Functional renormalization group equation for strongly correlated fermions

collective degrees of freedom

Hubbard model

- Electrons on a cubic lattice

  here : on planes (d = 2)

- Repulsive local interaction if two electrons are on the same site
- Hopping interaction between two neighboring sites

In solid state physics : “ model for everything “

- Antiferromagnetism
- High T_c superconductivity
- Metal-insulator transition
- Ferromagnetism

Hubbard model

Functional integral formulation

$$
\begin{aligned}
Z[\eta] \;=\; & \int_{\hat{\psi}(\beta)=-\hat{\psi}(0),\hat{\psi}^*(\beta)=-\hat{\psi}^*(0)} \mathcal{D}(\hat{\psi}^*(\tau),\hat{\psi}(\tau)) \\
& \exp\Big(-\int_0^\beta d\tau\big(\sum_{\mathbf{x}} \hat{\psi}^\dagger_{\mathbf{x}}(\tau)(\frac{\partial}{\partial\tau}-\mu)\hat{\psi}_{\mathbf{x}}(\tau) \\
& \quad + \sum_{\mathbf{xy}} \hat{\psi}^\dagger_x(\tau)\mathcal{T}_{\mathbf{xy}}\hat{\psi}_{\mathbf{y}}(\tau) \\
& \quad + \frac{1}{2}U\sum_{\mathbf{x}}(\hat{\psi}^\dagger_{\mathbf{x}}(\tau)\hat{\psi}_{\mathbf{x}}(\tau))^2 \\
& \quad - \sum_{\mathbf{x}}(\eta^\dagger_{\mathbf{x}}(\tau)\hat{\psi}_{\mathbf{x}}(\tau)+\eta^T_{\mathbf{x}}(\tau)\hat{\psi}^*_{\mathbf{x}}(\tau))\big)\Big)
\end{aligned}
$$

next neighbor interaction

$$
\mathcal{T}_{xy} = \begin{cases} -t & \text{, if } \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ are nearest neighbors} \\ 0 & \text{, else} \end{cases}
$$

External parameters
T : temperature
μ : chemical potential
(doping)

U > 0 :
repulsive local interaction

Fermi surface

Fermion quadratic term

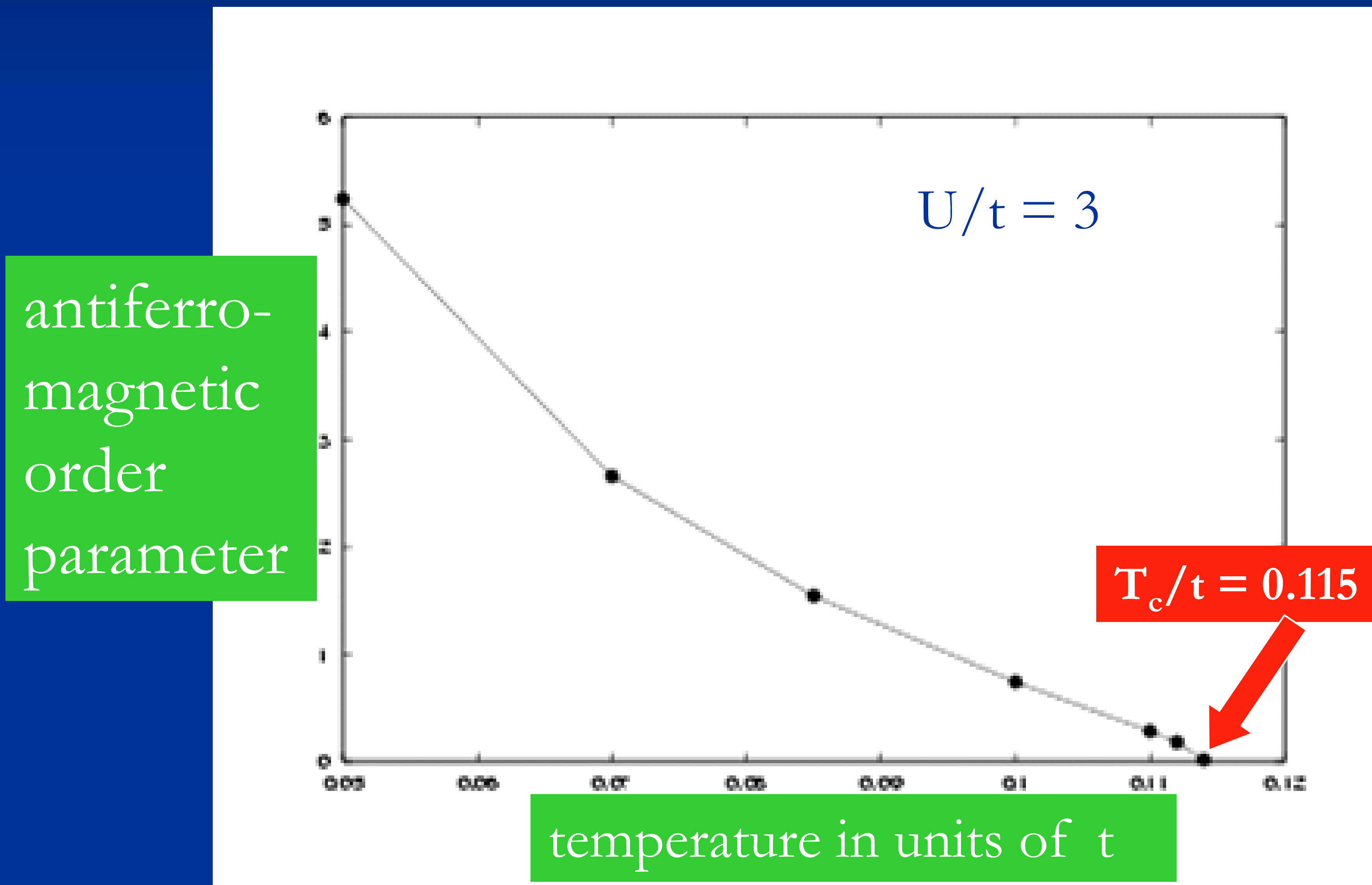
$$S_{F,\mathrm{kin}} = \sum_Q \hat{\psi}^\dagger(Q) P(Q) \hat{\psi}(Q)$$

$$P(Q) = i\omega_F - \mu - 2t(\cos q_1 + \cos q_2)$$

$$\omega_F = (2n+1)\pi T$$

Fermi surface : zeros of P for T=0

Antiferromagnetism in d=2 Hubbard model



Collective degrees of freedom are crucial !

for $T < T_c$

- nonvanishing order parameter

$$\tilde{\vec{m}}(X) = \hat{\psi}^\dagger(X)\vec{\sigma}\hat{\psi}(X)$$

$$\hat{\vec{m}}(Q) \rightarrow \vec{a}\delta(Q - \Pi)$$

- gap for fermions

- low energy excitations: antiferromagnetic spin waves

QCD :
Short and long distance degrees of freedom are different !

Short distances : quarks and gluons
Long distances : baryons and mesons

How to make the transition?

confinement/chiral symmetry breaking

Nambu Jona-Lasinio model

$$S = \int d^4x \left\{ i \bar{\psi}^i_a \gamma^\mu \partial_\mu \psi^i_a \right.$$

$$\left. + 2\lambda_\sigma \left(\bar{\psi}^i_{Lb} \psi^i_{Ra} \right)\left(\bar{\psi}^j_{Ra} \psi^j_{Lb} \right) \right\}$$

$$\psi_{L,R} = \frac{1 \pm \gamma^5}{2} \psi$$

$i, j = 1 \ldots N_c$ color $(N_c = 3)$

$a, b = 1 \ldots N_F$ flavor $(N_F = 3, 2)$

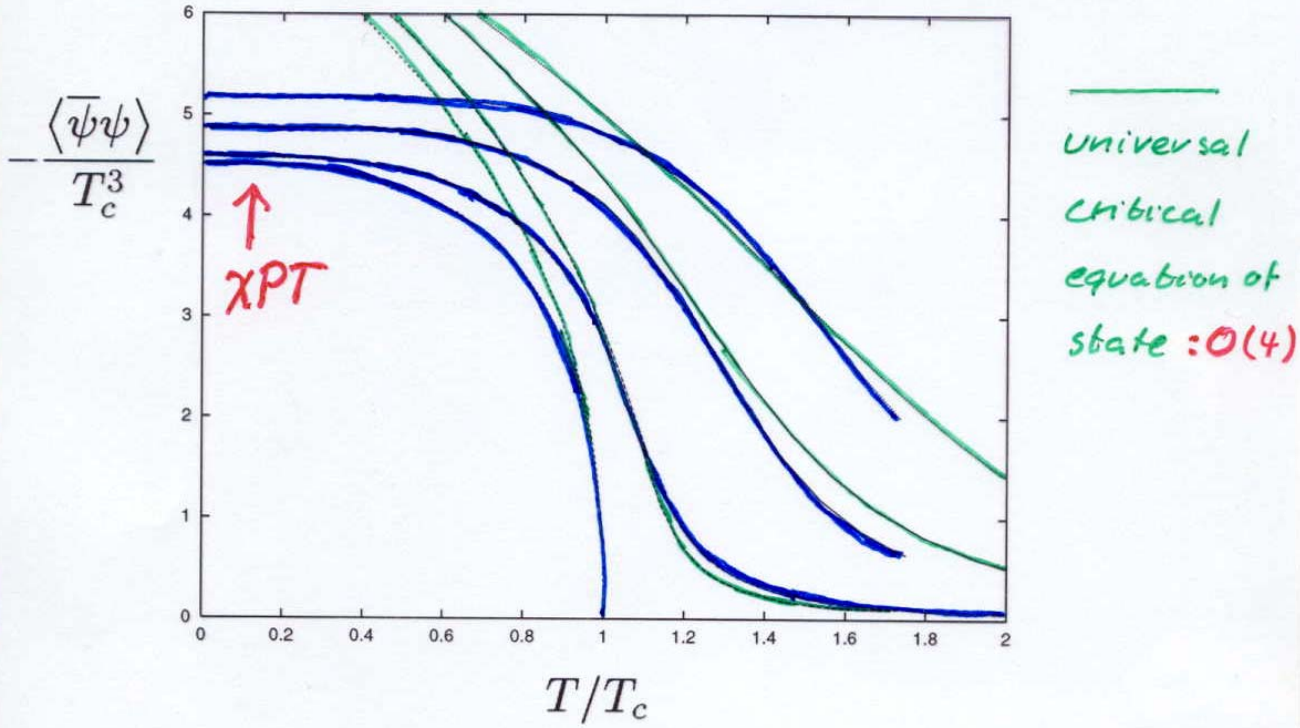
chiral flavor symmetry :

$$SU_L(N_F) \times SU_R(N_F)$$

...and more general quark meson models

Chiral condensate (N_f=2)

2nd order PT (expected for $O(4)$ Heisenberg model)



⟹ Explicit link between χPT domain of validity (4d) and critical (universal) domain near T_c (3d)

Functional Renormalization Group

from small to large scales

How to come from quarks and gluons to baryons and mesons ?

How to come from electrons to spin waves ?

