Introduction to String Theory

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Contents

Literature

This is the write-up of my course "Introduction to String Theory", held in the Master Programme at the University of Heidelberg during winter term 2011/12. The course comprises 28 lectures, each lasting for 100 minutes.
The material discussed in this lecture course is introductory and necessarily covers only a small fraction of the wide range of topics studied in modern textbooks. It is therefore highly recommended to consult the literature in addition to following the course. A list of classic references and some useful open-soruce material accessible online is collected here:

[GSW] Green, Schwarz, Witten: String Theory, Vol. 1& 2, Cambridge University Press 1987

[P] Polchinski: String Theory, Vol. 1& 2, Cambridge University Press 1998

[BBS] Becker, Becker, Schwarz: String Theory and M-Theory - A modern introduction, Cambridge University Press 2007

[BLT] Blumenhagen, Lüst, Theisen: Basic Concepts of String Theory, Springer 2012

[Z] Zwiebach: A first course in String Theory, Cambridge University Press 2004

[BP] Blumenhagen, Plauschinn: Introduction to Conformal Field Theory, Springer 2009

[T] Tong: Lectures on String Theory, http://arxiv.org/pdf/0908.0333

[PC] Polchinski, Chaudhuri, Johnson: Notes on D-branes, http://arXiv.org/pdf/hep-th/9602052

[U] Uranga: Introduction to String Theory http://www.ift.uam.es/paginaspersonales/angeluranga/firstpage.html

In addition, I will give more specific references at various places.
The first part of these lecture notes, in particular the choice of conventions and notation, is inspired mostly by [BLT]. In the second half we follow in addition [P].

Chapter 1

Motivation and Overview

1.1 Why do we study String Theory?

There are many reasons to study String Theory. Here are just a few:

- **String theory for idealists**
  String Theory is the leading proposal for a fundamental unified theory of Quantum Gravity *and* Quantum Field Theory. What distinguishes it from other approaches to quantising gravity is that String Theory is currently the only theory that is able to provide a **unified quantum description of all interactions of Nature**, both gravitational and non-gravitational ones. String Theory is a universal theory that deals both with the most fundamental questions of Relativity - such as the nature of spacetime singularities, black hole physics or the history of the Universe - and provides a theoretically well-founded guideline for particle physics beyond the Standard Model. Its aim is none less than to provide a theory of everything. However, one must keep in mind that at this stage the String Theory approach to Quantum Gravity and Quantum Field Theory is speculative as it is not (yet?) proven (or disproven) if this is the path chosen by Nature. In following this proposal theorists aim high...

- **String Theory for pragmatists**
  While the question whether or not string theory is the fundamental theory of Nature is certainly controversial, it is widely accepted that String Theory provides a powerful tool to study strongly coupled field theories via the concept of holography. If a theory inside a volume V obeys the **holographic principle** this means that the information about its spectrum and dynamics is encoded in the degrees of freedom residing on the boundary ∂V. That this principle is obviously at work in Quantum Gravity is reflected already in the famous Bekenstein-Hawking entropy law, according to which the entropy of a black hole is given by $S = \frac{A}{4G_N}$ with A the area of the horizon surface. More spectacularly, considerations of String Theory have revealed a much more radical manifestation of the holographic principle in the disguise of the **AdS/CFT correspondence**: String Theory on a space of negative constant curvature, an Anti-deSitter (AdS) space, is dual to a conformal field theory (CFT) on the boundary of the AdS space. The power of this remarkable insight lies in the fact this is a **strong-weak coupling duality**: Weakly coupled, perturbative string theory on AdS is equivalent to a *strongly coupled* field theory. This way, perturbative string computations can give insights into the boundary field theory in the strongly coupled regime, for which no perturbative techniques exist. This has lead to many applications and

has made String Theory a tool of interest even for physicists primarily (or exclusively) interested in aspects of field theory. The ultimate hope would be to learn something about confining gauge theories such as QCD. More recent developments include applications to condensed matter theory phenomena (AdS/CMT correspondence) or fluid dynamics (AdS/Hydrodynamics). In calling this a pragmatic approach we recall that here String Theory is primarily viewed as a tool to learn about strongly coupled field theory and not necessarily as a fundamental theory of Nature. Even if it turned out that Nature has chosen a different UV completion of gravity and Yang-Mills theory, the field theoretic insights derived via the AdS/CFT correspondence will always remain valid.

- **String theory for aesthetes**
  String Theory is a theory of fascinating mathematical complexity and beauty. It has lead to an exuberantly fruitful **interplay between mathematics and physics**, which actually goes both ways: Not only does String Theory offer a natural arena for the application of modern mathematical, in particular geometric concepts. The intuition of String Theorists has also lead to completely new developments and insights within pure mathematics. The most famous example of this interplay is the concept of Mirror Symmetry, a one-to-one correspondence of pairs of certain complex manifolds (so-called Calabi-Yau manifolds), which was first discovered in String Theory and subsequently studied within mathematics. In fact, while Mirror Symmetry is rather surprising from the purely mathematical perspective, it is almost a triviality (and was discovered as such) from the point of the view of the underlying string theoretic conformal field theory. Other examples of stringy explanations for highly non-trivial mathematical facts are the group theoretic Monster Moonshine or the classification of singularities by ADE Dynkin diagrams.

- **String theory for agnosticists**
  For these and many more reasons, String Theory has developed as an entire **framework** (rather than a specific theory) from which numerous developments of High Energy physics depart. It is fair to say that for young theoretical physicists at least a general knowledge of String Theory is almost as indispensible as a solid foundation of Quantum Field Theory if they want to follow the recent current developments. **Love it or hate it, but you must know it!**

1.2 The need for a quantum theory of gravity

We now analyse in slightly greater detail the motivation to study a theory of strings as a fundamental quantum theory of interactions. Obviously the pillars of modern physics are on the one hand Quantum Mechanics or, rather, Quantum Field Theory (QFT) and on the other hand General Relativity (GR). Both have lead to a triumphant description of many observed phenomena of Nature, but both theories are in a sense incomplete and cannot be considered fundamental theories:

- The incompleteness of QFT, at least in its perturbative formulation, becomes evident in the appearance of **ultra-violet (UV) loop divergencies** in the computation of perturbative scattering amplitudes. Such divergencies are present already for as simple theories as QED and cast serious doubt on the validity of the theory at high energies.

- The incompleteness of classical GR is exemplified in the context of black hole physics: According to the classical Einstein equations, the center of a black hole contains a **spacetime singularity** where the curvature blows up. This is particularly scandalous as black holes can form dynamically starting from a perfectly well-defined initial distribution of matter. Thus there exist *dynamical* processes that take us outside the regime of validity of the classical theory.

It is important to keep in mind that the appearing divergencies or singularities in QFT and GR are not a problem at a practical or computational level: In GR the singularity at the center of a black hole is hidden behind the black hole horizon (in fact, according to the *Cosmic Censorship Hypothesis*, this is conjectured to hold for all singularities - no naked singularities are supposed to exist). In QFT, the UV divergencies can be "hidden" at a practical level by the powerful machinery of regularisation and renormalisation (at least for renormalisable theories). But irrespective of our ability to argue away the divergencies in applications, their very appearance remains highly unsatisfactory.
In fact the situation becomes not only unsatisfactory, but even technically unaccpetable as soon as combined effects of GR and QFT are taken into account. Instances where quantum aspects in gravity cannot be neglected include the study of (small) black holes and, most notably, a successful treatment of very early cosmology. The problem is that **Einstein gravity is not renormalisable at a perturbative quantum level**.

In principle two different ways out of this dilemma are conceivable:

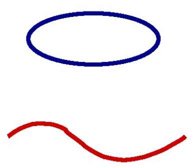
- Stick to the dynamical degrees of freedom of classical Einstein gravity, but modify the quantisation procedure. This is the philosophy underlying, among others, Loop Quantum Gravity: The aim is to quantise gravity as such, and no unification of gravity and the remaining forces in Nature is implied or even aimed at.

- Alternatively, we may stick to the conventional methods of quantisation, but change the dynamics of gravity in such a way as to recover the degrees of freedom of Einstein gravity at low energies while at the same time arriving at a consistent UV completion. This is the approach taken by String Theory. In particular, the modifications of the dynamics automatically affect GR and QFT in a way that leads to a unified description of all forces.

In fact, String Theory represents in some sense the simplest, most economical and most conservative modification of QFT and GR. The **only new dynamical input** can be summarised as follows:

The fundamental objects in Nature are not pointlike, but 1-dimensional.

This axiom is combined with the **standard kinematics of general covariance (Relativity) and the usual procedure of quantisation**. As we will see, quantisation of a generally covariant theory of strings inevitably results in a consistent unified description of gravity and Yang-Mills theory.



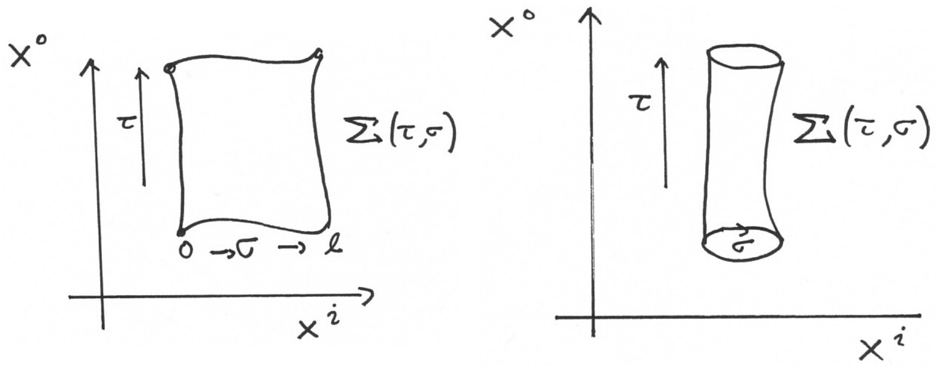
The two sectors arise from the elementary fact that a string can have two possible topologies: It can be open or closed. Open strings describe Yang-Mills theory, closed strings describe Gravity. Since open strings can close up and vice versa, Gravity and YM theory are automatically related dynamically.

What is important is that all other outcomes of the theory follow without further assumptions or arbitrariness from application of the laws of general covariance and quantisation to a theory strings.

1.3 Programme

The general roadmap, which is also the one followed in this course, can be summarised as follows:

- Define the classical action of a string propagating in d-dimensional spacetime. Generalising the concept of the (0+1)-dimensional worldline of a point particle (which is a 0-dimensional object), a string traces out a (1+1)-dimensional surface - the **string worldsheet (WS)**. The following picture shows the string worldsheet of an open and a closed string propagating in spacetime.



- Application of the rules of quantisation provides us with the **Fock space of string excitations**. The massless modes of the bosonic sector include (among others):

open string:	A_μ	spin 1	vector boson of Yang-Mills Theory
closed string:	$h_{\mu\nu}$	spin 2	graviton

  In addition one finds a **tower of massive string excitiations** of mass

$$M^2 = \frac{1}{\alpha'}(N-1) \quad \text{(open bosonic)}, \qquad M^2 = \frac{4}{\alpha'}(N-1) \quad \text{(closed bosonic)} \quad N = 0,1,2,\ldots, \tag{1.1}$$

  The appearance of the tachyon, the lowest lying state of negative mass^2 will be overcome by moving from the bosonic string theory to **superstring theory**, in which no tachyons are present. This is the theory we study eventually.

- Note the appearance of the **dimensionful parameter** α' of dimension

$$[\alpha'] = [\text{length}]^2. \tag{1.2}$$

  The intrinisic length and mass scales of the theory are thus

$$\ell_s = 2\pi\sqrt{\alpha'} : \text{string length} \qquad M_s = \frac{1}{\sqrt{\alpha'}} : \text{string mass scale.} \tag{1.3}$$

Importantly, α' is the only free paramenter of String Theory. In principle it can take any value in the range

$$\underbrace{10^{-33}\,\mathrm{cm}}_{\text{Planck length}} \leq \sqrt{\alpha'} \leq \underbrace{10^{-17}\,\mathrm{cm}}_{\text{TeVscale}}. \tag{1.4}$$

- Apart from this dimensionful parameter there are **no free dimensionless parameters**. The situation is thus fundamentally different from GR or QFT, where all coupling constants (particle masses, Yukawa couplings...) are input parameters of the action with no fundamental explanation. As we will see, the couplings in String Theory are expectation values of dynamical fields (so-called moduli) which take their value dynamically.
- The **interactions follow uniquely** and without further extra input. Essentially this is because there is a unique way in which strings can join and split.

To analyse the properties of the theory it is useful to distinguish two different regimes:

- The **UV regime** involves energies of the order or bigger than the string scale M_s or, equivalently, distances of the order of the string length ℓ_s. In this regime the extended nature of the string becomes important. As a consequence it turns out that the theory is non-local with important consequences for the nature of interactions.
- The **low-energy dynamics** kicks in at energies $\ll M_s$ or, equivalently, at distances $\gg \ell_s$, strings appear as effectively pointlike. One can deduce the low-energy effective theory by integrating out the massive string tower and focussing, to first order, on the massless string excitations.

Let us consider both regimes in turn:

Low-energy dynamics:

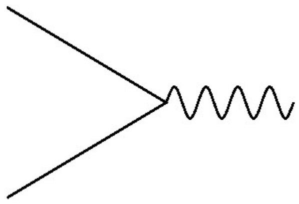
In the low-energy limit the effective dynamics reduces to gauge interactions and gravity. Consistency of the worldsheet teory *predicts* explicitly that to lowest order in $\sqrt{\alpha'}$ the gravitational laws are given by

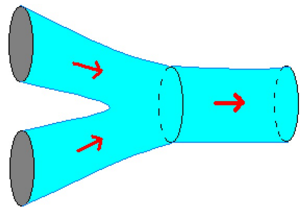
$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \tag{1.5}$$

Indeed we will see in this course how to **derive Einstein's equations as a corollary** of self-consistency of the worldsheet theory at the quantum level. In addition one finds higher curvature corrections, which can be computed systematically order by order in $\sqrt{\alpha'}/r$, with r a typical length scale in the problem.

