^2\; d^{\,2}}{\hbar^2}\right\} \tag{3}$$

with the constants $A, d, p_0 \in \mathbb{R}$.

c) Use this Gaussian amplitude function in (2) and perform the required integration. Then determine $|\psi(x, t)|^2$. **(4 points)**
Use the shortened notation $v = \frac{p_0}{m}$ and $\delta_t = \frac{t\hbar}{2md^2}$.

d) Determine the constant A, so that $\int_{-\infty}^{\infty} dx|\psi(x, t)|^2 = 1$. **(1 point)**

e) Calculate $< x >$ and $\Delta x = \sqrt{< (x- < x >)^2 >}$ as time dependent functions. Sketch $|\psi(x, 0)|^2$ and $|\psi(x, t)|^2$ for $t > 0$ and describe how the form of $|\psi(x, t)|^2$ varies with time. **(4 points)**

from Kurt Baumann, Roman U. Sexl: Die Deutungen der Quantentheorie, Vieweg 1984, p.4