Find effective description where relevant degrees of freedom depend on momentum scale or resolution in space.

Microscope with variable resolution:

- High resolution , small piece of volume: quarks and gluons
- Low resolution, large volume : hadrons

From

Microscopic Laws

(Interactions, classical action)

↓

to Fluctuations!

Macroscopic Observation

(Free energy functional, effective action)

- block spins

  Kadanoff, Wilson

- exact renormalization group equations

  Wilson, Kogut
  Wagner, Houghton
  Weinberg
  Polchinski
  Hasenfratz^2

- Lattice finite size scaling

  Lüscher,...

- coarse grained free energy/average action

Effective potential includes all fluctuations

Average potential U_k

$\equiv$ scale dependent effective potential
$\equiv$ coarse grained free energy

Only fluctuations with momenta $q^2 > k^2$ included

k: infrared cutoff for fluctuations, "average scale"

Λ: characteristic scale for microphysics

$$U_\Lambda \approx S \to U_0 \equiv U$$

Scalar field theory

linear sigma-model for
chiral symmetry breaking in QCD
or:
scalar model for antiferromagnetic spin waves
(linear O(3) – model)

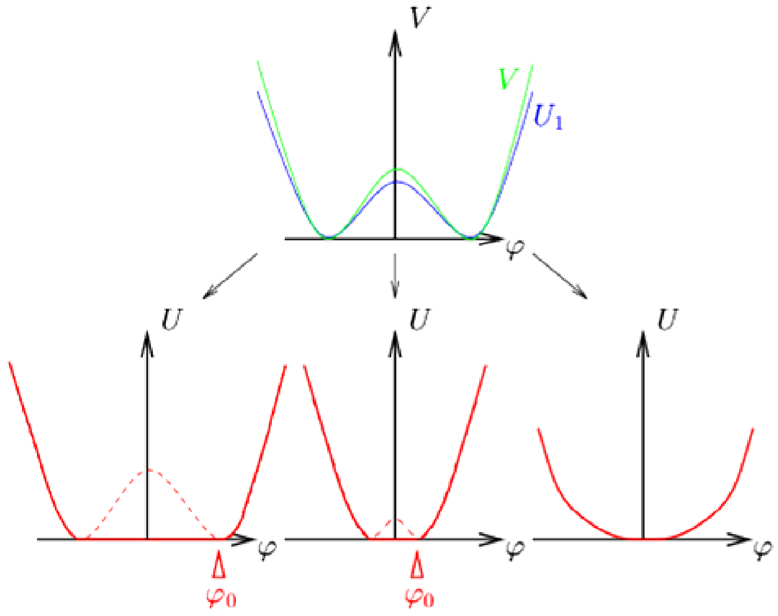
fermions will be added later

Scalar field theory

$\varphi_a(x)$: magnetization, density, chemical concentration, Higgs field, meson field, inflaton, cosmon

$O(N)$-symmetry:

$$S = \int d^d x \left\{ \frac{1}{2} \partial_\mu \varphi_a \partial_\mu \varphi_a + V(\rho) \right\}; \quad \rho = \frac{1}{2} \varphi_a \varphi_a$$



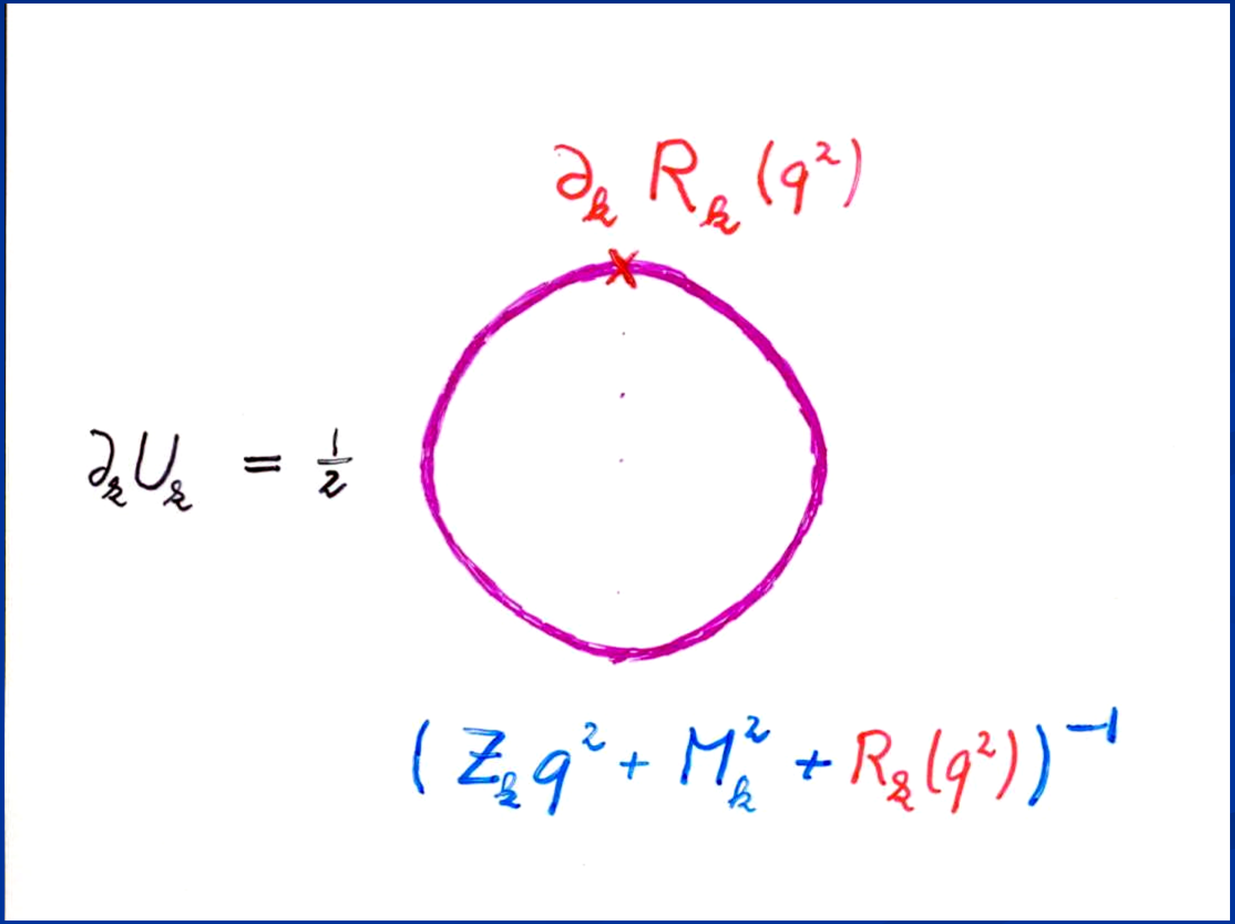
Flow equation for average potential

$$\partial_k U_k(\varphi) = \frac{1}{2} \sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

$\bar{M}^2_{k,ab} = \frac{\partial^2 U_k}{\partial\varphi_a \partial\varphi_b}$: Mass matrix

$\bar{M}^2_{k,i}$: Eigenvalues of mass matrix

Simple one loop structure –nevertheless (almost) exact



Infrared cutoff

R_k : IR-cutoff

e.g $\quad R_k = \dfrac{Z_k q^2}{e^{q^2/k^2} - 1}$

or $\quad R_k = Z_k(k^2 - q^2)\Theta(k^2 - q^2) \qquad$ (Litim)

$\lim_{k\to 0} R_k = 0$

$\lim_{k\to\infty} R_k \to \infty$

Flow equation for U_k

$$\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

'91

$\bar{M}^2_{k,ab} = \frac{\partial^2 U_k}{\partial\varphi_a \partial\varphi_b}$: Mass matrix

$\bar{M}^2_{k,i}$: Eigenvalues of mass matrix

R_k : IR-cutoff

e.g $R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$

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$\lim_{k\to 0} R_k = 0$

$\lim_{k\to\infty} R_k \to \infty$

Partial differential equation for function U(k,φ) depending on two (or more) variables

$Z_k = c\, k^{-\eta}$

Regularisation

For suitable R_k :

$$R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$$
$$R_k = Z_k (k^2 - q^2)\Theta(k^2 - q^2)$$

- Momentum integral is ultraviolet and infrared finite
- Numerical integration possible
- Flow equation defines a regularization scheme (ERGE –regularization)

$$\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

Integration by momentum shells

$$\partial_k U_k(\varphi) = \frac{1}{2}\sum_i \int \frac{d^d q}{(2\pi)^d} \frac{\partial_k R_k(q^2)}{Z_k q^2 + R_k(q^2) + \bar{M}^2_{k,i}(\varphi)}$$

Momentum integral
is dominated by
$q^2 \sim k^2$.

Flow only sensitive to
physics at scale k

Wave function renormalization and anomalous dimension

Z_k: wave function renormalization

$$k\partial_k Z_k = -\eta_k Z_K$$

η_k: anomalous dimension

$$t = \ln(k/\Lambda)$$

$$\partial_t \ln Z = -\eta$$

for Z_k (φ,q^2) : flow equation is **exact !**

Effective average action

and

exact renormalization group equation

Generating functional

generating functional for connected Green's functions in presence of quadratic infrared cutoff

$$W_k[j] = \ln \int \mathcal{D}\chi \, \exp\left(-S[\chi] - \Delta_k S[\chi] + \int d^d x \, j_a \chi_a \right)$$

$$\Delta_k S = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} R_k(q^2) \chi_a(-q) \chi_a(q)$$

$$\text{e.g.} \quad R_k = \frac{Z_k q^2}{e^{q^2/k^2} - 1}$$

$$\lim_{k \to 0} R_k = 0$$

$$R_{k \to \infty} \to \infty$$

Effective average action

$$\Gamma_k[\varphi] = -W_k[j] + \int d^d x\, j_a \varphi_a - \Delta_k S[\varphi]$$

$\Gamma_0[\varphi]$: quantum effective action

generates 1PI vertices

free energy: $F = \Gamma T + \mu n V$

Γ_k includes all fluctuations (quantum, thermal) with $q^2 > k^2$

Γ_Λ specifies microphysics

$$\varphi_a = \langle \chi_a \rangle = \frac{\delta W_k}{\delta j_a}$$

Loop expansion :
perturbation theory
with
infrared cutoff
in propagator

Quantum effective action

for $k \to 0$
all fluctuations (quantum + thermal)
are included

knowledge of $\Gamma_{k \to 0} \hat{=}$ solution of model

Exact renormalization group equation

Exact flow equation

for scale dependence of average action

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2}\mathrm{Tr}\left\{\left(\Gamma_k^{(2)}[\varphi] + R_k\right)^{-1} \partial_k R_k\right\}$$

'92

$\left(\Gamma_k^{(2)}\right)_{ab}(q, q') = \frac{\delta^2 \Gamma_k}{\delta\varphi_a(-q)\delta\varphi_b(q')}$

$\mathrm{Tr} : \sum_a \int \frac{d^d q}{(2\pi)^d}$

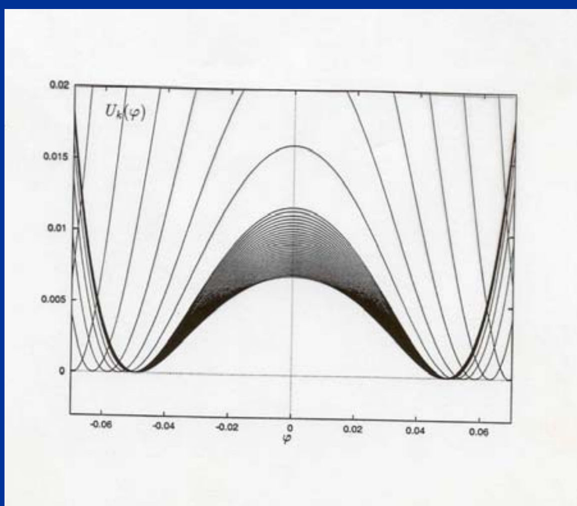
(fermions : STr)

Exact flow equation for effective potential

- Evaluate exact flow equation for homogeneous field φ .
- R.h.s. involves exact propagator in homogeneous background field φ.