High energy dynamics:

- Compared to a point particle theory, the string interaction is smoothened out. In fact there does no longer exist the notion of a sharply defined interaction vertex at which the interaction is localised in space and time. Rather, locally the string always look like it is freely propagating and the interactions are encoded only in the topology of the worldsheet, i.e. in global properties of the worldsheet. A schematic comparison of a point particle and a string vertex can be found in the following illustration.





This is an important feature: In point particle theories the sharp localisation of the interaction vertex is indeed responsible for the appearance of divergencies in the computation of amplitudes.

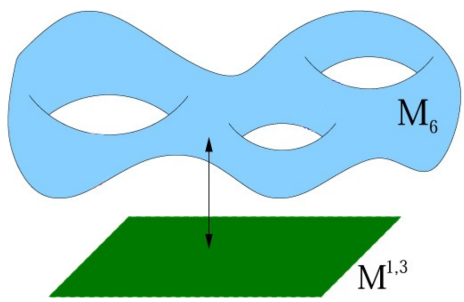
- The number of combinatorically relevant diagrams is greatly reduced compared to point particle QFT. At each loop order there exists only a single string diagram for a given process. This is to be contrasted with a factorially growing number of Feynman graphs in QFT.

These two properties are responsible for the improvement of the UV properties of scattering amplitudes. **UV finiteness** has been proven rigorously for the superstring up to 2-loop order. However, it is conjectured that all higher order amplitudes are likewise UV finite.

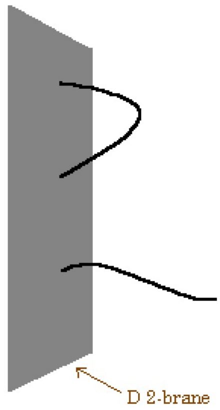
Another important property of the string interactions is that gravity and YM interactions are no longer independent: Since open strings can join to form closed strings, consistent YM interactions automatically imply the presence of gravity as well. This of course is as it must be in a quantum theory of fundamental interactions as the energy stored in, say, the electric field gravitates itself. Note that the situation is fundamentally different in other approaches to quantum gravity, where the gravitational and the YM sector are treated as indpendent building blocks. By contrast, String Theory provides ar truly **unified quantum theory of gravity and YM interactions**.

Further properties of the theory:

- Consistency of the worldsheet theory requires that the total **number of spacetime dimensions** be $d = 26$ in the bosonic string theory and $d = 10$ in the superstring, which is the one we will eventually study. String theoy is currently the only theory that predicts the number of spacetime dimensions as an outcome of the theory, not as an input.
- Contact with physics in 4 extended spacetime dimensions is made by **compactification of the extra dimensions** on a small internal manifold. The study of the resulting landscape of string vacua is the subject of string phenomenology.



- The theory predicts the presence of higher-dimensional objects called D-branes. These are hypersurfaces on which open strings can end. They play a crucial role for the dynamics of the theory.



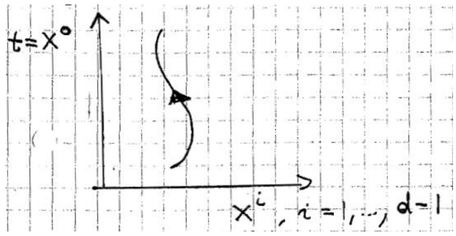
- In superstring theory the spectrum of string excitations enjoys a symmetry called **supersymmetry**, which is a symmetry between bosons and fermions. This symmetry is realised in the full ten-dimensional theory. Compactification can and, in general, does break supersymmetry. The scale at which supersymmetry is broken depends, among other things, on the geometric details of the compactification. It is important to keep in mind that low-energy supersymmetry is not a prediction of string theory. String theory is perfectly fine with supersymmetry broken at scales above the TeV scale probed currently by LHC.

Chapter 2

The classical bosonic string

2.1 The relativistic point particle

Before turning to the classical dynamics of a string, we recall some basic facts about the kinematics and dynamics of a point particle. The relevant concepts will then be generalised directly to the classical string.
The propagation of a particle in $\mathbb{R}^{1,d-1}$ is described by a **worldline** γ parametrised by $\tau \in \mathbb{R}$:



$$\gamma : \quad \tau \mapsto X^\mu(\tau) \in \mathbb{R}^{1,d-1} \ , \quad \mu = 0, i.$$

- We will stick to $\hbar = c = 1$ and work in signature (-1, 1,..., 1).
- The invariant length or line element of the metric $\eta_{\mu\nu}$ is ds with $ds^2 = -\eta_{\mu\nu} dX^\mu dX^\nu$.
- The dynamics of a massive particle with mass $m \neq 0$ is captured by the **Nambu-Goto-Action**

$$S_{NG} = -m \int_\gamma ds = -m \int_\gamma \sqrt{-\eta_{\mu\nu} dX^\mu dX^\nu}. \tag{2.1}$$

With $dX^\mu(\tau) = \frac{dX^\mu}{d\tau} d\tau \equiv \dot{X}^\mu d\tau$ this is

$$S_{NG} = -m \int_\gamma d\tau \sqrt{-\eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu} \equiv \int_\gamma d\tau L(\tau). \tag{2.2}$$

Note that τ is a priori an arbitrary parameter. For the special case that we identify τ with the proper time τ_0 of the particle S_{NG} reduces to the familiar action

$$\tilde{S} = -m \int dt \sqrt{1 - \vec{v}^2}. \qquad (\hbar = c = 1)$$

To see this recall that, reinstating c for illustrative reasons, we have $X^\mu = (ct, \vec{x})$ and the proper time τ_0 satisfies

$$\frac{dt}{d\tau_0} = \gamma = \frac{1}{\sqrt{1-\frac{\vec{v}^2}{c^2}}}, \quad \vec{v} = \frac{d\vec{x}}{dt} \Rightarrow \quad \frac{dX^\mu}{d\tau_0} = (\gamma c, \gamma\vec{v}), \quad \left(\frac{dX^\mu}{d\tau_0}\right)^2 = \gamma^2(-c^2+\vec{v}^2) = -c^2.$$

- S_{NG} is invariant under reparametrization $\tau \to \tilde{\tau}(\tau)$, which corresponds to **local diffeomorphism invariance** on the worldline.

  Under this transformation $\displaystyle \frac{dX^\mu}{d\tau} = \frac{dX^\mu}{d\tilde{\tau}}\frac{d\tilde{\tau}}{d\tau} \Rightarrow \tilde{S}_{NG} = -m\int d\tilde{\tau}\sqrt{-\eta_{\mu\nu}\frac{dX^\mu}{d\tilde{\tau}}\frac{dX^\nu}{d\tilde{\tau}}}.$

- The canonical momentum associated to X^μ is

$$P^\mu = \frac{\partial L}{\partial \dot{X}_\mu} = m\frac{\dot{X}^\mu}{\sqrt{-\dot{X}^2}}. \tag{2.3}$$

  This leads to the **constraint** $\Phi = P^2 + m^2 = 0$.

  At this stage we collect - without proof - some useful facts on constrained systems, following [BLT]:

  – A dynamical system is called singular if the matrix $M_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$ is not invertible. In this case the generalised velocities $\dot{q}^i$ cannot be expressed uniquely in terms of the q^i, p_i.

  – For a singular system each zero eigenvalue of M_{ij} gives rise to a primary constraint. A **primary constraint** is a constraint on the system that follows from the definition of the conjugate momentum *without use of the equations of motion*. Secondary constraints are further constraints introduced so that if the primary constraints are satisfied at a certain time, they remain to be so at all later times.

  – In systems with primary constraints, the Hamiltonian is not uniquely defined as one can always add terms that vanish upon imposing the constraints. If there are constraints ϕ_k the Hamiltonian is $H = H_{can} + c^k\phi_k$ where $H_{can} = p^i\dot{q}_i - L$ is the canonical Hamiltonian and c^k do not depend of q_i and p^i. Different choices of c^k depends to different choices of gauge.

  – A system with time reparametrisation invariance is always a system with a primary constraint. First, invariance under $\tau \to \tilde{\tau}(\tau)$ implies that $H_{can} = 0$ because of form invariance of the time evolution $\frac{df}{d\tau} = \frac{\partial f}{\partial \tau} + \{f, H\}_{P.B.}$ for an arbitrary function f. This implies $M_{ij}\dot{q}^j = \frac{\partial}{\partial q^i}H_{can} = 0$, i.e. the system is singular.

  – Indeed one can check explicitly that for the Nambu-Goto-action of the free particle $H_{can} \equiv 0$. The full Hamiltonian is thus

$$H = c\Phi = c(P^2 + m^2). \tag{2.4}$$

  From this,

$$\frac{dX\mu}{d\tau} = \{X^\mu, H\}_{P.B.} = c\frac{\dot{X}^\mu}{\sqrt{-\dot{X}^2}} \Longrightarrow c = \sqrt{-\dot{X}^2}, \tag{2.5}$$

  identifying the choice $c = 1$ with the gauge choice $\tau = \tau_0$, the proper time.

- The equations of motion (e.o.m.) are

$$0 = \frac{d}{d\tau}\frac{\partial L}{\partial \dot{X}^\mu} - \frac{\partial L}{\partial X^\mu} = \frac{d}{d\tau}\left(m\frac{\dot{X}^\mu}{\sqrt{-\dot{X}^2}}\right). \tag{2.6}$$

  For $\tau = \tau_0$ the proper time this reduces to $m\ddot{X}^\mu = 0$.

- The complete dynamics of the system is described by the e.o.m. **together with the constraints**. For $\tau = \tau_0$ we recover, as expected,

$$\ddot{X}^\mu = 0 \qquad \& \qquad \dot{X}^2 = -1. \tag{2.7}$$

- The disadvantage of S_{NG} is that it is not defined for $m = 0$.
  This is remedied by introducing the **Polyakov action**

$$S_P = \frac{1}{2} \int d\tau e(\tau) \left(\frac{1}{e^2(\tau)} \left(\frac{dX^\mu}{d\tau} \right)^2 - m^2 \right), \tag{2.8}$$

  with $e(\tau)$ a new independent degree of freedom.
  Invariance under **local diffeomorphisms** $\tau \to \tilde{\tau}(\tau) = \tau - \epsilon(\tau)$ is given provided X^μ and e transform as

$$\begin{aligned} X^\mu(\tau) &\rightarrow X^\mu(\tau) + \frac{dX^\mu}{d\tau}\epsilon(\tau) && (2.9)\\ e(\tau) &\rightarrow e(\tau) + \frac{d}{d\tau}(\epsilon(\tau)e(\tau)) && (2.10) \end{aligned}$$

  (Check this!)
  This means that from the worldline perspective X^μ transforms as a **scalar**, while e transforms as a **scalar density of weight 1**. More on the definition of tensor densities can be found on the first examples sheet.

- The equations of motion now follow by variation w.r.t. X^μ and $e(\tau)$.
  First, $e(\tau)$ is treated as an independent degree of freedom (d.o.f.) with associated e.o.m.

$$0 = \frac{\delta S_P}{\delta e(\tau)} \quad \Rightarrow \quad \frac{1}{e^2(\tau)} \left(\dot{X}^\mu \right)^2 + m^2 = 0. \tag{2.11}$$

$$\Rightarrow \quad e(\tau) = \frac{\sqrt{-(\dot{X}^\mu)^2}}{m} \qquad \text{if } m \neq 0. \tag{2.12}$$

  In addition we have the e.o.m. for X^μ,

$$0 = \frac{\delta S_P}{\delta X^\mu} \ \Rightarrow \ \frac{d}{d\tau} \left(\frac{1}{e(\tau)} \dot{X}^\mu \right) = 0, \tag{2.13}$$

  where we used $\delta_X \int e^{-1} \left(\frac{dX^\mu}{d\tau}\right)^2 = 2 \int \frac{d}{d\tau} \left(e^{-1} \frac{dX^\mu}{d\tau} \delta X_\mu\right) - 2 \int \delta X_\mu \frac{d}{d\tau} \left(e^{-1} \frac{dX^\mu}{d\tau}\right)$.

- To show classical equivalence of the Nambu-Goto and the Polyakov action we insert the e.o.m. of $e(\tau)$ into S_P. More generally, this procedure is called integrating out $e(\tau)$. Note, however, that classical equivalence does in general not imply equivalence also at the quantum level.

- In view of the transformation properties of $e(\tau)$ we can rewrite S_P as

$$S_P = \frac{1}{2} \int d\tau \sqrt{\det g_{\tau\tau}} \left(g^{\tau\tau} \frac{dX^\mu}{d\tau} \frac{dX_\mu}{d\tau} - m^2 \right), \tag{2.14}$$

  where $g_{\tau\tau} \equiv e^2(\tau)$ is identified as the worldline metric.

- One can use local diffeomorphism invariance to fix the gauge by setting e.g.

$$e(\tau) = \begin{cases} \frac{1}{m}, & m \neq 0. \\ 1, & m = 0. \end{cases} \tag{2.15}$$

Then **the e.o.m. for $e(\tau)$ translate into constraints in addition to the e.o.m. for X^μ:**

$$\ddot{X}^\mu = 0 \qquad \& \qquad \left\{ \begin{matrix} \dot{X}^2 = -1, & m \neq 0 \\ \dot{X}^2 = 0, & m = 0 \end{matrix} \right\} \text{ for } \tau = \tau_0 \text{ (proper time).} \tag{2.16}$$

This is a general lesson that will become crucial in the treatment of the string: **After integrating out an auxiliary field we must still impose its equations of motion in the form of constraints.**

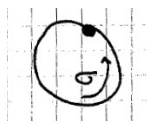
2.2 The bosonic string action

2.2.1 Nambu-Goto action

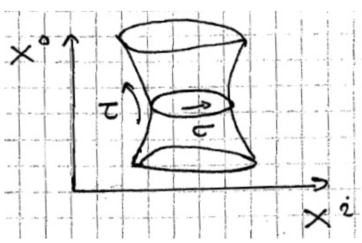
We now generalise these considerations to the classical string.

