8TH QUANTUM MECHANICS HOMEWORK SHEET

To be handed in on the 02.06.

Q 21: *Laguerre Polynomial* **(5 points)**

In the lectures you have learnt about the bound states of the Coulomb potential (without spin or relativistic corrections)

$$\psi_{nlm}(r,\theta,\phi) = \psi_{nl}(r)Y_{lm}(\theta,\phi)$$

where

$$\psi_{nl}(r) = -\frac{2}{n^2}\sqrt{\frac{(n-l-1)!}{((n+l)!)^3}}\left(\frac{Z}{a}\right)^{3/2}\left(\frac{2Z}{na}r\right)^{l} e^{-\frac{Z}{na}r}\, L_{n+l}^{2l+1}\left(\frac{2Z}{na}r\right)$$

with $a := \frac{\hbar^2}{me_0^2}$.

$L_n^m(x)$ are the associated Laguerre polynomials, that are given by

$$L_n^m(x) \;=\; \frac{d^m}{dx^m}L_n(x) \;=\; \frac{d^m}{dx^m}\left(e^x\frac{d^n}{dx^n}\left(x^n e^{-x}\right)\right)$$

with $m,n \in \mathbb{N}_0$ and the Laguerre polynomials $L_n(x) = e^x\frac{d^n}{dx^n}\left(x^n e^{-x}\right)$.

The associated Laguerre polynomials satisfy the orthogonality relation

$$\frac{(n-m)!}{(n!)^3}\int_0^\infty x^m e^{-x}\, L_n^m(x)\, L_p^m(x)\, dx \;=\; \delta_{pn}$$

as well as the differential equation

$$x\,\frac{d^2y}{dx^2} \;+\; (m+1-x)\,\frac{dy}{dx} \;+\; (n-m)y \;=\; 0\,.$$

a) Calculate the associated Laguerre polynomials $L_2^1(x)$,$L_2^2(x)$,$L_3^2(x)$, $L_3^4(x)$. **(2 points)**

b) Verify the orthogonality relation for $m = 2$, $n = 2$, $p = 3$, and again for the case $m = 1$, $n = 2$, $p = 2$. **(2 points)**

c) Verify that $L_3^2(x)$ satisfies the given differential equation for $m = 2, n = 3$. **(1 point)**

Q 22: *3D isotropic harmonic oscillator* **(15 points)**

Consider a particle in the 3D oscillator potential $V(\vec{r}) = \frac{1}{2}m\omega^2 r^2$ with $r^2 = |\vec{r}|^2$ (see the lectures). The Hamiltonian of the system is given by

$$\hat{H} = \sum_{i=x,y,z}\hat{H}_i \qquad \text{mit} \qquad \hat{H}_i = \frac{1}{2m}\left(\hat{P}_i^2 + m^2\omega^2\hat{Q}_i^2\right)$$

with $i = x, y, z$. If U_i is the state space belonging to the conjugate pairs $\hat{P}_i, \hat{Q}_i$, then the state space of the complete system is given by the tensor product $U = U_x \otimes U_y \otimes U_z$. One defines for every conjugate pair $\hat{Q}_i$, $\hat{P}_i$, $i = x, y, z$ (in analogue to the 1D case, see the lectures) the ladder operators $\hat{a}_i$, $\hat{a}_i^\dagger$:

$$\hat{a}_i = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{m\omega}\, \hat{Q}_i + \frac{i}{\sqrt{m\omega}} \hat{P}_i \right), \qquad \hat{a}_i^+ = \frac{1}{\sqrt{2\hbar}} \left(\sqrt{m\omega}\, \hat{Q}_i - \frac{i}{\sqrt{m\omega}} \hat{P}_i \right)$$

They satisfy the commutation relation

$$[\hat{a}_i, \hat{a}_j] = \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] = 0, \qquad \left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{i,j}$$

The corresponding number operators are $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$. If $|n_i\rangle$ are the Eigenvectors of the Hamiltonian $\hat{H}_i$, then $|n_x n_y n_z\rangle = |n_x\rangle |n_y\rangle |n_z\rangle$ form a complete orthonormal system in U.
If $|000\rangle$ is the Eigenvector of the ground state, then

$$\begin{aligned} \hat{a}_x|000\rangle &= \hat{a}_y|000\rangle = \hat{a}_z|000\rangle = 0 \\ |n_x n_y n_z\rangle &= (n_x! n_y! n_z!)^{-\frac{1}{2}}\, \hat{a}_x^{\dagger n_x} \hat{a}_y^{\dagger n_y} \hat{a}_z^{\dagger n_z} |000\rangle \end{aligned} \tag{1}$$

From the lectures you know that for a central potential $\hat{H}, \hat{\vec{L}}^2$ and $\hat{L}_z$ form a complete set of commuting observables as well. The complete set of Eigenvectors are labelled by the quantum numbers n, l and m with the corresponding Eigenvalues E_n, $\hbar^2 l(l+1)$ and $\hbar m$. The states $|nlm\rangle$ result from $|n_x n_y n_z\rangle$ via a unitary transformation.

(a) Write the operators $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ in terms of the operators $\hat{a}_i^\dagger$ and $\hat{a}_i$ ($i = x, y, z$). **(3 points)**

(b) Consider the states with the energy $E = \hbar\omega(1 + \frac{3}{2})$. The corresponding Eigenvectors of $\hat{H}$ in the $|n_x n_y n_z\rangle$ representation are then $|100\rangle$,$|010\rangle$,$|001\rangle$. These form a basis in the subspace of all Eigenvectors of $\hat{H}$ with Eigenvalue $E = \hbar\omega(1 + \frac{3}{2})$. Determine the matrix representation of $\hat{L}_z$ and calculate the corresponding Eigenvalues and Eigenvectors (as a linear combination of the states $|100\rangle$,$|010\rangle$,$|001\rangle$) from $\hat{L}_z$. **(5 points)**

(c) Show that the Eigenvectors constructed in (b) as Eigenvectors of $\hat{L}_z$ are also Eigenvectors of $\hat{\vec{L}}^2$ with Eigenvalue $2\hbar^2$ (i.e. $l = 1$). For doing this, write $\hat{\vec{L}}^2$ as a function of $\hat{a}_i^\dagger$ and $\hat{a}_i$ and let $\hat{\vec{L}}^2$ act directly on the Eigenvectors. **(3 points)**

(d) Determine the position space representation of the states $\langle\vec{r}|100\rangle$,$\langle\vec{r}|010\rangle$,$\langle\vec{r}|001\rangle$ and show that the Eigenvectors from (b) of $\hat{L}_z$, which are linear combinations of these functions, give

$$\psi_{nlm}(\vec{r}) = \text{const} \quad r e^{-\frac{1}{2}\alpha^2 r^2} Y_l^m(\theta, \phi)$$

with $l = 1$ und $m = 0, \pm 1$, $\alpha = \sqrt{\frac{m\omega}{\hbar}}$. **(4 points)**