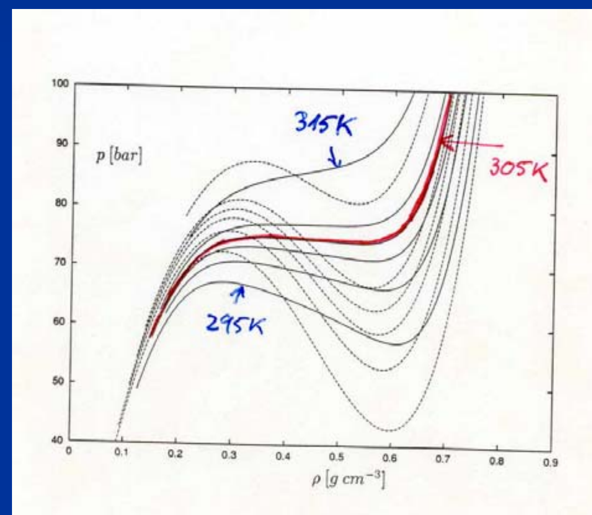
Flow of effective potential

Ising model



S.Seide …

CO_2



Experiment :

T_* =304.15 K
p_* =73.8.bar
ρ_* = 0.442 g cm-2

Critical exponents

$d = 3$

Critical exponents ν and η

N	ν			η
0	0.590	0.5878	0.039	0.0292
1	0.6307	0.6308	0.0467	0.0356
2	0.666	0.6714	0.049	0.0385
3	0.704	0.7102	0.049	0.0380
4	0.739	0.7474	0.047	0.0363
10	0.881	0.886	0.028	0.025
100	0.990	0.980	0.0030	0.003
		↑		↑

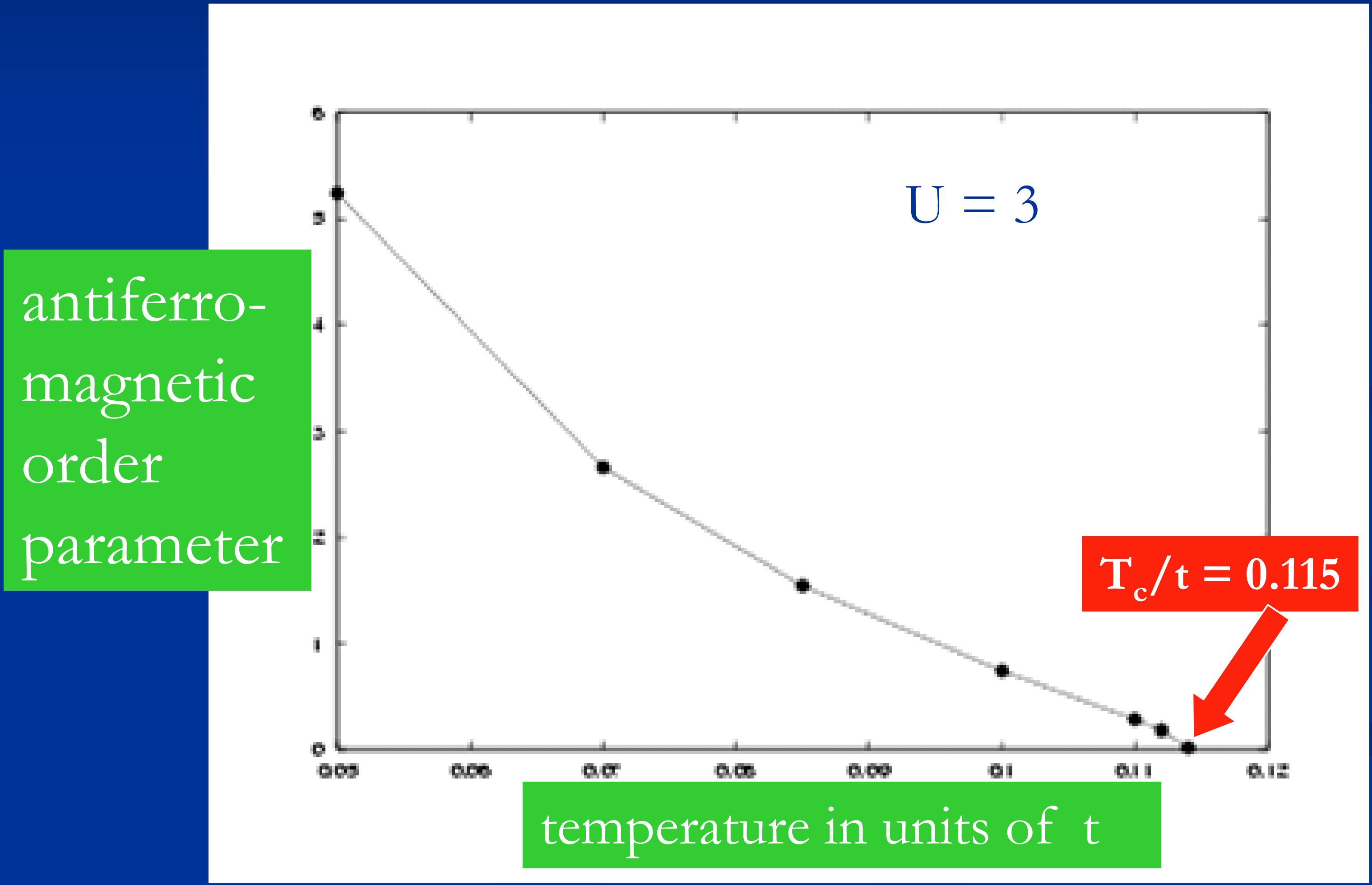
"average" of other methods
(typically $\pm(0.0010 - 0.0020)$)

Antiferromagnetic order in the Hubbard model

A functional renormalization group study

T.Baier, E.Bick, …

Temperature dependence of antiferromagnetic order parameter



Mermin-Wagner theorem ?

No spontaneous symmetry breaking of continuous symmetry in $d=2$!

Fermion bilinears

$$\tilde{\rho}(X) = \hat{\psi}^\dagger(X)\hat{\psi}(X),$$
$$\tilde{\vec{m}}(X) = \hat{\psi}^\dagger(X)\vec{\sigma}\hat{\psi}(X)$$

Introduce sources for bilinears

$$S_F = S_{F,\mathrm{kin}} + \frac{1}{2}U(\hat{\psi}^\dagger\hat{\psi})^2 - J_\rho\tilde{\rho} - \vec{J}_m\tilde{\vec{m}}$$

Functional variation with respect to sources J yields expectation values and correlation functions

$$Z = \int \mathcal{D}(\psi^*,\psi)\exp\left(-(S_F + S_\eta)\right)$$
$$S_\eta = -\eta^\dagger\psi - \eta^T\psi^*$$

Partial Bosonisation

- collective bosonic variables for fermion bilinears
- insert identity in functional integral
  (Hubbard-Stratonovich transformation)
- replace four fermion interaction by equivalent bosonic interaction (e.g. mass and Yukawa terms)
- problem : decomposition of fermion interaction into bilinears not unique (Grassmann variables)

$$(\hat{\psi}^\dagger(X)\hat{\psi}(X))^2 = \tilde{\rho}(X)^2 = -\frac{1}{3}\tilde{\vec{m}}(X)^2$$

Partially bosonised functional integral

$$Z[\eta,\eta^*,J_\rho,\vec{J}_m] = \int \mathcal{D}(\psi^*,\psi,\hat{\rho},\hat{\vec{m}})\exp\left(-(S+S_\eta+S_J)\right)$$

$$S = S_{F,\mathrm{kin}} + \frac{1}{2}U_\rho\hat{\rho}^2 + \frac{1}{2}U_m\hat{\vec{m}}^2 - U_\rho\hat{\rho}\tilde{\rho} - U_m\hat{\vec{m}}\tilde{\vec{m}},$$
$$S_J = -\ J_\rho\hat{\rho} - \vec{J}_m\hat{\vec{m}}$$

equivalent to fermionic functional integral

if $$U = -U_\rho + 3U_m$$

Bosonic integration is Gaussian

or:

solve bosonic field equation as functional of fermion fields and reinsert into action

$$\hat{\rho} = \tilde{\rho} + \frac{J_\rho}{U_\rho}, \qquad \hat{\vec{m}} = \tilde{\vec{m}} + \frac{\vec{J}_m}{U_m}$$

fermion – boson action

$$S = S_{F,\mathrm{kin}} + S_B + S_Y + S_J,$$

fermion kinetic term

$$S_{F,\mathrm{kin}} = \sum_Q \hat{\psi}^\dagger(Q)(i\omega_F - \mu - 2t(\cos q_1 + \cos q_2))\hat{\psi}(Q),$$

boson quadratic term ("classical propagator")

$$S_B = \frac{1}{2}\sum_Q \left(U_\rho \hat{\rho}(Q)\hat{\rho}(-Q) + U_m \hat{\vec{m}}(Q)\hat{\vec{m}}(-Q)\right),$$

Yukawa coupling

$$S_Y = -\sum_{QQ'Q''} \delta(Q - Q' + Q'') \times (U_\rho \hat{\rho}(Q)\hat{\psi}^\dagger(Q')\hat{\psi}(Q'') + U_m \hat{\vec{m}}(Q)\hat{\psi}^\dagger(Q')\vec{\sigma}\hat{\psi}(Q'')),$$

source term

$$S_J = -\sum_Q \Big(J_\rho(-Q)\hat{\rho}(Q) + \vec{J}_m(-Q)\hat{\vec{m}}(Q) \Big)$$

is now linear in the bosonic fields

Mean Field Theory (MFT)

Evaluate Gaussian fermionic integral in background of bosonic field , e.g.