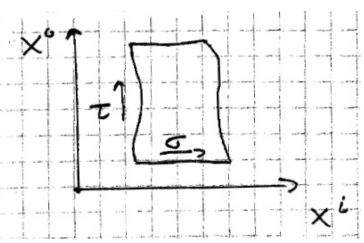
- A string is a 1-dimensional object. The position along the string is parametrised by a spatial coordinate σ, which we normalise to take values in the range $0 \leq \sigma \leq l$.
- There are 2 types of strings, corresponding to the 2 possible topologies of a 1-dimensional object:

closed open



- The propagation of a string in $\mathbb{R}^{1,d-1}$ now defines a 2-dimensional **worldsheet (WS)** Σ





Σ is parametrised by $\tau \in \mathbb{R}$ amd $0 \leq \sigma \leq l$ and can be viewed as the map

$$\Sigma \quad : \qquad (\tau,\ \sigma) \mapsto X^\mu(\tau,\ \sigma) \in \mathbb{R}^{1,d-1}. \tag{2.17}$$

This map is called the **string map**. It provides an embedding of the WS into the ambient space.
Oftentimes we write $(\tau,\ \sigma) \equiv \xi^a, \qquad a = 0,\ 1$ for the worldsheet coordinates.

- In complete analogy to point particles one defines the Nambu-Goto action

$$S_{NG} = -T \int_\Sigma dA, \tag{2.18}$$

where dA is the area element of Σ

$$dA = \sqrt{-\det\left\{\frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\eta_{\mu\nu}\right\}}d^2\xi. \tag{2.19}$$

(Note: For point-particle $ds = \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}\eta_{\mu\nu}}\, d\tau$.)

The object

$$G_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b}\eta_{\mu\nu} \tag{2.20}$$

is called the **induced metric** on Σ or the pullback of the ambient space metric $\eta_{\mu\nu}$ onto Σ. Therefore

$$S_{NG} = -T \int_\Sigma \sqrt{-\det G_{ab}}d^2\xi. \tag{2.21}$$

- X^μ and ξ^a have mass dimension (-1)

$$\Rightarrow [T] = [\text{mass}^2] = [\text{length}^{-2}].$$

T is called the string tension. One further defines the quantities

$$T = \frac{1}{2\pi\alpha'} \qquad \alpha' \equiv \text{Regge slope.} \tag{2.22}$$

$$\ell_s = 2\pi\sqrt{\alpha'} \qquad \ell_s \equiv \text{string length.} \tag{2.23}$$

$$M_s = \frac{1}{\sqrt{\alpha'}} \qquad M_s \equiv \text{ string (mass) scale.} \tag{2.24}$$

2.2.2 The Polyakov action

As for the point particle it is more convenient to eliminate the the square root in S_{NG}. To this end we introduce an auxiliary field, the WS metric $h_{ab}(\xi^a)$, and define the **Brink-DiVecchia-Howe (BDH)** or **Polyakov action**

$$S_P = -\frac{T}{2}\int_\Sigma d^2\xi\sqrt{-\det h}\, h^{ab}(\xi)\, \partial_a X^\mu(\xi)\partial_b X^\nu(\xi)\eta_{\mu\nu} = -\frac{T}{2}\int_\Sigma d^2\xi\sqrt{-\det h}\, h^{ab}\, G_{ab}\,. \tag{2.25}$$

Comments:

- $h_{ab}(\xi^a)$ is a (symmetric) 2-tensor on the WS, which plays the role of the **intrinsic WS metric**. A priori it is independent of the inherited or pullback metric $G_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$.

- $X^\mu(\xi^a)$ is a **scalar on the WS** (since it carries no "a, b" index), but a **spacetime vector** (as evident due to its"μ" index).

- The metric h_{ab} is in general curved and endowes the WS with the Levi-Civita-connection. The associated covariant derivative ∇_a on the WS is defined w.r.t. $\Gamma^a{}_{bc} = \frac{1}{2}h^{ad}(\partial_b h_{cd} + \partial_c h_{bd} - \partial_d h_{ab})$. Recall that for a scalar field the partial and the covariant derivative agree. In particular $\nabla_a X^\mu(\xi) = \partial_a X^\mu(\xi)$.

- The metric h has signature $(-,+)$. This explains the overall minus sign in S_P (compared to the expression for the point particle).

It is crucial to appreciate that in String Theory the **spacetime coordinates** X^μ of the string are promoted to **dynamical fields**[1] in the 2-dimensional field theory on the worldsheet defined by the Polyakov action. In a sense, **spacetime becomes a derived concept**. The fundamental object is the field theory on the worldsheet.
Furthermore we consider a 2-dimensional field theory on a curved worldsheet whose metric we treat dynamically. In other words:

The Polyakov action for propagation of a string in $\mathbb{R}^{1,d-1}$ describes d two-dimensional scalar fields $X^\mu(\xi)$ coupled to the dynamical WS metric $h_{ab}(\xi)$. Studying (bosonic) string theory is equivalent to studying 2-dimensional gravity coupled to scalars.

How general is S_P?

One could add two types of terms to S_P:

i) A 2-dimensional "cosmological constant" term $S_1 = \lambda_1 \int_\Sigma \sqrt{-\det h}$. This would spoil the crucial consistency condition of conformal invariance discussed in detail later. Thus we require $\lambda_1 = 0$.

ii) A 2-dimensional Einstein-Hilbert term $S_2 = \frac{\lambda_2}{4\pi}\int_\Sigma \sqrt{-\det h}\, R^{(2)}$ with $R^{(2)}$ the Ricci scalar of the WS connection associated with $\Gamma^a{}_{bc}$. In 2-dimensions this term has no dynamics as it is given by a total derivative. This corresponds to the fact that 2-dimensional gravity is dynamically trivial (see the exercises for details). In fact: $\chi(\Sigma) \equiv \frac{1}{4\pi}\int_\Sigma \sqrt{-\det h}\, R^{(2)}$ is a topological invariant called **Euler characteristic** - it is an integer invariant under continuous deformations of the WS metric. While it has no local dynamics it will weigh different WS topologies differently in the path integral. We ignore this term for now and revisit it when setting up the perturbation series in the path integral approach to quantisation.

For completeness, let us stress that the above expression for $\chi(\Sigma)$ refers to closed strings. For open strings the WS has a boundary $\partial\Sigma$ and the expression for the Euler characteristic is

$$\chi = \frac{1}{4\pi}\int_\Sigma d^2\xi\sqrt{-h}\mathcal{R} + \frac{1}{2\pi}\int_{\partial\Sigma} ds\,\mathcal{K}. \tag{2.26}$$

Here the extrinisic curvature $\mathcal{K}$ is defined as

$$\mathcal{K} = \pm t^a\, n_b\, \nabla_a\, t^b \tag{2.27}$$

with t^a a unit vector tangent to the boundary and n^a an outward unit vector orthogonal to t^a. The upper/lower sign refer to timelike/spacelike boundaries.

The energy momentum tensor

An important quantity in field theory in the energy momentum tensor.

[1]Formally, the same holds of course the for the spacetime coordinates of a point particle. The resulting 1-dimensional "field theory", however, is trivial and does not add any new perspective. This will be seen to be completely different in 2 dimensions.

- The energy momentum tensor T_{ab} measures the response of S_P w.r.t. a metric variation δh^{ab}, while keeping all other fields unchanged. Our conventions for its normalisation are[2]

$$\delta S_P = \frac{1}{4\pi}\int d^2\xi\sqrt{-h}\,T_{ab}\,\delta h^{ab}, \tag{2.28}$$

$$\text{i.e.} \qquad T_{ab} = \frac{4\pi}{\sqrt{-h}}\frac{\delta S_P}{\delta h^{ab}} \qquad (h := \det h). \tag{2.29}$$

Using

$$\delta h = -h\,h_{ab}(\delta h^{ab}), \tag{2.30}$$

an identity proven in the exercises, one finds

$$T_{ab} = -\frac{1}{\alpha'}\left(G_{ab} - \frac{1}{2}h_{ab}\,h^{cd}\,G_{cd}\right), \tag{2.31}$$

where again $G_{ab} = \partial_a X \cdot \partial_b X$.

- The equations of motion for h_{ab} are $T_{ab} = 0$, i.e.

$$G_{ab} = \frac{1}{2}(h \cdot G)\,h_{ab}. \tag{2.32}$$

Thus on-shell, the pullback of the ambient space metric, G_{ab}, and the induced metric h_{ab} are proportional.

- As the for the point particle one finds that on-shell for h_{ab}, S_P and S_{NG} agree,

$$S_P[X, h_{ab}\Big|_{T_{ab}=0}] = S_{NG}. \tag{2.33}$$

This will be proven in the exercises.

- **Properties of T_{ab}**

  i) T_{ab} is **traceless**,

$$T^a{}_a = -\frac{1}{\alpha'}h_{ab}(G^{ab} - \frac{1}{2}h^{ab}(h \cdot G)) = 0. \tag{2.34}$$

  This holds **without** use of the e.o.m. for X^μ. Note that the two terms in $T^a{}_a$ cancel because the WS is 2-dimensional and thus $h_{ab}\,h^{ab} = 2$. The special significance of the dimensionality of the WS will become clearer momentarily.

  ii) T_{ab} is **conserved**,

$$\nabla^a T_{ab} = 0. \tag{2.35}$$

  This holds **upon use** of e.o.m. for X^μ. The fact that we need the e.o.m. for X^μ is a consequence of the specific definition of T_{ab} as describing response to a variation only of h_{ab}, not of X. Thus $\frac{\delta S}{\delta X} \equiv 0$ must be assumed to show conservation.

Both properties are rooted in the

[2]Beware that the normalisation differs from textbook to textbook, and in some context also the variation of fields other than the metric is included.

2.2.3 Symmetries of the Polyakov action

Corresponding to the fact that we have a dual interpretation of the X^μ - either as the string coordinates in spacetime or as dynamical scalar fields on the 2-dimensional WS - we must carefully distinguish between the

- spacetime symmetries acting on $\mathbb{R}^{1,d-1}$ and the
- WS symmetries acting on Σ.

i) **Spacetime symmetries**
The Polyakov action enjoys manifest d-dimensional Poincaré-invariance,

$$X^\mu(\xi) \mapsto \Lambda^\mu{}_\nu X^\nu(\xi) + V^\mu, \qquad \Lambda \in SO(1, d-1). \tag{2.36}$$

Note that Poincaré-invariance can be interpreted as a **global internal symmetry** from the perspective of the 2-dim. field theory.

ii) **WS symmetries**

1. Local diffeomorphism invariance under

$$\xi^a \mapsto \tilde{\xi}^a(\xi) = \xi^a - \epsilon^a(\xi). \tag{2.37}$$

The various fields transform according to their tensorial nature:

- $X^\mu(\xi)$ is a scalar field from the WS perspective and thus transforms as

$$\begin{aligned} X^\mu(\xi) \mapsto \tilde{X}^\mu(\tilde{\xi}) &= X^\mu(\xi(\tilde{\xi})) = X^\mu(\xi) + \epsilon^c \partial_c X^\mu(\xi) && (2.38)\\ &\equiv X^\mu + \delta X^\mu + \mathcal{O}(\epsilon^2), && (2.39)\\ \delta X^\mu &= \epsilon^c \partial_c X^\mu. && (2.40) \end{aligned}$$

- The metric $h_{ab}(\xi)$ transforms like a WS 2-tensor, i.e.

$$h_{ab}(\xi) \mapsto \tilde{h}_{ab}(\tilde{\xi}) = \frac{\partial \xi^c}{\partial \tilde{\xi}^a}\frac{\partial \xi^d}{\partial \tilde{\xi}^b} h_{cd}(\xi(\tilde{\xi})) = h_{ab} + \delta h_{ab} + \mathcal{O}(\epsilon^2), \tag{2.41}$$

$$\begin{aligned} \delta h_{ab} &= \epsilon^c \partial_c h_{ab} + (\partial_a \epsilon^c) h_{cb} + (\partial_b \epsilon^c) h_{ac}, && (2.42)\\ &= \nabla_a \epsilon_b + \nabla_b \epsilon_a. && (2.43) \end{aligned}$$

- The object $\sqrt{-\det h}$ turns out to transform like a scalar density of weight 1,

$$\delta\sqrt{-\det h} = \partial_c(\epsilon^c \sqrt{-\det h}). \tag{2.44}$$

More details are presented in the exercises.

2. Local conformal, i.e. Weyl invariance,

$$\begin{aligned} \delta X^\mu &= 0, && (2.45)\\ h_{ab} &\to \exp(2\,\Lambda(\xi))\, h_{ab} = h_{ab} + \delta h_{ab} + \mathcal{O}(\Lambda^2), && (2.46)\\ \delta h_{ab} &= 2\,\Lambda(\xi) h_{ab}. && (2.47) \end{aligned}$$

Crucially this extra symmetry is special to the fact that the WS is 2-dimensional. To appreciate this, let us pretend the WS were D-dimensional and compute

$$\sqrt{-\det h} \mapsto (e^{2\Lambda})^{\frac{D}{2}} \sqrt{-\det h}, \tag{2.48}$$

$$h^{ab} \mapsto (e^{2\Lambda})^{-1} h^{ab}. \tag{2.49}$$

It follows that $\int d^D\xi \sqrt{-\det h}\, h^{ab}\, G_{ab}$ is invariant iff $D = 2$.

Comments

- Weyl invariance will become pivotal for a consistent quantisation in the sequel. Note that the cosmological constant term $\lambda \int \sqrt{-\det h}$ spoils Weyl invariance at the classical level. Since our whole quantisation scheme depends on Weyl invariance we therefore discard this term for now.
- The appearance of Weyl invariance for 2-dimensional worldhseets identifies *String* Theory as a very special generalisation of the point particle theory.

  Indeed one might wonder about even higher dimensional generalisations, e.g. to worldsheets of dimensions $D = 3$ corresponding to so-called membranes, or $D > 3$. It turns out that already the case $D = 3$ cannot be treated with the same methods that are successful in the string case. Part of the reason is the lack of conformal invariance. In addition, the gauge fixed action is no longer free for $D \geq 3$. A detailed acount of these complications can be found in the review by Helling and Nicolai, http://arXiv.org/pdf/hep-th/9809103. At this stage it is not known how to consistently quantise membranes perturbatively. Nonetheless they play a crucial role as non-perturbative objects in M-theory.

Finally note that under combined diffeomorphism and Weyl rescaling the metric transforms as

$$\begin{aligned} \delta h_{ab} &= \nabla_a \epsilon_b + \nabla_b \epsilon_a + 2\Lambda h_{ab} && (2.50)\\ &= \nabla_a \epsilon_b + \nabla_b \epsilon_a - \nabla^c \epsilon_c h_{ab} + 2(\Lambda + \frac{1}{2}\nabla^a \epsilon_a) h_{ab}, && (2.51)\\ \delta h_{ab} &\equiv (P \cdot \epsilon)_{(ab)} + 2\tilde{\Lambda} h_{ab}. && (2.52) \end{aligned}$$

The linear operator P maps vectors to symmetric traceless 2-tensors. For transformations ϵ_a such that $(P \cdot \epsilon)_{ab} = 0$ the effect on h_{ab} can be undone by Weyl rescaling. The corresponding ϵ_a are called **conformal Killing vectors.**

The presence of the above spacetime and WS symmetries leads to **conserved quantities** via **Noether's theorem**, as revised in the exercises.

i) Spacetime invariance

- Global Poincaré-invariance of spacetime,

$$X^\mu(\xi) \mapsto X^\mu(\xi) + V^\mu, \tag{2.53}$$

  implies conservation of the energy-momentum current

$$\begin{aligned} P^a_\mu &= -T\sqrt{-\det h}\, h^{ab}\, \partial_b X_\mu, && (2.54)\\ \partial_a P^a_\mu &= \nabla_a P^a_\mu = 0. && (2.55) \end{aligned}$$

  In the last step we used that P^a_μ is a vector density of weight 1 from the WS perspective and thus $\partial_a P^a_\mu = \nabla_a P^a_\mu$.

- Invariance under global Lorentz transformation, $X^\mu \mapsto \Lambda^\mu{}_\nu X^\nu(\xi)$, implies conservation of the angular momentum current

$$\begin{aligned} J^a_{\mu\nu} &= -T\sqrt{-h} h^{ab}(X_\mu \partial_b X_\nu - X_\nu \partial_b X_\mu) && (2.56)\\ &= X_\mu P^a_\nu - X_\nu P^a_\mu, && (2.57)\\ \nabla_a J^a_{\mu\nu} &= 0. && (2.58) \end{aligned}$$

ii) Worldsheet symmetries

- T_{ab} is the conserved current w.r.t. local WS diffeomorphism invariance,

$$\left.\begin{array}{l}\xi \mapsto \tilde{\xi} = \xi - \epsilon(\xi) \\ h_{ab} \mapsto h_{ab} + \nabla_a \epsilon_b + \nabla_b \epsilon_a\end{array}\right\} \Rightarrow \nabla^a T_{ab} = 0 \quad \text{on-shell for} \quad X. \tag{2.59}$$

  This is easy to prove:

$$0 = \frac{1}{4\pi} \int d^2\xi \sqrt{-h}\, T_{ab}\, \delta h^{ab} = \frac{1}{2\pi} \int d^2\xi \sqrt{-h}\, T_{ab}\, \nabla^a \epsilon^b \tag{2.60}$$

$$= \frac{1}{2\pi} \int d^2\xi\, \nabla^a \Big(\sqrt{-h}\, T_{ab}\, \epsilon^b \Big) - \frac{1}{2\pi} \int d^2\xi\, \nabla^a (\sqrt{-h} T_{ab})\, \epsilon^b. \tag{2.61}$$

  Since $\nabla^a \sqrt{-h} = 0$ for the Levi-Civita metric we conclude $\nabla^a T_{ab} = 0$.
  Note again that this holds only on-shell for X^μ because by definition T_{ab} only measures effect of diffeomorphism on h_{ab} while $\delta X = 0$.
- Tracelessness $T^a{}_a = 0$ is a consequence of Weyl invariance as proven in the exercises.

Important generalisation

Every conformal Killing vector, i.e. every ϵ_a satisfying $(P \cdot \epsilon)_{ab} = 0$, yields a conserved current

$$J^a_\epsilon = T^{ab} \epsilon_b \quad \text{with} \quad \nabla_a J^a_\epsilon = 0. \tag{2.62}$$

The proof goes as follows: Since T^{ab} is conserved we have

$$\nabla_a J^a = \nabla_a (T^{ab} \epsilon_b) = T^{ab} \nabla_a \epsilon_b. \tag{2.63}$$

Now we solve the conformal Killing equation $0 = (P \cdot \epsilon)_{ab}$ for $\nabla_a \epsilon_b + \nabla_b \epsilon_a = h_{ab}(\nabla^c \epsilon_c)$, which we plug in, using symmetry of T^{ab},

$$\nabla_a J^a = \frac{1}{2} T^{ab} (\nabla_a \epsilon_b + \nabla_b \epsilon_a) = \frac{1}{2} T^{ab} h_{ab} (\nabla^c \epsilon_c) = 0. \tag{2.64}$$

In the last equation tracelessness $T^a{}_a = 0$, which is a consequence of Weyl invariance, is crucial.

2.2.4 Flat worldsheet coordinates

On a 2-dimensional worldsheet the local diffeomorphism and Weyl invariance can be used to "fix the metric" h_{ab} by locally gauging away all its parameters.

- That this is possible can be anticipated already by counting degrees of freedom. For a D-dimensional worldsheet this counting goes as follows:

$$\left.\begin{array}{rl} h_{ab}: & \frac{1}{2} D(D+1) \quad \text{d.o.f.} \\ \text{diffeomorphism + Weyl}: & D+1 \quad \text{d.o.f} \end{array}\right\} \quad \begin{array}{l}\frac{D}{2}(d+1) - (D+1) \text{ d.o.f. can-} \\ \text{not be gauged away locally.}\end{array}$$

  Thus we see that precisely if $D = 2$ we have a chance to locally gauge away all parameters.

- In fact, in the exercises we will show that under a Weyl transformation $h_{ab} \mapsto \exp(2\omega(\xi))h_{ab}$ the Ricci scalar on the worldsheet transforms as

$$\sqrt{-h}R \mapsto \sqrt{-h}(R - 2\nabla^2\omega). \tag{2.65}$$

  By solving this differential equation for ω we can achieve that locally $R = 0$. In two dimensions this implies that also the Riemann tensor vanishes locally, as discussed in the exercises. Thus, locally the metric can be made flat $R = 0$. The remaining diffeomorphism invariance can now be used to bring this flat metric into the form

$$h_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.66}$$

- This is true locally. There may exist topological obstructions to setting $h_{ab} = \eta_{ab}$ globally on the worldsheet. In this case there remain parameters in the metric, so-called moduli, which cannot be removed by a conformal rescaling and diffeomorphisms. Such effects will be discussed later; for now only local considerations are relevant.

Sometimes it is useful to work in *conformal gauge*

$$h_{ab} = \Omega^2(\sigma,\tau)\,\eta_{ab}. \tag{2.67}$$

For now we work in *flat gauge* by setting

$$h_{ab} = \eta_{ab} \tag{2.68}$$

In flat gauge the Polyakov action reduces to the action of a set of free scalar fields,

$$S_P = \frac{T}{2}\int_\Sigma d\tau d\sigma \Big((\partial_\tau X)^2 - (\partial_\sigma X)^2 \Big). \tag{2.69}$$

A convenient set of coordinates are the **lightcone coordinates**:

- These are defined as

$$\xi^+ = \tau + \sigma\,, \qquad \xi^- = \tau - \sigma. \tag{2.70}$$

- The metric in light cone coordinates follows from

$$ds^2 = -d\tau^2 + d\sigma^2 = -d\xi^+ d\xi^- \tag{2.71}$$

  as

$$h_{++} = h_{--} = 0\,; \quad h_{+-} = h_{-+} = -\frac{1}{2}\,; \quad h^{+-} = h^{-+} = -2. \tag{2.72}$$

  For example, raising and lowering indices with this metric implies reolation of the type

$$V_+ = h_{+-}V^- = -\frac{1}{2}V^-. \tag{2.73}$$

- The partial derivatives follow as

$$\partial_\pm = \frac{\partial}{\partial\xi^\pm} = \frac{1}{2}(\partial_\tau \pm \partial_\sigma). \tag{2.74}$$

- The measure in lightcone gauge reads

$$d\tau d\sigma = d\xi^+ d\xi^- \ \det\frac{\partial(\sigma,\tau)}{\partial(\xi^+,\xi^-)} = \frac{1}{2}d\xi^+ d\xi^-. \tag{2.75}$$

The Polyakov action in lightcone gauge

- The Polyakov action takes the form

$$S_P = T \int d^2\xi^{\pm}\, \partial_+ X^\mu \partial_- X^\nu\, \eta_{\mu\nu}. \tag{2.76}$$

- The energy momentum tensor

$$T_{ab} = -\frac{1}{\alpha'}(\partial_a X \cdot \partial_b X - \frac{1}{2} h_{ab}\, h^{cd} \partial_c X \cdot \partial_d X) \tag{2.77}$$

  has the following properties in lightcone gauge:

  - $T_{+-} = T_{-+} = 0$, which is the statement of tracelessness $T_{ab}h^{ab} = 0$.
  - The non-vanishing components are

$$T_{++} = -\frac{1}{\alpha'}\partial_+ X \cdot \partial_+ X, \qquad T_{--} = -\frac{1}{\alpha'}\partial_- X \cdot \partial_- X. \tag{2.78}$$

  - Conservation $\nabla^a T_{ab} = 0$ implies $\partial_- T_{++} = \partial_+ T_{--} = 0$ such that

$$T_{++} = T_{++}(\xi^+) \quad \text{and} \quad T_{--} = T_{--}(\xi^-)$$

.

Two crucial remarks

- Before going to flat gauge, the e.o.m. for h_{ab} gave $T_{ab} = 0$. This condition has to be implemented as a **constraint in flat coordinates**. In lightcone gauge the constraints therefore take the form

$$T_{++} = 0 = T_{--}.$$

  We will see that that a proper treatment of these constraints will save the day when it comes to quantising the string.

- Even after gauge fixing there is left a large **residual symmetry** in that setting $h_{ab} = \eta_{ab}$ does not completely remove the gauge freedom. The generators of this residual gauge symmetry are the conformal Killing vectors

$$(P\epsilon)_{ab} = 0 \quad \Leftrightarrow \nabla_a \epsilon_b + \nabla_b \epsilon_a = h_{ab}(\nabla_c \epsilon^c),$$

  whose effect on the metric can be undone by a Weyl rescaling.

  In flat gauge the conformal Killing vector equation takes the form $\partial_+ \epsilon_+ = \partial_- \epsilon_- = 0$ and thus

$$\begin{array}{lcl} \partial_+ \epsilon^- = 0 & \Rightarrow & \epsilon^- \equiv \epsilon^-(\xi^-) \\ \partial_- \epsilon^+ = 0 & & \epsilon^+ \equiv \epsilon^+(\xi^+) \end{array} \tag{2.79}$$

Conformal Killing transformations (CKT)
Since the last observation is so important in what follows, let us expand on it once more: A transformation

$$\xi^+ \mapsto \tilde{\xi}^+(\xi^+) = \xi^+ - \epsilon^+(\xi^+)\,, \quad \xi^- \mapsto \tilde{\xi}^-(\xi^-) = \xi^- - \epsilon^-(\xi^-) \tag{2.80}$$

can be undone by a Weyl transformation and is not fixed by the gauge $h_{ab} = \eta_{ab}$.