$$
\begin{aligned}
\hat{\rho}(Q) &\rightarrow \rho\delta(Q) \\
\hat{\vec{m}}(Q) &\rightarrow \vec{a}\delta(Q-\Pi)
\end{aligned}
$$

$$
\begin{aligned}
Z_{\mathrm{MF}} &= \int \mathcal{D}(\hat{\psi}^*, \hat{\psi}) \exp(-S_{\mathrm{MF}}), \\
S_{\mathrm{MF}} &= \sum_Q \hat{\psi}^\dagger(Q)(i\omega_F - \mu - 2t(\cos q_1 + \cos q_2))\hat{\psi}(Q) \\
&\quad - \sum_Q (U_\rho \rho \hat{\psi}^\dagger(Q)\hat{\psi}(Q) + U_m \vec{a}\hat{\psi}^\dagger(Q+\Pi)\vec{\sigma}\hat{\psi}(Q)) \\
&\quad + \frac{V_2}{2T}(U_\rho \rho^2 + U_m \vec{a}^2) - J_\rho(0)\rho - \vec{J}_m(-\Pi)\vec{a}
\end{aligned}
$$

$$
\Gamma_{\mathrm{MF}} = -\ln Z_{\mathrm{MF}} + J_\rho(0)\rho + \vec{J}_m(-\Pi)\vec{a}
$$

Effective potential in mean field theory

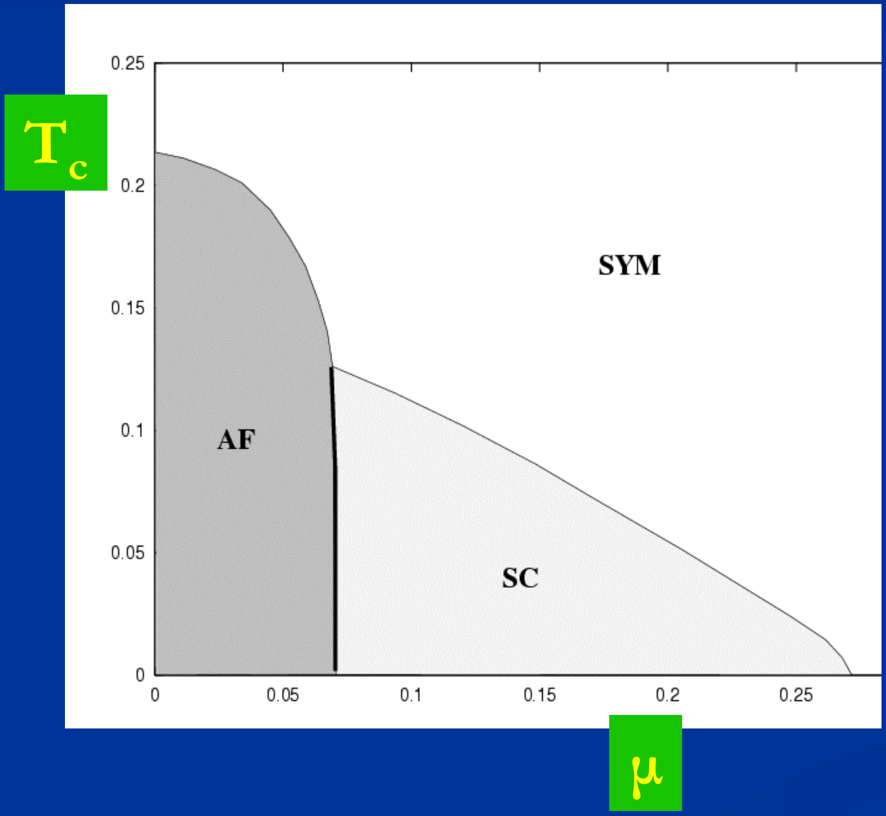
$$U(\rho,\vec{a}) = \frac{T\Gamma}{V_2} = \frac{1}{2}(U_\rho \rho^2 + U_m \vec{a}^2) + \Delta U(\rho,\vec{a})$$

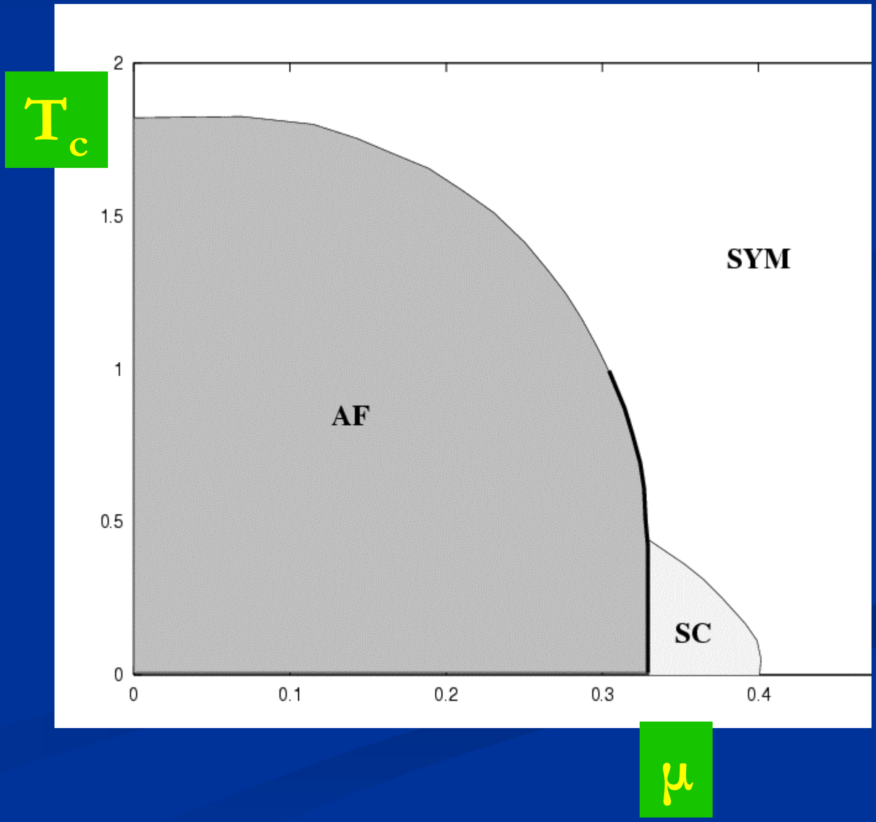
$$\Delta U(\rho,\vec{a}) = -\frac{T}{V_2} \ln \int \mathcal{D}(\hat{\psi}^*,\hat{\psi}) \exp(-S_\Delta),$$

$$S_\Delta = \sum_Q \left(\hat{\psi}^\dagger(Q) P(Q) \hat{\psi}(Q) - U_m \vec{a} \hat{\psi}^\dagger(Q+\Pi) \vec{\sigma} \hat{\psi}(Q) \right)$$

$$\begin{aligned} P(Q) &= i\omega_F - \mu_{\text{eff}} - 2t(\cos q_1 + \cos q_2), \\ \mu_{\text{eff}} &= \mu + U_\rho \rho. \end{aligned}$$