We already saw before that the Conformal Killing Vector transformations imply infinitely many conserved currents. In lightcone gauge this fact manifests itself as follows,

$$\nabla^a(T_{ab}\epsilon^b) = 0 \quad \Rightarrow \quad \begin{array}{l} \partial_-(T_{++}(\xi^+)\epsilon^+(\xi^+)) = 0 \\ \partial_+(T_{--}(\xi^-)\epsilon^-(\xi^-)) = 0 \end{array} \tag{2.81}$$

The conserved charges associated with these currents are[3]

$$L_{\epsilon^+} = \frac{\ell}{4\pi^2}\int d\sigma \epsilon^+(\epsilon^+)T_{++}(\xi^+), \tag{2.82}$$

$$L_{\epsilon^-} = \frac{\ell}{4\pi^2}\int d\sigma \epsilon^-(\epsilon^-)T_{--}(\xi^-). \tag{2.83}$$

2.2.5 Hamiltonian formalism

So far we have been working in a lagrangian formulation with action, in flat gauge,

$$S_P = -\frac{T}{2}\int d\tau d\sigma\, \partial_a X \cdot \partial_b X \eta^{ab} = \int d\tau d\sigma \mathcal{L}. \tag{2.84}$$

- We can move to a Hamiltonian formulation with canonical fields $X^\mu(\tau,\sigma)$ and conjugate momenta
$$\Pi^\mu(\tau,\sigma) = \frac{\partial \mathcal{L}}{\partial \dot{X}_\mu(\tau,\sigma)} = T\dot{X}^\mu(\tau,\sigma)$$
by defining the Hamiltonian
$$H = \int_0^\ell d\sigma(\dot{X}^\mu(\tau,\sigma)\Pi_\mu(\tau,\sigma) - \mathcal{L}) = T\int_0^\ell d\sigma[(\partial_+X)^2 + (\partial_-X)^2]. \tag{2.85}$$

- To define the symplectic structure via the Poisson brackets one takes all fields at equal time τ. For fields $F(\tau,\sigma)$, $G(\tau,\sigma')$ the Poisson brackets are then defined as
$$\{F,\, G\} := \int d\tilde{\sigma}\left(\frac{\partial F(\tau,\sigma)}{\partial X^\mu(\tau,\tilde{\sigma})}\frac{\partial G(\tau,\sigma')}{\partial \Pi_\mu(\tau,\tilde{\sigma})} - \frac{\partial G(\tau,\sigma')}{\partial X^\mu(\tau,\tilde{\sigma})}\frac{\partial F(\tau,\sigma)}{\partial \Pi_\mu(\tau,\tilde{\sigma})}\right). \tag{2.86}$$

- This leads to the canonical equal time Poisson bracket relations
$$\{X^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')\} = \eta^{\mu\nu}\delta(\sigma-\sigma') \tag{2.87}$$
$$\{X, X\} = 0 = \{\Pi,\Pi\}. \tag{2.88}$$

- **The Poisson brackets of the conserved charges generate the associated symmetries.** E.g. for the residual conformal symmetries $\xi^+ \mapsto \tilde{\xi}^+(\xi^+) = \xi^+ + f(\xi^+)$ and $\xi^- \mapsto \tilde{\xi}^-(\xi^-) = \xi^- + g(\xi^-)$:
$$\{L_f, X(\tau,\sigma)\} = -\frac{\ell}{2\pi}f(\xi^+)\partial_+X(\tau,\sigma) \tag{2.89}$$
$$\{L_g, X(\tau,\sigma)\} = -\frac{\ell}{2\pi}g(\xi^-)\partial_-X(\tau,\sigma) \tag{2.90}$$
where $L_f = -\frac{\ell}{4\pi^2}\int_0^l d\sigma f(\xi^+)T_{++}(\xi^+)$ etc.. This will be proven in the exercises.

[3]Our conventions are that if we define $\xi^\pm \to \xi^\pm - \epsilon^\pm(\xi^\pm)$, $L\epsilon^\pm$ is defined with a plus as given here.

2.3 Oscillator expansions

2.3.1 Equations of motion and boundary conditions

It is time to analyse the equations of motion of the string fields $X^\mu(\tau,\sigma)$. We noted already that in flat gauge the action

$$S_P = 2T\int d\tau d\sigma\,\partial_+ X\cdot\partial_- X = \frac{T}{2}\int d\tau d\sigma\left((\partial_\tau X)^2 - (\partial_\sigma X)^2\right)$$

takes the form of a theory for d *free* scalars $X^\mu(\tau,\sigma)$.
To obtain the e.o.m. for X^μ we vary S_P such that $\delta X^\mu(\tau=-\infty)$ and $\delta X^\mu(\tau=\infty)$ vanish. This gives

$$\delta S_P = \frac{T}{2}\int d\tau d\sigma\,[2\partial_\tau X\cdot\partial_\tau\delta X - 2\partial_\sigma X\cdot\partial_\sigma\delta X] \tag{2.91}$$

$$= -T\int d\tau d\sigma\,[(\partial_\tau^2-\partial_\sigma^2)X\cdot\delta X] \tag{2.92}$$

$$+\,T\int d\sigma\,\partial_\tau X\cdot\delta X|_{\tau=-\infty}^{\infty} - T\int d\tau\,\partial_\sigma X\cdot\delta X|_{\sigma=-\ell}^{\ell}. \tag{2.93}$$

- The τ-boundary term vanishes by the assumption $\delta X|_{\tau=-\infty}^{\infty}=0$.
- **If** also σ-boundary term vanishes, then the e.o.m take the form of a **free wave equation**

$$(\partial_\tau^2-\partial_\sigma^2)X^\mu = 0, \qquad \Leftrightarrow \qquad \partial_+\partial_- X^\mu = 0. \tag{2.94}$$

We must now impose boundary conditions such that indeed the boundary terms vanish.[4] The consistent boundary conditions are classified as follows:

1. **Closed string**
   By definition, a closed string propagating in $\mathbb{R}^{1,d-1}$ is subject to periodic boundary conditions
   $$X^\mu(\tau,\ \sigma=0) = X^\mu(\tau,\ \sigma=\ell). \tag{2.95}$$
   Thus, the boundary terms at $\sigma=0$ and $\sigma=\ell$ cancel each other.
   More generally, cancellation of the boundary terms only requires
   $$X^\mu(\tau,\ \sigma=0) = M^\mu{}_\nu X^\nu(\tau,\ \sigma=\ell)\ , \tag{2.96}$$
   with $M^\mu{}_\nu$ an $O(1,\ d-1)$ matrix.
   However, if we want to interpret $X^\mu(\tau,\sigma)$ as string embedding coordinates and preserve Poincaré invariance in $\mathbb{R}^{1,d-1}$, then we need (2.95). For more general $M^\mu{}_\nu\in O(1,\ d-1)$ the string target space is an orbifold, a space describable as flat space modded out by the action of a group.

2. **Open string**
   The two string endpoints are independent. Thus the boundary terms at $\sigma=0$ and $\sigma=l$ have to vanish separately. For each X^μ and for each $\sigma=0,\ l$ we can impose either
   $$a)\ \text{Neumann boundary conditions:}\qquad \partial_\sigma X^\mu|_{\sigma=0\text{ and/or }\sigma=l}=0 \tag{2.97}$$

[4]Otherwise it would not be possible to define a local 2-dimensional worldsheet theory as the e.o.m would involve non-local boundary contributions in addition to the second-order PDE characterising the free wave equation.

$$b)\text{ Dirichlet boundary conditions:} \qquad \delta X^\mu|_{\sigma=0 \text{ and/or } \sigma=l} = 0. \tag{2.98}$$

Dirichlet boundary conditions for X^μ correspond to the string endpoint being fixed in the $\mu-$direction. For each μ the possible combinations of boundary conditions at the endpoints are therefore (NN), (DD), (ND).

2.3.2 Closed string expansion

We now give the most general solution of the string equations of motion subject to periodic boundary conditions. The free wave equation

$$\partial_+\partial_- X^\mu = 0 \tag{2.99}$$

implies that X^μ is the sum of a left- and a right-moving wave along the string,

$$X^\mu = X_L^\mu(\xi^+) + X_R^\mu(\xi^-), \tag{2.100}$$

such that $X^\mu(\tau,\ \sigma=0) = X^\nu(\tau,\ \sigma=\ell)$.
The most general solution is given by the Fourier expansion

$$X_R^\mu(\tau-\sigma) = \frac{1}{2}(x^\mu + c^\mu) + \frac{1}{2}\frac{2\pi\alpha'}{\ell}p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z},\ n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-\frac{2\pi}{\ell}in(\tau-\sigma)}, \tag{2.101}$$

$$X_L^\mu(\tau+\sigma) = \frac{1}{2}(x^\mu - c^\mu) + \frac{1}{2}\frac{2\pi\alpha'}{\ell}p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z},\ n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-\frac{2\pi}{\ell}in(\tau+\sigma)}, \tag{2.102}$$

where

- α_n^μ, $\tilde{\alpha}_n^\mu$ represent **independent** right-/left-moving Fourier modes with the convention that the positive frequency modes correspond to $n < 0$ and
- we take $c^\mu = 0$ for the time being.
- The left-/right-moving zero mode piece p^μ is coupled by the boundary conditions.
- Reality $X^\mu(\tau,\sigma) = [X^\mu(\tau,\sigma)]^*$ implies

$$x^\mu = (x^\mu)^* \ ; \quad p^\mu = (p^\mu)^* \ ; \quad (\alpha_n^\mu)^* = \alpha_{-n}^\mu \ ; \quad (\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu \ . \tag{2.103}$$

With the definition

$$\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu \tag{2.104}$$

one obtains

$$\partial_- X^\mu = \quad \dot{X}_L^\mu = \quad \frac{2\pi}{l}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-\frac{2\pi}{l}in(\tau-\sigma)}, \tag{2.105}$$

$$\partial_+ X^\mu = \quad \dot{X}_R^\mu = \quad \frac{2\pi}{l}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\tilde{\alpha}_n^\mu e^{-\frac{2\pi}{l}in(\tau+\sigma)}. \tag{2.106}$$

Later we will understand that it is the derivatives $\partial_\pm X^\mu$ rather than X^μ themselves which represent good fields on the worldsheet.
The above prefactors were chosen such that the following interpretation of the coefficients can be given:

- x^μ is the center-of-mass position of the string at $\tau = 0$ because

$$q^\mu = \frac{1}{\ell}\int_0^\ell d\sigma X^\mu = x^\mu + \frac{2\pi\alpha'}{\ell} p^\mu \tau. \tag{2.107}$$

  This already implies that p^μ is the total spacetime momentum of the string.

- Indeed this is backed up by noting that

$$\int_0^\ell d\sigma \Pi^\mu(\tau,\sigma) = \frac{1}{2\pi\alpha'}\int_0^\ell d\sigma \dot{X}^\mu = p^\mu. \tag{2.108}$$

- The total angular momentum is

$$J^{\mu\nu} = \int_0^\ell d\sigma \left(X^\mu \Pi^\nu - X^\nu \Pi^\mu\right) = l^{\mu\nu} + E^{\mu\nu} + \tilde{E}^{\mu\nu} \tag{2.109}$$

  with a center-of-mass contribution

$$l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \tag{2.110}$$

  and an oscillator contribution

$$E^{\mu\nu} = -i\sum_{n=1}^{\infty} \frac{1}{n}\left(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n\right). \tag{2.111}$$

The most general solution thus corresponds to a string moving with center-of-mass momentum p^μ while carrying a general superposition of left-and right-moving vibrational modes with excitation $\tilde{\alpha}^\mu_n, \alpha^\mu_n$. Note that the momentum and the oscillation modes are further subject to the primary constraints $T_{++} = 0 = T_{--}$, as will be detailed further in the next chapter.

Of prime importance for quantisation of the theory are the **Poisson brackets**.
From $\{X^\mu(\tau,\sigma),\ \Pi^\nu(\tau,\sigma')\} = \eta^{\mu\nu}\delta(\sigma-\sigma')$ we find by explicit computation

$$\{\alpha^\mu_m,\ \alpha^\nu_n\} = -im\delta_{m+n,0}\ \eta^{\mu\nu}\ , \tag{2.112}$$

$$\{\tilde{\alpha}^\mu_m,\ \tilde{\alpha}^\nu_n\} = -im\delta_{m+n,0}\ \eta^{\mu\nu}\ , \tag{2.113}$$

$$\{\alpha^\mu_m,\ \tilde{\alpha}^\nu_n\} = 0\ , \qquad \{x^\mu,\ p^\nu\} = \eta^{\mu\nu}\ . \tag{2.114}$$

The **Hamiltonian** enjoys the oscillator expansion

$$\begin{aligned} H &= T\int_0^\ell d\sigma \left[(\partial_+ X)^2 + (\partial_- X)^2\right] \\ &= \frac{\pi}{\ell}\sum_{n\in\mathbb{Z}} \left(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n\right)\ . \end{aligned} \tag{2.115}$$

2.3.3 Open string expansion

For the open string we can impose, for each X^μ independently, Neumann-Neumann (NN) boundary conditions, Dirichlet-Dirichlet (DD) boundary conditions or mixed (DN) or (ND) boundary conditions. Let us discuss these in turn.

1. **Neumann boundary conditions** at both ends $\sigma = 0$ and $\sigma = \ell$, i.e.

$$X'^{\mu}(\tau, \sigma) = 0 \quad \text{for} \quad \sigma = 0,\ \ell, \tag{2.116}$$

lead to the most general solution

$$\text{(NN)} \quad X^{\mu}(\tau,\sigma) = x^{\mu} + \frac{2\pi\alpha'}{\ell} p^{\mu}\tau + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z},\ n\neq 0} \frac{1}{n}\alpha_n^{\mu} e^{-i\frac{\pi}{\ell}n\tau} \cos\left(\frac{n\pi\sigma}{\ell}\right). \tag{2.117}$$

Note that the boundary conditions relate X_L and X_R such that independence of the left- and right-moving oscillators is lost, $\alpha_n \equiv \tilde{\alpha}_n$.
Defining

$$\alpha_0^{\mu} = \sqrt{2\alpha'} p^{\mu} \tag{2.118}$$

allows us to write

$$\partial_{\pm} X^{\mu} = \frac{\pi}{\ell}\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_n^{\mu} e^{-i\frac{\pi}{\ell}n(\tau\pm\sigma)}. \tag{2.119}$$

Interpretation:

- The (NN) string can move freely in that the endpoints at $\sigma = 0,\ \ell$ are free, but the (NN) boundary conditions implement that there is no momentum flow off the string,

$$P_{\mu}^{\alpha} = -T\sqrt{-h}\partial^{\alpha} X_{\mu} \ \Rightarrow \ P_{\mu}^{\sigma}|_{\sigma=0,\ell} = -T(X')_{\mu}|_{\sigma=0,\ell} = 0.$$

- x^{μ}, p^{μ} are again the c.o.m. position and momentum and the angular momentum is $J^{\mu\nu} = \frac{1}{2\pi\alpha'}\int_0^l d\sigma \left(X^{\mu}\dot{X}^{\nu} - X^{\nu}\dot{X}^{\mu}\right) = l^{\mu\nu} + E^{\mu\nu}$ as before.
- The Poisson brackets are $\{x^{\mu}, p^{\nu}\} = \eta^{\mu\nu}$, $\qquad \{\alpha_m^{\mu},\ \alpha_n^{\nu}\} = -im\delta_{m+n,0}\ \eta^{\mu\nu}$.
- Reality implies $(\alpha_n^{\mu})^* = \alpha_{-n}^{\mu}$.
- The Hamiltonian can be expanded as

$$H = \frac{\pi}{2l}\sum_{n\in\mathbb{Z}} (\alpha_{-n}\cdot\alpha_n).$$

2. **Dirichlet boundary conditions** at both ends, $\delta X^{\mu} = 0$, $\sigma = 0,\ \ell$, imply that the string endpoints are **fixed**,

$$\dot{X}^{\mu} = 0\ ,\ \sigma = 0,\ \ell. \tag{2.120}$$

Consequently we define

$$X^{\mu}(\tau, \sigma = 0) = x_0^{\mu}\ , \qquad X^{\mu}(\tau, \sigma = \ell) = x_1^{\mu}, \tag{2.121}$$

while the momentum vanishes, $p^{\mu} = 0$. The general solution is of the form

$$\text{(DD)} \quad X^{\mu}(\tau,\sigma) = x_0^{\mu} + \frac{1}{\ell}(x_1^{\mu} - x_0^{\mu})\sigma + \sqrt{2\alpha'} \sum_{n\in\mathbb{Z},\ n\neq 0} \frac{1}{n}\alpha_n^{\mu} e^{-i\frac{\pi}{l}n\tau} \sin\left(\frac{n\pi\sigma}{\ell}\right). \tag{2.122}$$

With

$$\alpha_0^\mu = \frac{1}{\sqrt{2\alpha'}}\frac{1}{\pi}(x_1^\mu - x_0^\mu) \tag{2.123}$$

we obtain

$$\partial_\pm X^\mu = \pm\frac{\pi}{l}\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-i\frac{\pi}{\ell}n(\tau\pm\sigma)}. \tag{2.124}$$

Comments:

- The c.o.m. position of the (DD) string is

$$q^\mu = \frac{1}{\ell}\int_0^\ell d\sigma X^\mu(\tau,\sigma) = \frac{1}{2}(x_0^\mu + x_1^\mu). \tag{2.125}$$

- The Hamiltonian acquires an extra contribution from the potential energy of the stretched string,

$$H = \frac{T}{2l}(x_1^\mu - x_0^\mu)^2 + \frac{\pi}{2l}\sum_{n\in\mathbb{Z}}(\alpha_{-n}\cdot\alpha_n). \tag{2.126}$$

  This is a consequence of the fact that a string has tension - stretching it costs energy.

3. Finally, we can also impose **mixed (ND) boundary conditions**, e.g.

$$X'^\mu = 0 \quad \sigma = 0\ , \qquad \dot{X}^\mu = 0\ \sigma = \ell, \tag{2.127}$$

(or the other way round). Such boundary conditions enforce the solution

$$\text{(ND)}\quad X^\mu(\tau,\sigma) = x^\mu + i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{1}{n}\alpha_n^\mu e^{-i\frac{\pi}{\ell}n\tau}\cos\left(\frac{n\pi\sigma}{\ell}\right). \tag{2.128}$$

Comments:

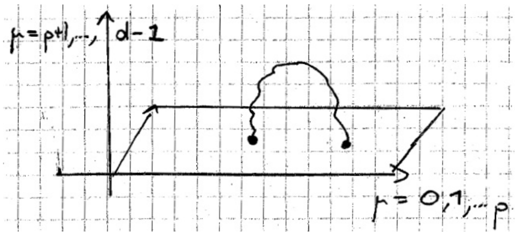
- The Fourier modes are now half-integer, $n \in \mathbb{Z} + \frac{1}{2}$.
- The string is fixed on one end, here at $\sigma = \ell$ at position x^μ.
- Consequently the c.o.m. momentum vanishes.

2.3.4 The concept of D-branes

Having acheived the general open string solutions corresponding to the various boundary conditions we now give a deeper physical interpretation. As stated several times in each dimension μ we can choose (NN), (DD) or (ND) boundary conditions independently.
As a first example we consider the following configuration of boundary conditions:

- (NN) for X^μ, $\mu = 0,\ldots,p$,
- (DD) for X^μ, $\mu = p+1,\ldots,d-1$.

Suppose further that in the (DD) directions $x_0^\mu = x_1^\mu$. Then the open string moves freely along the (p+1)-dimensional hypersurface extending along $\mu = 0, \dots, p$ and is fixed in the directions normal to it.



This leads us to the important concept of a D-brane:

A **Dp-brane** is a (p+1)-dim. hypersurface of spacetime on which open strings can end.

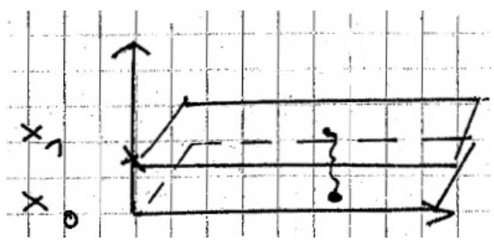
At this stage of our analysis a Dp-brane is really just what is written in the previous sentence. In fact, however, a Dp-brane is by itself a **dynamical object**. While it will take more preparation to make this precise at a quantitative level, we can already guess this from the observation that there is non-zero momentum flow off the string in the (DD) directions normal to it. Total momentum conservation implies **momentum exchange between the string and the Dp-brane**.

One might object that momentum conservation might not hold because the presence of the D-brane Dp-brane, or equivalently the choice of boundary conditions, partially breaks translational invariance. However, it turns out that a D-brane only breaks translational invariance *spontaneously*. A spontaneous breakdown of a continuous symmetry does not affect the assocated conservation laws.

The realisation by Joseph Polchinski in 1996 that the Dp-brane itself carries non-trivial dynamics has revolutionised the way we think about string theory. We will discuss the dynamics of D-branes later in the course. Suffice it here to state that the dynamics of the D-brane is described by an analogue of the Nambu-Goto action. We will find that the tension scales like $\frac{1}{g_s}$ with g_s the string coupling. This means that D-branes are **non-perturbative** objects.

More general brane configurations include

- parallel branes, e.g.



with

- (NN) for $\mu = 0, \dots, p$
- (DD) for $\mu = p+1, \dots, d-1$ with $x_0^\mu \neq x_1^\mu$.

- branes of different dimension, e.g. a Dp-brane and a Dq-brane corresponding to a string subject to (DN)-boundary conditions (see Ass. sheet 3 for an example).

- branes at angles.

We furthermore identify (NN) boundary conditions in all d dimensions as the special case of a $D(d-1)$-brane, i.e. a D-brane filling all of $\mathbb{R}^{1,d-1}$.

2.3.5 The Virasoro generators

With the mode expansion at hand we are finally in a position to introduce the Virasoro generators, which will be key players in what follows.
Consider first the **closed string** and recall that the conformal Killing transformations

$$\xi^+ \mapsto \xi^+ + f(\xi^+), \quad \xi^- \mapsto \xi^- + g(\xi^-) \tag{2.129}$$

are generated by the Noether charges

$$L_f = -\frac{\ell}{4\pi^2}\int_0^\ell T_{++}(\xi^+)f(\xi^+)d\sigma, \qquad L_g = -\frac{\ell}{4\pi^2}\int_0^l T_{--}(\xi^-)g(\xi^-)d\sigma \tag{2.130}$$

in the sense that

$$\{L_f\ ,\ X^\mu\}_{\text{P.B.}} = -\frac{\ell}{2\pi}\, f(\xi^+)\ \partial_+ X^\mu, \qquad \{L_g\ ,\ X^\mu\}_{\text{P.B.}} = -\frac{\ell}{2\pi}\, g(\xi^-)\ \partial_- X^\mu. \tag{2.131}$$

Now we can pick a basis $\left\{\exp\left(in\frac{2\pi}{l}\xi^\pm\right)\right\}$ for the functions $g(\xi^-),\ f(\xi^+)$ and define

$$\begin{aligned} L_m &:= -\frac{\ell}{4\pi^2}\int_0^\ell d\sigma T_{--}\exp\left(im\frac{2\pi}{\ell}\xi^-\right) & (2.132)\\ &\equiv -\frac{\ell}{4\pi^2}\int_0^\ell d\sigma T_{--}\exp\left(-im\frac{2\pi}{\ell}\sigma\right) \\ &= \frac{1}{2}\sum_n \alpha_{m-n}\cdot\alpha_n & (2.133)\end{aligned}$$

$$\begin{aligned} \tilde{L}_m &= -\frac{\ell}{4\pi^2}\int_0^l d\sigma T_{++}\exp\left(im\frac{2\pi}{\ell}\sigma\right) \\ &= \frac{1}{2}\sum_n \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n. & (2.134)\end{aligned}$$

Note that in going from the first to the second line in (2.132) we were using that the Noether charges are conserved and thus essentially time independent. Furthermore $L_m^* = L_{-m}$, $\tilde{L}_m^* = \tilde{L}_{-m}$.
The L_m and $\tilde{L}_m$ are called **Virasoro generators**. They are **the generators of the conformal Killing transformations** in the sense that

$$\{L_m, X^\mu\}_{\mathbb{P}.B.} = -\frac{\ell}{2\pi}\ \exp\left(in\frac{2\pi}{\ell}\xi^-\right)\ \partial_- X \qquad \left\{\tilde{L}_m, X^\mu\right\}_{\mathbb{P}.B.} = -\frac{\ell}{2\pi}\ \exp\left(in\frac{2\pi}{\ell}\xi^+\right)\ \partial_+ X. \tag{2.135}$$

Explicit computation shows that the Virasoro generators satisfy the classical **Witt algebra**

$$\boxed{\{L_m, L_n\}_{\text{P.B.}} = -i(m-n)L_{m+n}} \tag{2.136}$$

$$\boxed{\left\{\tilde{L}_m, \tilde{L}_n\right\}_{\text{P.B.}} = -i(m-n)\tilde{L}_{m+n}}. \tag{2.137}$$

To fully appreciate the geometric significance of the Witt algebra, we note that the conformal Killing transformations satisfy an algebra generated by the differential operators

$$T_m = -\frac{\ell}{2\pi}\exp\left(in\frac{2\pi}{l}\xi^-\right)\partial_-, \qquad \tilde{T}_m = -\frac{\ell}{2\pi}\exp\left(in\frac{2\pi}{l}\xi^+\right)\partial_+. \tag{2.138}$$

It is easy to check that these classical geometric operators satisfy the commutation relations

$$[T_m, T_n] = i(m-n)T_{m+n}. \tag{2.139}$$

As established above, the $L_m/\tilde{L}_m$ are the Noether charges associated with the transformations $T_m/\tilde{T}_m$ in that

$$\{L_m, X\}_{\text{P.B.}} = T_m X\ , \qquad L_m \equiv Q_{T_m}. \tag{2.140}$$

The Witt algebra relation can thus be written very suggestively as

$$\{Q_{T_m}, Q_{T_n}\}_{\mathbb{P}.B.} = -Q_{[T_m,T_n]}. \tag{2.141}$$

Indeed this connection between the Poisson bracket relations of the Noether charges of a symmetry and the Lie algebra relations of its generators holds very generally.

Of special significance are the Virasoro generators L_0 and $\tilde{L}_0$. This is because the **Hamiltonian** $H = \frac{1}{2}\frac{2\pi}{\ell}\sum_{n=-\infty}^{\infty}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n)$ is related to the Virasoro generators as

$$\boxed{H = \frac{2\pi}{\ell}\left(L_0 + \tilde{L}_0\right)}. \tag{2.142}$$

This connection is no wonder because the Hamiltonian generates time reparametrisations, i.e. translations in the coordinate τ. This is precisely what $L_0 + \tilde{L}_0$ does.

This brings us to the Virasoro constraints. Recall that on each string solution we must **impose** the constraints $T_{\alpha\beta} \stackrel{!}{=} 0$ which arose as the equations of motion of the worldsheet metric. Since the Virasoro generators are nothing but the Fourier modes of the energy-momentum tensor, the constraints can be written as

$$L_m \stackrel{!}{=} 0, \qquad \tilde{L}_m \stackrel{!}{=} 0 \quad \forall m. \tag{2.143}$$

This has to be understood as a set of constraint on the choice of string oscillation modes $\alpha_m, \tilde{\alpha}_m$ making up the string solution.
In particular, the Hamiltonian must vanish, $H \stackrel{!}{=} 0$. Geometrically this constraint can be viewed as implementing time reparametrization invariance. Since $p^2 = \frac{2}{\alpha'}\frac{1}{2}(\alpha_0^2 + \tilde{\alpha}_0^2)$ we can solve $H = 0$ to obtain the important **spacetime mass shell relation**

$$M^2 = -p^2 \stackrel{H=0}{=} \frac{2}{\alpha'}\sum_{n=1}^{\infty}(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n)\,. \tag{2.144}$$

Note that the complementary "level zero" Virasoro constraint $(L_0 - \tilde{L}_0) = 0$ implements diffeomorphism invariance along the worldsheet. This and many other cool things are the subject of the examples sheet.

For the **open string** the analogous story goes through, except that T_{++} and T_{--} are not independent as a consequence of the boundary conditions. The Virasoro generators are now defined as

$$L_m := -\frac{\ell}{2\pi^2}\int_0^\ell d\sigma\left[e^{im\frac{\pi}{l}\sigma}T_{++}+e^{-im\frac{\pi}{l}\sigma}T_{--}\right] \tag{2.145}$$

$$= \frac{1}{2}\sum_{n\in\mathbb{Z}}\alpha_{m-n}\cdot\alpha_n \tag{2.146}$$

and again satisfy

$$\{L_m, L_n\} = -i(m-n)L_{m+n}. \tag{2.147}$$

The vanishing of the Hamiltonian

$$H = \frac{\pi}{\ell}L_0 \tag{2.148}$$

as one of the Virasoro constraints $L_m = 0$ implies the open string mass shell relation

$$M^2 = -p^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n. \tag{2.149}$$

This is correct for open strings with (NN) boundary conditions in all dimensions. We will discuss more general cases in the context of the quantum string.