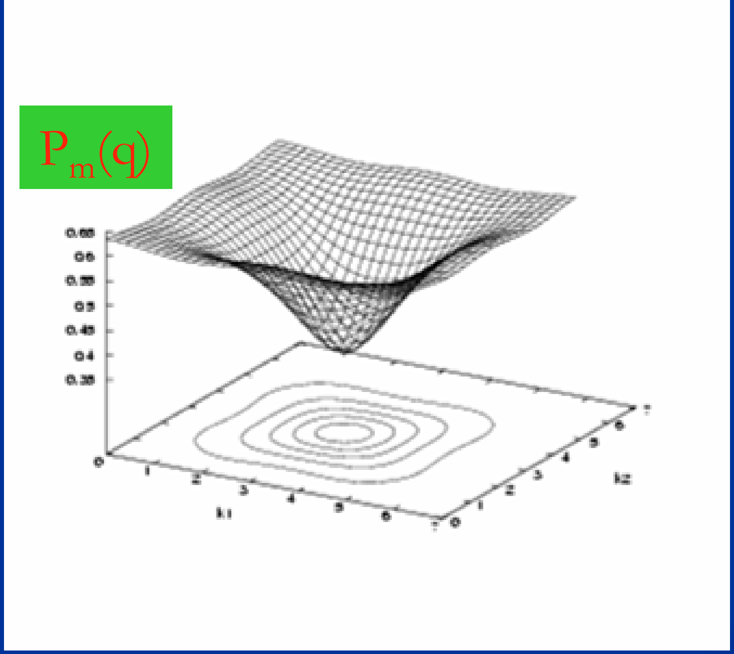
Mean field phase diagram



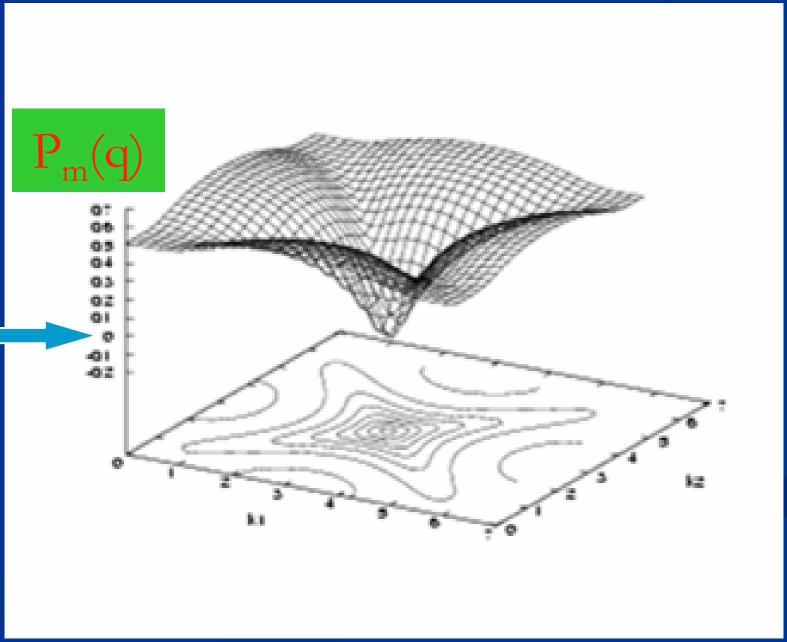


Mean field inverse propagator for spin waves

$T/t = 0.5$

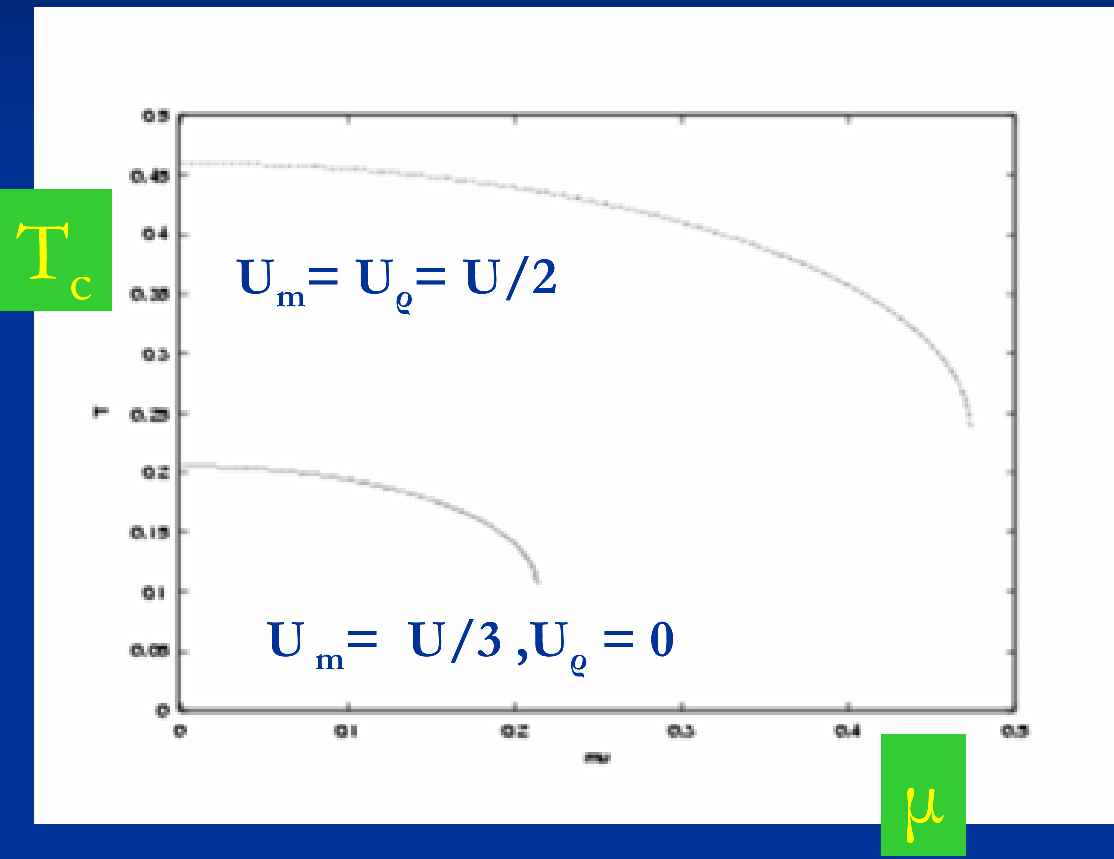


$T/t = 0.15$



Baier,Bick,…

Mean field ambiguity



Artefact of approximation …

cured by inclusion of bosonic fluctuations

J.Jaeckel,…

mean field phase diagram

$$U = -U_\rho + 3U_m$$

Flow equation for the Hubbard model

T.Baier , E.Bick , …

Truncation

Concentrate on antiferromagnetism

$$\vec{a}(Q) = \vec{m}(Q + \Pi)$$

Potential U depends only on $\alpha = a^2$

$$\Gamma_{\psi,k}[\psi, \psi^*] = \sum_Q \psi^\dagger(Q) P_F(Q) \psi(Q),$$

$$P_F(Q) = i\omega_F + \epsilon - \mu, \quad \epsilon(\boldsymbol{q}) = -2t(\cos q_x + \cos q_y),$$

$$\Gamma_{Y,k}[\psi, \psi^*, \vec{a}] = -\bar{h}_{a,k} \sum_{KQQ'} \vec{a}(K)\psi^*(Q)\vec{\sigma}\psi(Q') \times \delta(K - Q + Q' + \Pi)$$

$$\Gamma_{a,k}[\vec{a}] = \frac{1}{2}\sum_Q \vec{a}(-Q) P_a(Q) \vec{a}(Q) + \sum_X U[\vec{a}(X)]$$

$$\begin{aligned}
\text{SYM} : \sum_X U[\vec{a}] &= \sum_K \bar{m}_a^2 \alpha(-K, K) + \\
&+ \frac{1}{2} \sum_{K_1 \ldots K_4} \bar{\lambda}_a \delta(K_1 + K_2 + K_3 + K_4) \\
&\quad \times \alpha(K_1, K_2)\alpha(K_3, K_4), \\
\text{SSB} : \sum_X U[\vec{a}] &= \frac{1}{2} \sum_{K_1 \ldots K_4} \bar{\lambda}_a \delta(K_1 + K_2 + K_3 + K_4) \\
&\quad \times (\alpha(K_1, K_2) - \alpha_0 \delta(K_1)\delta(K_2)) \\
&\quad \times (\alpha(K_3, K_4) - \alpha_0 \delta(K_3)\delta(K_4))
\end{aligned}$$

$$\alpha(K, K') = \frac{1}{2}\vec{a}(K)\vec{a}(K')$$

scale evolution of effective potential for antiferromagnetic order parameter

$$\begin{aligned}\partial_k U\ (\alpha) &= \partial_k U^B(\alpha) + \partial_k U^F(\alpha)\\ &= \frac{1}{2}\sum_{Q,i}\tilde{\partial}_k \ln[P_a(Q) + \hat{M}_i^2(\alpha) + R_k^a(Q)]\\ &\quad -2T\int_{-\pi}^{\pi}\frac{d^2q}{(2\pi)^2}\tilde{\partial}_k \ln\cosh y(\alpha).\end{aligned}$$

boson contribution

fermion contribution

$$\hat{M}^2_{1,2,3}(\alpha) = \\ = \begin{cases}(\bar{m}_a^2 + 3\bar{\lambda}_a\alpha, \bar{m}_a^2 + \bar{\lambda}_a\alpha, \bar{m}_a^2 + \bar{\lambda}_a\alpha) & \text{SYM}\\ (\bar{\lambda}_a(3\alpha-\alpha_0), \bar{\lambda}_a(\alpha-\alpha_0), \bar{\lambda}_a(\alpha-\alpha_0)) & \text{SSB}\end{cases}$$

effective masses depend on α !

$$y(\alpha) = \frac{1}{2T_k}\sqrt{\epsilon^2(\boldsymbol{q}) + 2\bar{h}_a^2\alpha}.$$

gap for fermions ~α

running couplings

SYM:

$$\partial_k \bar{m}_a^2 = \frac{\partial}{\partial \alpha}(\partial_k U(\alpha))|_{\alpha=0},$$

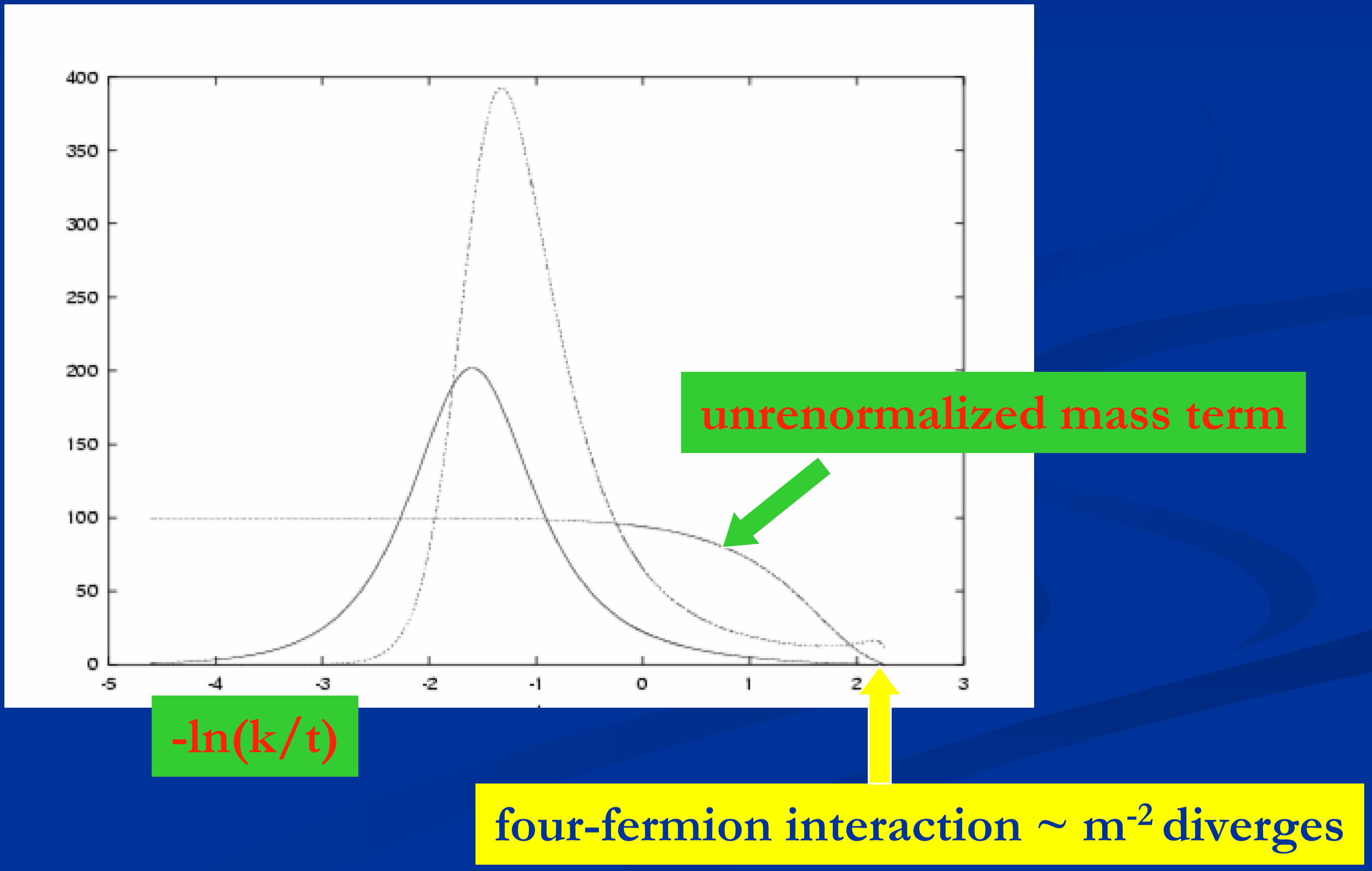
$$\partial_k \bar{\lambda}_a = \frac{\partial^2}{\partial \alpha^2}(\partial_k U(\alpha))|_{\alpha=0},$$

SSB:

$$\partial_k \alpha_0 = -\frac{1}{\bar{\lambda}_a}\frac{\partial}{\partial \alpha}(\partial_k U(\alpha))|_{\alpha=\alpha_0},$$

$$\partial_k \bar{\lambda}_a = \frac{\partial^2}{\partial \alpha^2}(\partial_k U(\alpha))|_{\alpha=\alpha_0}.$$

Running mass term



dimensionless quantities

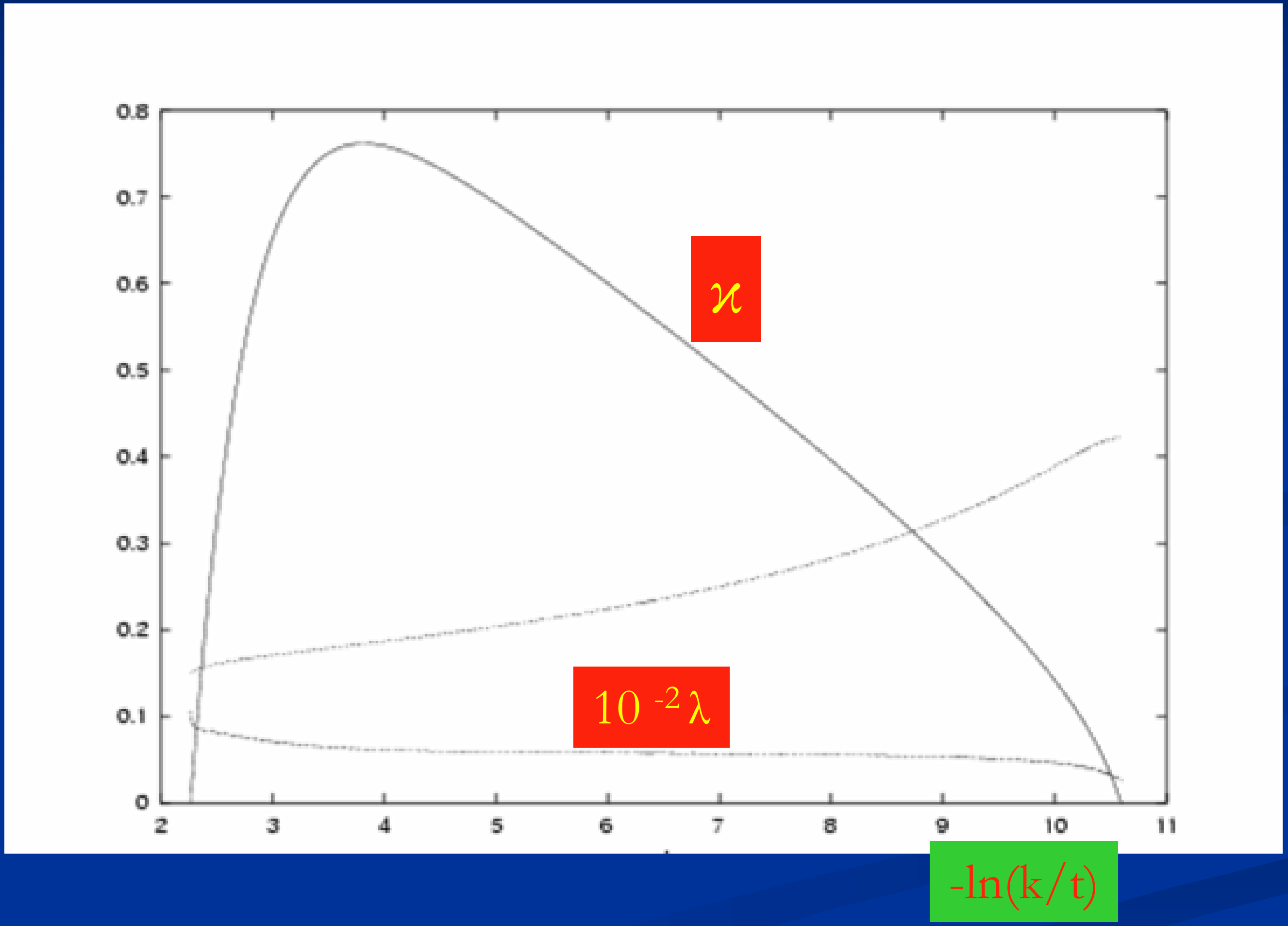
$$u = \frac{Ut^2}{Tk^2}, \quad \tilde{\alpha} = \frac{Z_a t^2 \alpha}{T}$$

$$m_a^2 = \frac{\bar{m}_a^2}{Z_a k^2} = \frac{\partial u}{\partial \tilde{\alpha}}, \quad \kappa_a = \frac{Z_a t^2}{T} \alpha_0,$$

$$\lambda_a = \frac{T}{Z_a^2 t^2 k^2} \bar{\lambda}_a = \frac{\partial^2 u}{\partial \tilde{\alpha}^2}, \quad h_a^2 = \frac{T}{Z_a t^4} \bar{h}_a^2$$

renormalized antiferromagnetic order parameter κ

evolution of potential minimum

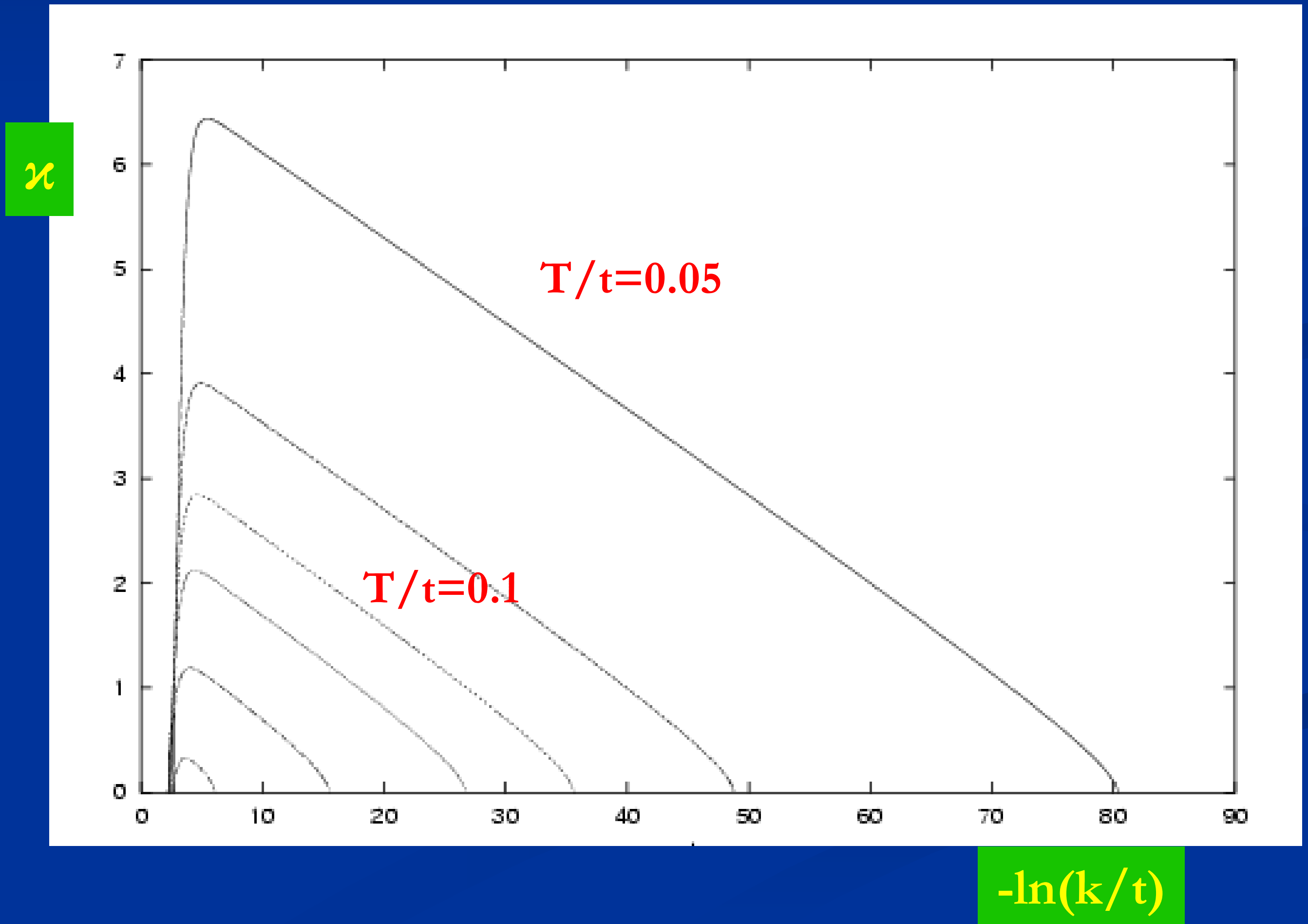


U/t = 3 , T/t = 0.15

Critical temperature

For $T<T_c$: $\varkappa$ remains positive for $k/t > 10^{-9}$

size of probe > 1 cm



$T_c=0.115$

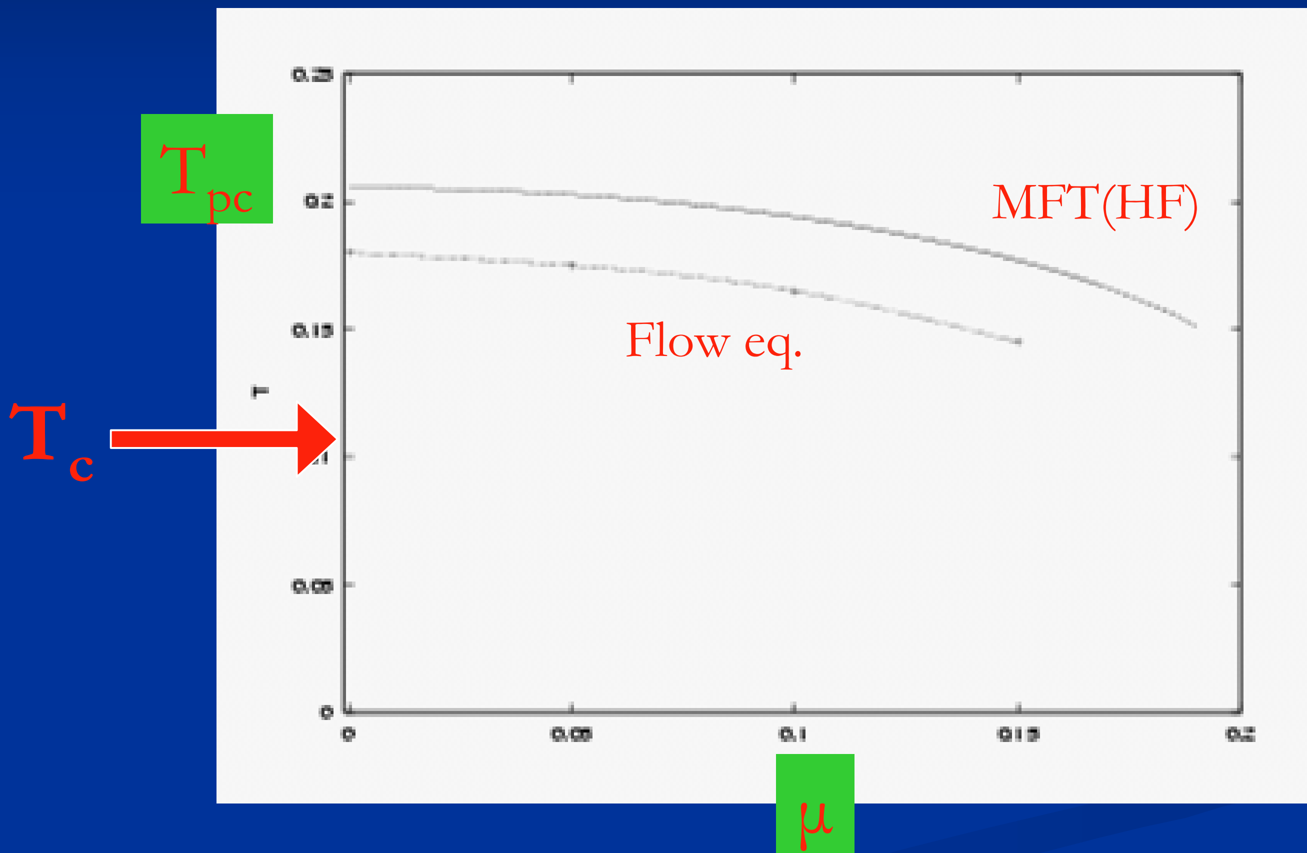
Pseudocritical temperature T_{pc}

Limiting temperature at which bosonic mass term vanishes ($\varkappa$ becomes nonvanishing)

It corresponds to a diverging four-fermion coupling

This is the "critical temperature" computed in MFT !