Chapter 3

Quantisation of the bosonic string

We will discuss 3 alternative ways to quantise the bosonic string, each shedding different light on the significance of the critical number of spacetime dimensions as a consistency condition for existence of a well-defined quantum theory.

- **Old covariant quantisation (OCQ)**:
  The Virasoro constraints are implemented at a quantum level. This procedure is manifestly Lorentz invariant, but unitarity only holds in the "critical" number of spacetime dimensions.
- **Light-cone quantisation (LCQ)**:
  The Virasoro constraints are implemented classically before quantisation. This leads to a manifestly unitary quantisation scheme, but Lorentz invariance holds only in the "critical" dimension.
- **Path-integral quantisation**:
  This method uses the celebrated Faddeev-Popov gauge fixing procedure as appropriate for gauge theories. Criticality is equivalent to closure of the BRST algebra.

3.1 Old canonical quantisation

3.1.1 Canonical quantisation

In canonical quantisation the classical fields $X^\mu(\tau,\sigma)$ and their canonical momenta $\Pi^\mu(\tau,\sigma)$ are promoted to operators $\hat{X}^\mu(\tau,\sigma)$, $\hat{\Pi}^\mu(\tau,\sigma)$ and we replace

$$\{\ ,\ \}_{\text{P.B.}} \quad \longrightarrow \quad \frac{1}{i}[\ ,\ ]. \tag{3.1}$$

This leads to the canonical equal time commutation relations

$$\left[\hat{X}^\mu(\tau,\sigma),\ \hat{\Pi}^\nu(\tau,\sigma')\right] \quad = \quad i\,\eta^{\mu\nu}\,\delta(\sigma-\sigma'), \tag{3.2}$$

$$\left[\hat{X}^\mu(\tau,\sigma),\ \hat{X}^\nu(\tau,\sigma')\right] \ = 0 = \ \left[\hat{\Pi}^\mu(\tau,\sigma),\ \hat{\Pi}^\nu(\tau,\sigma')\right]. \tag{3.3}$$

Also the oscillator modes are promoted to operators - we will do without the little pretentious ˆ

- with commutation relations

$$\begin{aligned}
[x^\mu,\ p^\nu] &= i\,\eta^{\mu\nu}, && (3.4)\\
[\alpha^\mu_m,\ \alpha^\nu_n] &= m\,\delta_{m+n,0}\,\eta^{\mu\nu} \ = \ [\tilde\alpha^\mu_m,\ \tilde\alpha^\nu_n]\,, && (3.5)\\
[\tilde\alpha^\mu_m,\ \alpha^\nu_n] &= 0. && (3.6)
\end{aligned}$$

Hermiticity $\hat X^\mu = (\hat X^\mu)^\dagger$, $\hat\Pi^\mu = (\hat\Pi^\mu)^\dagger$ implies

$$\alpha^\mu_m = (\alpha^\mu_{-m})^\dagger. \tag{3.7}$$

As always this quantisation procedure is not completely well-defined because it not clear a priori what is the correct order of products of non-commuting operators. This leads to nasty normal ordering ambiguities, and string theory is no exception here. To study these ambiguities we define **normal ordering** as

$$:\alpha^\mu_m \alpha^\nu_n := \begin{cases} \alpha^\mu_m\alpha^\nu_n, & \text{if } m \leq n \\ \alpha^\nu_n\alpha^\mu_m, & \text{if } n < m. \end{cases} \tag{3.8}$$

In particular we **define** the Virasoro operators in normal-ordered form,

$$\boxed{L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{m-n}\cdot\alpha_n:} \tag{3.9}$$

$$\boxed{\tilde L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\tilde\alpha_{m-n}\cdot\tilde\alpha_n:} \tag{3.10}$$

Note that an actualy ambiguity arises only for L_0, $\tilde L_0$. The above definition of the normal ordered form of $L_0, \tilde L_0$ means that

$$L_0 = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n \tag{3.11}$$

and correspondingly for $\tilde L_0$. If we quantise classical expressions involving L_0, we introduce a **normal ordering constant** a,

$$L_0 \longrightarrow L_0 - a, \tag{3.12}$$

to be fixed later. Indeed we will see that consistency of quantisation scheme will soon force a definite value of a upon us, with remarkable consequences for the theory.

3.1.2 The quantum Virasoro algebra

Classically the Virasoro generators satisfy the Witt algebra $\{L_m, L_n\}_{\text{P.B.}} = -i(m-n)L_{m+n}$. Naively we would expect this to translate into straightforward commutation relations for the quantum operators following the procedure (3.1). It turns out, though, that special care has to be applied to compute $[L_m, L_n]$ in the quantum case due to normal ordering issues. As a result of a lengthy computation that starts from the definition of the L_m and the commutation relations for the α^μ_m, presented in detail on examples sheet 4, the Virasoro generators satisfy the quantum **Virasoro Algebra**

$$\boxed{[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \equiv (m-n)L_{m+n} + \Delta(m,n).} \tag{3.13}$$

Comments:

- c is called **central charge**. Direct computation shows that it is given by the **number of spacetime dimensions**, i.e. the number of scalar fields X^μ

$$c = \eta^\mu{}_\mu = d. \tag{3.14}$$

- The Virasoro algebra is a **central extension by** $\mathbb{C}$ of the (classical) Witt algebra.

  Loosely speaking, given a Lie algebra g, its central extension by $\mathbb{C}$, denoted as $\hat{g}$, is the Lie algebra spanned by the elements of g as well as a new element c with commutation relations

$$[x, y]_{\hat{g}} = [x, y]_g + c \cdot p(x, y) \ , \qquad x,\ y \in g, \tag{3.15}$$

$$[x, c]_{\hat{g}} = 0 = [c, c]_{\hat{g}}. \tag{3.16}$$

  That is, we extend g by a new element c commuting with g, i.e. in the center of g. The map $p:\ g \times g \to \mathbb{C}$ is bilinear and antisymmetric.

- Only the term m^3-term in $\Delta(m, n)$ carries physical information. Namely, under a redefintion $L_0 \to L_0 - \alpha$ the central term $\Delta(m, n)$ transforms as

$$\Delta(m, n) \to \Delta(m, n) + 2\alpha \cdot m. \tag{3.17}$$

- **Important:** The fact that $c \neq 0$ indicates a **quantum anomaly of the conformal symmetry of the WS theory**. This quantum anomaly is key to understanding the structure of the quantum theory in the sequel.

3.1.3 Fock space

Let us now construct the Hilbert space of our quantum theory. That is, we seek the irreducible representations of the operators

- $x^\mu = (x^\mu)^\dagger, \qquad p^\mu = (p^\mu)^\dagger$ as well as
- $\alpha^\mu_m = (\alpha^\mu_m)^\dagger, \quad \tilde{\alpha}^\mu_m = (\tilde{\alpha}^\mu_m)^\dagger$, which form an infinite family of harmonic oscillators.

Let us recall some basics facts about the **harmonic oscillator**:

- The harmonic oscillator is defined in terms of the operators $a, a^\dagger$ with commutation relations $[a^\dagger, a] = -1$.
- The number operator $N = a^\dagger a$ is hermitian and diagonalisable with eigenstates such that $N\,|n\rangle = n\,|n\rangle$ and commutation relations

$$\left.\begin{aligned} [N, a] &= -a \\ [N, a^\dagger] &= a^\dagger \end{aligned}\right\} \quad \begin{aligned} &a : \text{lowering operator} \\ &a^\dagger : \text{raising operator} \end{aligned} \tag{3.18}$$

- The Fock space is constructed form the vacuum $|0\rangle$ with the property $a\,|0\rangle = 0$ as the space of states of the form $a^\dagger \dots a^\dagger\,|0\rangle$.

Returning to the bosonic string, the key insight is that for each mode number m and for each dimension μ we are facing a separate harmonic oscillator defined by the commutation relations

$$[\alpha_m^\mu, \alpha_n^\nu] = m\ \delta_{m+n,0}\eta^{\mu\nu} \qquad \eta^{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1), \tag{3.19}$$

and similarly for $\tilde{\alpha}_m^\mu$ in the closed string. For spacelike $\mu = i$ we have $[\alpha_m^i, \alpha_{-m}^i] = +m$.
Thus, the operators $a_{|m|}^i = \frac{1}{\sqrt{|m|}}\alpha_{|m|}^i$ and $(a_{|m|}^i)^\dagger$ satisfy $[a_{|m|}^i, (a_{|m|}^i)^\dagger] = 1$, which identifies them as annihilation and creation operators. We prefer to work with the unnormalised α_m^i, though, i.e. with

$$\begin{aligned} \alpha_{-|m|}^i &: \quad \text{creation operators,} \\ \alpha_{|m|}^i &: \quad \text{annihilation operators.} \end{aligned}$$

In addition, we need to furnish a representation of the Heisenberg algebra formed by x^μ, p^μ. Combining everything we define a **ground state** $|0; p^\mu\rangle$ with the following properties

- $\hat{\Pi}^\mu\,|0; p\rangle = p^\mu\,|0; p^\mu\rangle$,
- $\alpha_m^\mu\,|0; p^\mu\rangle = 0 \quad \forall m > 0$.

That is the state $|0; p^\mu\rangle$ has momentum p^μ. The Fock space is then generated by action of the independent creation operators α_{-m}^μ, $m > 0$, on this state.
It is spanned by the set of states

$$\{\prod_m (\alpha_{-m}^\mu)^{n_{m,\mu}}\,|0; p\rangle\}. \tag{3.20}$$

Note that a priori there is a ground state for each value of p^μ, and p^μ and the oscillators and independent. We will soon see that this is not true in the physical Hilbert space any longer.
To characterise a particular state one introduces its **polarisation tensor**, e.g. $\xi_\mu\,\alpha_{-1}^\mu\,|0; p\rangle$ or $\zeta_{\mu\nu}\alpha_{-1}^\mu\alpha_{-1}^\nu\,|0; p\rangle$ etc.
This poses an immediate problem. Take e.g. $|\psi\rangle = \xi_\mu\,\alpha_{-k}^\mu\,|0; p\rangle$ with $k > 0$ and $\xi_\mu = (1, 0, \ldots, 0)$ and compute the norm

$$\langle\psi|\psi\rangle = \langle 0; p|\,a_k^0 a_{-k}^0\,|0; p\rangle = \langle 0; p|\,[a_k^0, a_{-k}^0]\,|0; p\rangle = -k\langle 0; p|0; p\rangle < 0.$$

The appearance of such negative norm states or "ghosts" is unacceptable in a quantum theory as they spoil unitarity.

To appreciate the problem and find its solution we recall that exactly the same issue arises in the **Gupta-Bleuler quantisation of QED**.

- To define a canonical formulation of $U(1)$ gauge theory it is necessary to start not from the gauge invariance Lagrangian, but rather from

  $$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}(\partial \cdot A)^2.$$

  This corresponds to (partially) fixing the $U(1)$ gauge symmetry by imposing the gauge fixing constraint $\partial \cdot A = 0$.

- While the naive Fock space suffers from ghosts, these are absent from the set of physical states defined by imposing the gauge fixing constraint at the quantum level.

In string theory we have likewise (partially) fixed the underlying diffeomorphism and Weyl symmetry by imposing the gauge fixing condition $h_{ab} = \eta_{ab}$. But even at the classical level we still have to impose the Virasoro Virasoro constraints $T_{ab} = 0$, i.e. $\Leftrightarrow L_m = 0 \; \forall \, m \in \mathbb{Z}$ (for open strings and similarly including $\tilde{L}_m$ for he closed string).

- Thus we must impose the constraints $L_m = 0$ as an operator equation. Our first guess $L_m \, |\varphi\rangle = 0 \; \forall \, m$ is inconsistent due to the central term in Virasoro algebra. But this would be too strict anyways as the analogue of the classical condition would be, in the spirit of Ehrenfests's theorem, rather to impose the vanishing of the expectation value of the Virasoro generators.
- Therefore it is sufficient to require, for the **open string**,

$$L_m \, |\varphi\rangle = 0 \qquad m > 0, \tag{3.21}$$
$$(L_0 - a) \, |\varphi\rangle = 0. \tag{3.22}$$

  I.e. **a state is called $|\varphi\rangle$ physical if and only if it satisfies**

$$\boxed{(L_m - a \, \delta_{m,0}) \, |\varphi\rangle = 0 \qquad \forall \, m \geq 0}. \tag{3.23}$$

- Indeed physical states satisfy $\langle \varphi | \, L_m - a\delta_{m,0} \, |\varphi\rangle = 0 \; \forall \, m$.
- Note that in implimenting the zero level Virasoro constraint we allow for a yet-to-be determined normal ordering constant a, following the logic discussed around (3.12).
- For the **closed string** we require

$$\boxed{(L_m - a \, \delta_{m,0}) \, |\varphi\rangle = 0, \qquad \forall \, m \geq 0,} \tag{3.24}$$
$$\boxed{(\tilde{L}_m - \tilde{a} \, \delta_{m,0}) \, |\varphi\rangle = 0, \qquad \forall \, m \geq 0.} \tag{3.25}$$

- A priori, a and $\tilde{a}$ might be different. However, we insist that invariance under $\sigma \to \sigma + \Delta$ continues to hold in the quantum theory. Otherwise the theory would suffer from a gravitational anomaly, i.e. an anomaly of spatial diffeomorphism invariance of the worldsheet.[1] Then

$$(L_0 - \tilde{L}_0) \, |\varphi\rangle = 0 \tag{3.26}$$

  requires

$$a = \tilde{a}. \tag{3.27}$$

Mass shell condition

As in the classical theory the quantum mass shell condition arises as the level-zero Virasoro constraint involving L_0. Thus the normal ordering constant a effects the mass of the string states.