Pseudocritical temperature



critical behavior

for interval $T_c < T < T_{pc}$
evolution as for classical Heisenberg model

cf. Chakravarty,Halperin,Nelson

$$k\partial_k\kappa = \frac{1}{4\pi} + \frac{1}{16\pi^2\kappa} + 0(\kappa^{-2})$$

$$\kappa(k) = \kappa_m(T) - \frac{1}{4\pi}\ln\frac{k_m(T)}{k}$$

critical correlation length

$$\xi t = c(T) \exp\left\{ 20.7 \beta(T) \frac{T_c}{T} \right\}$$

c,β : slowly varying functions

exponential growth of correlation length compatible with observation !

at T_c : correlation length reaches sample size !

$$\begin{aligned} \beta(T) &= \frac{\hat{\alpha}_0(T)\hat{Z}_a(T)}{\hat{\alpha}_0(T_c)\hat{Z}_a(T_c)}, \\ c(T) &= C_{\mathrm{SR}} \frac{k_m(T_c)}{k_m(T)} \left(\frac{k_m(T_c)}{t} \right)^{\delta(T)}, \\ \delta(T) &= \beta(T)\frac{T_c}{T} - 1 \end{aligned}$$

$$\xi = \frac{C_{\mathrm{SR}}}{k_m(T)} \exp\left(4\pi\kappa_m(T)\right)$$

$$\xi = \tilde{C} \exp\left(\frac{\gamma}{T}\right)$$

$$\gamma = 4\pi\hat{\alpha}_0(T)\hat{Z}_a(T)t^2.$$

$$T_c(k) = \frac{\gamma(T_c)}{\ln\left(k_m(T_c)/k\right)}$$

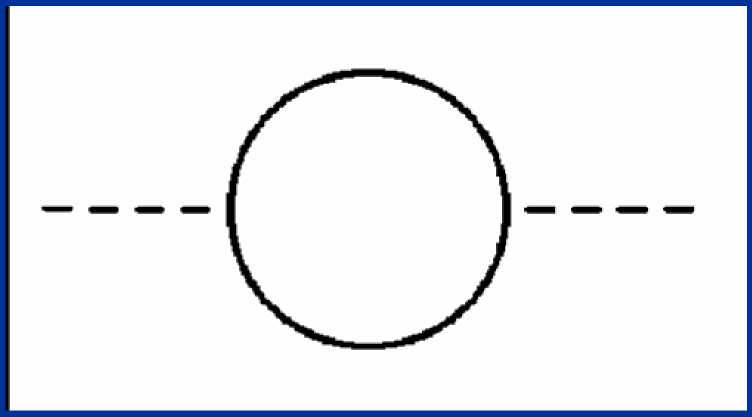
critical behavior for order parameter and correlation function

$$\kappa_a(T) = \left(\frac{\gamma(T)T_c}{T} - 1\right)\kappa_m(T_c) + \frac{1}{4\pi}\ln\frac{k_m(T_c)}{k_m(T)}.$$

$$G(q^2) = (Z_a(k = \sqrt{q^2})q^2)^{-1} \sim (q^2)^{-1+\eta_a/2}$$

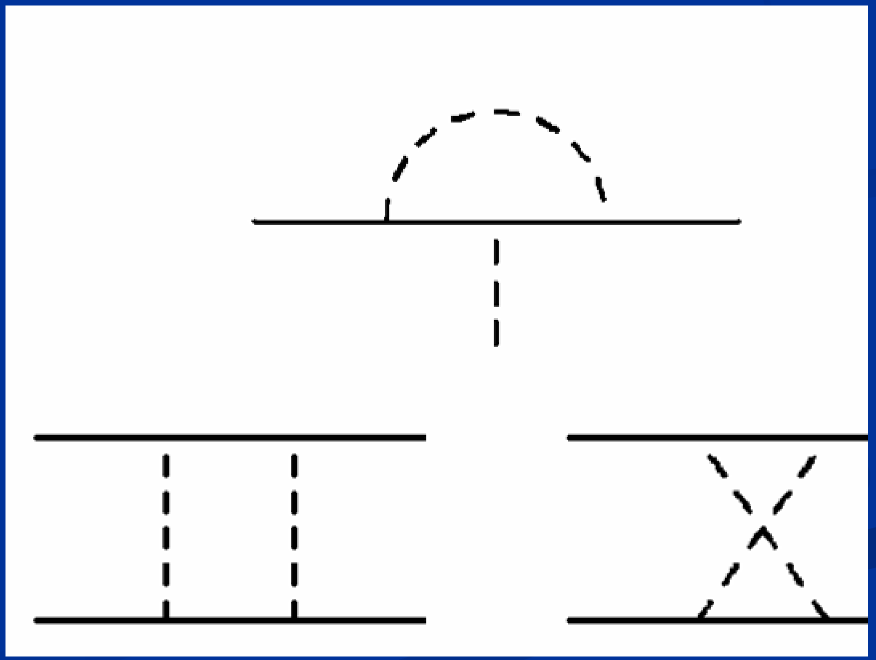
Bosonic fluctuations

fermion loops



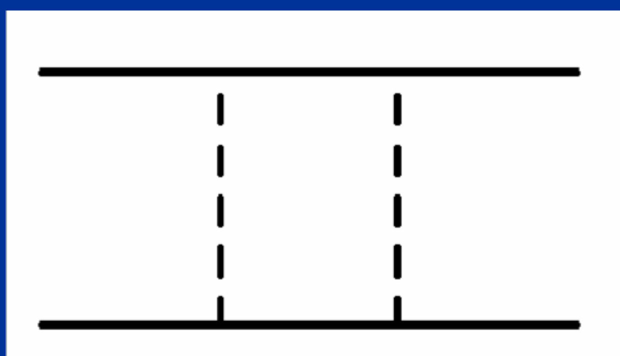
mean field theory

boson loops



Rebosonisation

- adapt bosonisation to every scale k such that



is translated to bosonic interaction

H.Gies , …

$$\begin{aligned}\Gamma_k[\psi,\psi^*,\phi] &= \sum_Q \psi^*(Q) P_{\psi,k}\psi(Q) \\ &+\frac{1}{2}\sum_Q \phi(-Q) P_{\phi,k}(Q)\phi(Q) \\ &-\sum_Q h_k(Q)\phi(Q)\tilde{\phi}(-Q) \\ &+\sum_Q \lambda_{\psi,k}(Q)\tilde{\phi}(Q)\tilde{\phi}(-Q)\end{aligned}$$

k-dependent field redefinition

$$\phi_k(Q) = \phi_{\bar{k}}(Q) + \Delta\alpha_k(Q)\tilde{\phi}(Q)$$

$$\partial_k\phi_k(Q) = -\,\partial_k\alpha_k(Q)\tilde{\phi}(Q)$$

absorbs four-fermion coupling

Modification of evolution of couplings …

Evolution with k-dependent field variables

$$\partial_k \Gamma_k[\psi,\psi^*,\phi_k] = \partial_k \Gamma_k[\psi,\psi^*,\phi_k]|_{\phi_k} + \sum_Q \left(\frac{\delta}{\delta\phi_k}\Gamma[\psi,\psi^*,\phi_k]\right)\partial_k\phi_k$$
$$= \partial_k \Gamma_k[\psi,\psi^*,\phi_k]|_{\phi_k} + \sum_Q \big(-\partial_k\alpha_k(Q)P_{\phi,k}(Q)\phi_k(Q)\tilde{\phi}(-Q) + h_k(Q)\partial_k\alpha_k(Q)\tilde{\phi}(Q)\tilde{\phi}(-Q)\big)$$

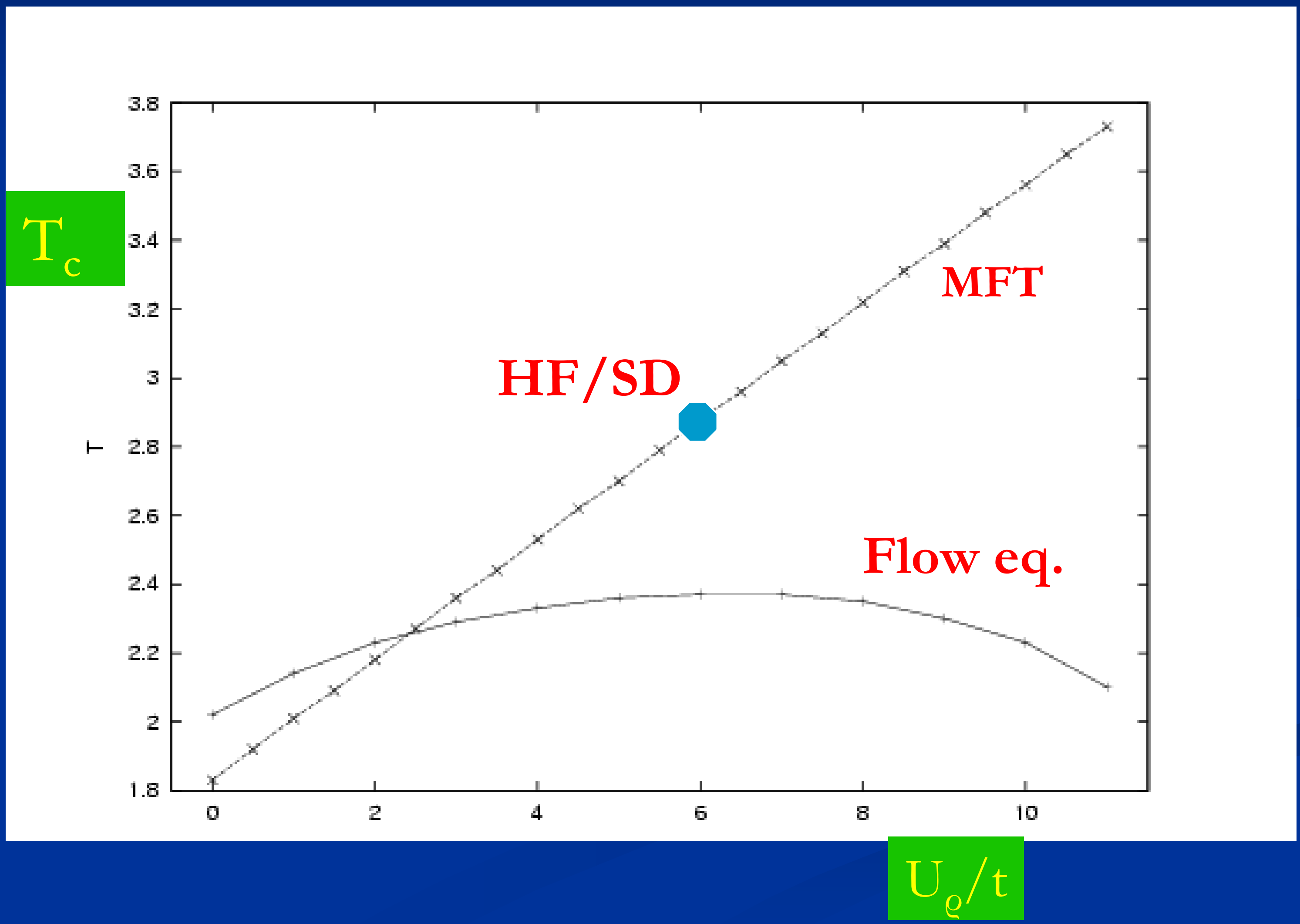
Rebosonisation

$$\begin{aligned}\partial_k h_k(Q) &= \partial_k h_k(Q)|_{\phi_k} + \partial_k\alpha_k(Q)P_{\phi,k}(Q),\\ \partial_k\lambda_{\psi,k}(Q) &= \partial_k\lambda_{\psi,k}(Q)|_{\phi_k} + h_k(Q)\,\partial_k\alpha_k(Q).\end{aligned}$$

Choose α_k such that no four fermion coupling is generated ➡

$$\partial_k h_k(Q) = \partial_k h_k(Q)|_{\phi_k} - \frac{P_{\phi,k}(Q)}{h_k(Q)}\partial_k\lambda_{\psi,k}(Q)|_{\phi_k}$$

…cures mean field ambiguity



Nambu Jona-Lasinio model

$$S = \int d^4x \left\{ i\,\bar{\psi}^i_a \gamma^\mu \partial_\mu \psi^i_a \right.$$

$$\left. + 2\lambda_\sigma \left(\bar{\psi}^i_{Lb}\, \psi^i_{Ra} \right)\left(\bar{\psi}^j_{Ra}\, \psi^j_{Lb} \right) \right\}$$

$$\psi_{L,R} = \frac{1 \pm \gamma^5}{2}\, \psi$$

$i, j = 1 \ldots N_c$ color $(N_c = 3)$

$a, b = 1 \ldots N_F$ flavor $(N_F = 3, 2)$

chiral flavor symmetry :

$$SU_L(N_F) \times SU_R(N_F)$$

Critical temperature , $N_f = 2$

$\frac{m_\pi}{\text{MeV}}$	0	45	135	230
$\frac{T_{pc}}{\text{MeV}}$	100.7	$\simeq 110$	$\simeq 130$	$\simeq 150$

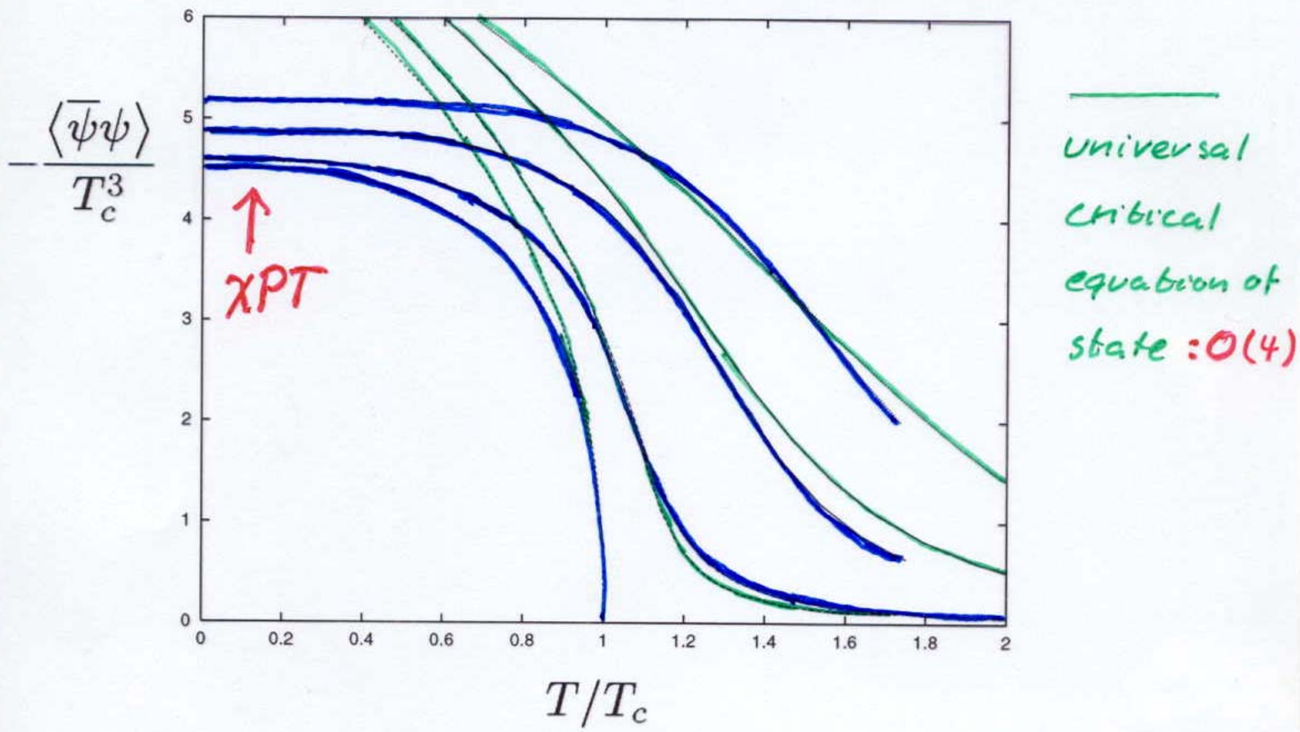
for $f_\pi = 93$ MeV

Lattice simulation

J.Berges,D.Jungnickel,…

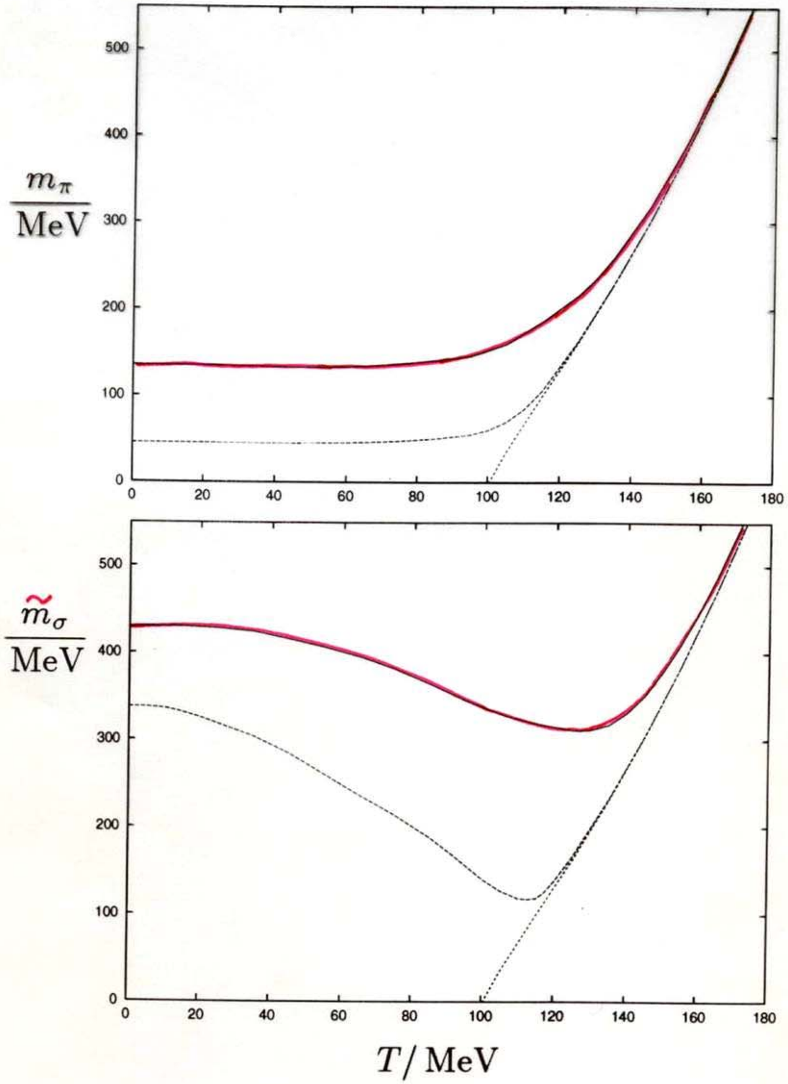
Chiral condensate

2nd order PT (expected for $O(4)$ Heisenberg model)



⟹ Explicit link between χPT domain of validity (4d) and critical (universal) domain near T_c (3d)

temperature dependent masses



pion mass

sigma mass

? $m_\sigma < 2m_\pi$ for $T \gtrsim 100\,\mathrm{MeV}$?

No long pion correlation length in thermal equilibrium!

Critical equation of state

Critical behavior for second order phase transitions :

correlation length $\xi = m_R^{-1}$

<u>only</u> relevant length scale

φ_R : renormalized field variable

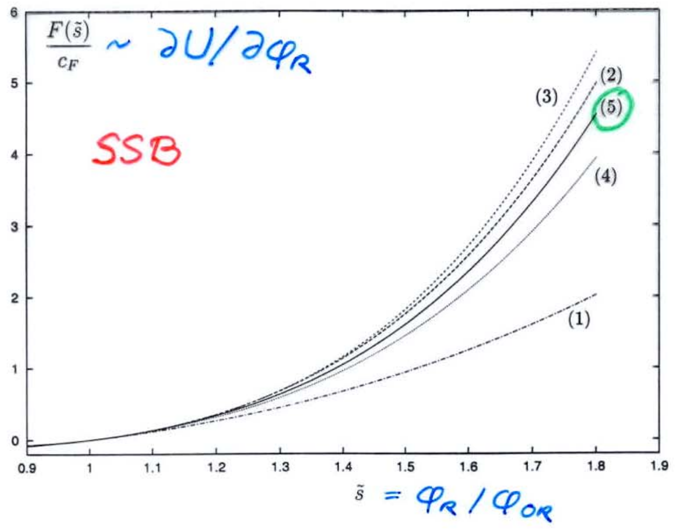
$U(\varphi_R)$ depends only on m_R

$$B \sim \frac{\partial U}{\partial \varphi_R} = F\left(\frac{\varphi_R}{\sqrt{m_R}}\right) m_R^{5/2}$$

↑

Widom scaling function

Scaling form of equation of state



critical equation of state

(2) ERGE, lowest order derivative exp.; Berges, Tetradis, …

(5) ERGE, first order derivative exp.; Seide, …

(1) mean field

(4) high-T-series, loop expansion, ε-expansion

(3) Monte Carlo

Berges, Tetradis,…

Universal critical equation of state
is valid near critical temperature
if the only light degrees of freedom
are pions + sigma with
O(4) – symmetry.

Not necessarily valid in QCD, even
for two flavors !

end