[1] Note that in canonical quantisation the time coordinate τ is singled out from spatial coordinates - here just σ. Unlike the gravitational anomaly affecting the latter, an anomaly in τ-reparametrisation invariance by $(L_0 + \tilde{L}_0 - 2a) \, |\varphi\rangle = 0$ is accepatable,

i) For the **open string** we consider (NN) boundary conditions in directions μ, (DD) boundary conditions in directions i and (ND) or (DN) boundary conditions in directions a and compute

$$L_0 = \sum_{n>0} \alpha_{-n} \cdot \alpha_n + \frac{1}{2}\alpha_0^2 = \sum_{n=1}^{\infty} [\alpha^{\mu}_{-n}(\alpha_n)_{\mu} + \alpha^{i}_{-n}(\alpha_n)_i] + \sum_{r \in \mathbb{N}_0 + \frac{1}{2}} \alpha^{a}_{-r}(\alpha_r)_a$$
$$+ \alpha'\, p^{\mu} p_{\mu} + \alpha'(T\Delta x^i)^2,$$

i.e.

$$\boxed{L_0 := N + \alpha'\, p^2 + \alpha'(T\Delta x)^2}$$

with N **the number operator** and $p^{\mu}p_{\mu} = -M^2$ the invariant mass. Then solving the level-zero Virasoro constraint $(L_0 - a)\,|\varphi\,\rangle = 0$ for $p^2 = -M^2$ yields

$$\boxed{\alpha'\, M^2\, |\varphi\rangle = (N + \alpha'\,(T\,\Delta x)^2 - a)\, |\varphi\rangle,} \tag{3.28}$$

with $N = \sum_{n>0} N_n + \sum_{r \in \mathbb{N}_0 + \frac{1}{2}} N_r \ldots$ the number of excitations. The contribution $\alpha'(T\,\Delta x)^2$ takes into account the energy from the tension of the string stretched in the (DD) directions.

ii) For the **closed string** we obtain

$$L_0 = \sum_{n>0} \alpha_{-n} \cdot \alpha_n + \frac{\alpha'}{4} p^2 = N + \frac{\alpha'}{4} p^2,$$
$$\tilde{L}_0 = \sum_{n>0} \alpha_{-n} \cdot \alpha_n + \frac{\alpha'}{4} p^2 = \tilde{N} + \frac{\alpha'}{4} p^2.$$

Level matching (3.26),

$$(N - \tilde{N})\,|\varphi\rangle = 0, \tag{3.29}$$

and the Virasoro constraints $(L_0 - a)\,|\varphi\rangle = 0 = (\tilde{L}_0 - a)\,|\varphi\rangle$ yield

$$\boxed{\alpha'\, M^2 = 4(N - a).} \tag{3.30}$$

3.1.4 Explicit open (NN) Fock space and criticality

The structure of the physical Hilbert space is that of a **tower of string excitations with increasing mass** according to the oscillator number N.
However, we still have not proven that imposing the Virasoro constraints really removes all negative norm states from the string spectrum. We will see now that this leads to the concept of **criticality**. While we do not present a full proof of the no-ghost theorem in the formalism of Old Canonical Quantisation, we outline the logic by analysing the impact of the Virasoro constraints for the open string spectrum with (NN) boundary conditions for all X^{μ} and construct the physical states at increasing excitation levels. The proof of the no-ghost theorem will be given in the context of the Lightcone Quantisation scheme, along with an analysis of the string spectrum also of the closed string and of the open string with more general boundary conditions.

i) **Vacuum** ($N = 0$): $|0; p^{\mu}\rangle$
The Virasoro constraints are of the following form:

- At level zero we find

$$(L_0 - a)\,|0; p^\mu\rangle \stackrel{!}{=} 0 \Rightarrow M^2 = -\frac{a}{\alpha'}. \tag{3.31}$$

  So for a is undetermined, but we see that if $a > 0$ the vacuum is tachyonic.

- The higher constraints $L_k\,|0;p\rangle \stackrel{!}{=} 0, k > 0$ turn out to be vacuous. Indeed

$$\begin{aligned} L_k\,|0;p^\mu\rangle &= \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{k-n}\cdot\alpha_n\,|0;p\rangle \stackrel{\alpha_{|n|}|0;p\rangle=0}{=} \frac{1}{2}\sum_{n=-\infty}^{0} \alpha_{k-n}\cdot\alpha_n\,|0;p\rangle \\ &= \sum_{n=-\infty}^{0} \frac{1}{2}[\alpha_n\cdot\alpha_{k-n} + (k-n)\delta_{k-n,n}]\,|0;p\rangle. \end{aligned}$$

  Thus $L_k\,|0;p\rangle \equiv 0$ identically for $k > 0$, i.e. the higher constraints are automatically satisfied.

  For future purposes we normalise the vacuum state as $\langle 0;p|0;p\rangle \equiv 1$.

ii) **First excited level ($N = 1$)**: $|\varphi\rangle = \xi_\mu\,\alpha^\mu_{-1}\,|0;p\rangle$ with polarisation vector ξ

The constraints give the following restrictions:

- To evaluate $(L_0 - a)\,|\varphi\rangle = (\alpha' p^2 + \sum_{n>0}\alpha_{-n}\cdot\alpha_n - a)\xi_\mu\alpha^\mu_{-1}\,|0;p\rangle$ we commute α_n through α_{-1}. In this process only the oscillator $n = 1$ picks up a commutator term, i.e. $(L_0 - a)\,|\varphi\rangle = (\alpha' p^2 + 1 - a)\,|\varphi\rangle \stackrel{!}{=} 0$˙ Thus $p^2 = \frac{a-1}{\alpha'}$ or

$$M^2 = \frac{1}{\alpha'}(1 - a). \tag{3.32}$$

- Next we evaluate

$$0 \stackrel{!}{=} L_1\,|\varphi\rangle = \frac{1}{2}(\sum_{n=-\infty}^{\infty} \alpha_{1-n}\cdot\alpha_n - a)\xi_\mu\alpha^\mu_{-1}\,|0;p\rangle \tag{3.33}$$

$$= \frac{1}{2}(\ldots + \alpha_2\alpha_{-1} + \alpha_1\alpha_0 + \alpha_0\alpha_1 + \alpha_{-1}\alpha_2 + \ldots)\xi_\mu\alpha^\mu_{-1}\,|0;p\rangle. \tag{3.34}$$

  We pick up a contribution only from the terms involving α_1, and if we remember that α_0 is proportional to p we conclude that $(\xi\cdot p)\,|0;p\rangle \stackrel{!}{=} 0$, i.e.

$$\xi^\mu\,p_\mu \stackrel{!}{=} 0. \tag{3.35}$$

- The higher constraints are vacuous: From

$$[L_m, \alpha^\mu_n] = -n\,\alpha^\mu_{m+n} \tag{3.36}$$

  it follows that all higher $L_m\,|\varphi\rangle \equiv 0$ are automatically satisfied for $m > 1$.

- The norm of the first excited state is found to be

$$\langle\varphi|\varphi\rangle = \langle 0;p|\,(\xi\cdot\alpha_{-1})^\dagger(\xi\cdot\alpha_{-1})\,|0;p\rangle = \xi\cdot\xi. \tag{3.37}$$

We see that depending on the value of the normal ordering constant a we must distinguish 3 physically inequivalent situations.

a) $\mathbf{a > 1}$
In this case $p^2 > 0$, i.e. $M^2 < 0$. This alone does not shock us. What is a killer, though, is that the spectrum still contains negative norm states. To see this we pick w.l.o.g. $p^\mu = (0, p^1, 0, \ldots, 0)$ and observe that we can find a polaristion vector ξ^μ satisfying the constraint $\xi \cdot p = 0$, but which is of negative norm $\xi \cdot \xi < 0$. (E.g we can take. $\xi^\mu = (\xi^0, \underline{0})$.) We conclude that $a > 1$ is inconsistent as it does not lead to a unitary quantum theory.

b) $\mathbf{a = 1}$
Now $p^2 = 0$ and w.l.o.g. we pick $p^\mu = (\omega, \omega, 0, \ldots, 0)$. The constraint $\xi \cdot p = 0$ implies that

$$\xi^\mu = a\, p^\mu + \xi_T^\mu, \qquad \xi_T^\mu = (0, 0, \underline{\xi}). \tag{3.38}$$

Thus the polarisation vector ξ^μ describes (d-2) transverse degrees of freedom ξ_T^μ and 1 longitudinal degree of freedom $a\, p^\mu$. The latter is **null**, i.e. of zero norm, because $p^2 = 0$ and furthermore orthogonal to any transverse vector.

Note: At this stage, this is all we can say. However, an in-depth analysis of the string interactions reveals that if $d = 26$ the longitudinal state decouples from all scattering processes. In that case it is not a physical degree of freedom. This is analogous to the situation of the photon in QED. Recall that the $U(1)$ gauge invariance implies

$$\begin{aligned} A'_\mu &= A_\mu + \partial_\mu \Lambda, \\ \xi'_\mu &= \xi_\mu + c\, p_\mu. \end{aligned}$$

Thus the null d.o.f. is pure gauge.
To conclude, if $a = 1$, then spectrum contains a massless vector boson with $d-2$ transverse d.o.f.. If $d = 26$ this particle corresponds to a gauge boson with consistent intersections.

c) $\mathbf{a < 1}$
Now $p^2 < 0$ and $M^2 > 0$ so w.l.o.g. we pick $p^\mu = (p^0, 0)$, $\xi^\mu = (0, \xi^i)$. This describes a massive vector of positive norm with $d-1$ d.o.f.. We know from QFT that massive vector theories are well-defined in principle so at this stage we have no objections to $a < 1$.

iii) **Second excited level ($N = 2$)**

The most general state is of the form

$$|\varphi\rangle = \left(\zeta_{\mu\nu} \alpha_{-1}^\mu \alpha_{-1}^\nu + \tilde{\zeta}_\mu \alpha_{-2}^\mu\right) |0; p\rangle. \tag{3.39}$$

Evaluation of the Virasoro constraints along the above lines reveals that $d \leq 26$ is a necessary condition for absence of ghosts. This will be discussed in the examples classes.

Let us summarise our findings. Analysis of the string spectrum up to the second excited level in Old Covariant Quantisation shows that

$$a \leq 1, \qquad d \leq 26$$

is a necessary condition for unitarity.

Criticality in OCQ

- One can prove, with considerably more efforts, the OCQ **no-ghost-theorem**:
  If $a = 1$, $d = 26$, than every physical state is of the form
  $$|\phi\rangle = |\phi_T\rangle + |s\rangle$$
  such that $\langle\phi_T|\phi_T\rangle > 0$ and $|s\rangle$ decouples from all physical processes.
- In OCQ it is not possible to prove at tree-level that $a < 1$ and/or $d < 26$ is inconsistent. But at one-loop-level unitarity does require $a = 1$, $d = 26$ if the structure of the worldsheet theory as free bosonic theory is to be maintained.

This leads to the concept of criticality:

$$\boxed{a = 1,\ d = 26 \text{ defines the critical string.}}$$

As stated already, while in OCQ we cannot prove the necessity of $a = 1$, $d = 26$ without looking at one-loop interactions, we will find a very simple proof in the Lightcone Quantisation approach.

The tachyon

We have seen that for $a = 1$ the vacuum state $|0; p\rangle$ is tachyonic, $M^2 = -\frac{1}{\alpha} < 0$.

- The existence of a tachyon is an artifact of the bosonic string theory. Eventually this feature will be overcome in the superstring theory.
- Note that in QFT tachyonic states are not inconsistent but indicate an instability of vacuum. The most celebrated example of a tachyon in everyday physics is of course the Higgs particle with potential
  $$V(\Phi) = \lambda(|\Phi|^2 - v)^2 \tag{3.40}$$
  $$\Rightarrow V''(\Phi)\big|_{\Phi=0} < 0 \Leftrightarrow M^2 < 0 \tag{3.41}$$
  The instability triggers tachyon condensation into a stable vacuum at $|\Phi|^2 = v$.
- An interesting question is if the same thing happens in string theory in the sense that the bosonic string is related, by tachyon condensation, to the superstring theory. This requires methods of string field theory is yet to be decided.

3.2 Light-cone quantisation (LCQ)

3.2.1 Light-cone gauge

This scheme works, like OCQ, with the flat worldsheet metric $h_{ab} = \eta_{ab}$, but uses the residual reparametrisation invariance (conformal symmetry)

$$\xi^+ \mapsto \tilde{\xi}^+(\xi^+), \qquad \xi^- \mapsto \tilde{\xi}^-(\xi^-) \tag{3.42}$$

to implement the Virasoro constraints $T_{ab} = 0$ *before quantisation.*

- The key insight is that we can use the residual conformal symmetry (3.42) to transform $(\tau, \sigma) \to (\tilde{\tau}, \tilde{\sigma})$ with

$$\tilde{\tau} = \frac{1}{2}(\tilde{\xi}^+(\xi^+) + \tilde{\xi}^-(\xi^-)), \tag{3.43}$$

$$\tilde{\sigma} = \frac{1}{2}(\tilde{\xi}^+(\xi^+) - \tilde{\xi}^-(\xi^-)). \tag{3.44}$$

This implies that $\tilde{\tau}$ satisfies the free wave equation

$$\partial_+\partial_-\tilde{\tau} = 0, \tag{3.45}$$

which is the same equation governing the dynamics of $X^\mu(\tau, \sigma)$, $\partial_+\partial_- X^\mu(\tau, \sigma) = 0$. We can therefore use (3.42) to identify $\tilde{\tau}$ with one of the X^μ.

Cautionary note: What we have just presented is a quick and dirty way to arrive at the correct conclusion. However, imposing the equation of motion before quantisation is generally problematic. There is a more thorough way to introduce light-cone quantisation without relying on this line of argument just by exploiting the local Weyl and diffeomorphism invariance of the action. This is described in detail in volume 1 of [P], p. 17-19.[2] In any case, invariance under Weyl transformations plays a key role.

- To actually use this identification to our advantage we introduce **lightcone coordinates for spacetime**

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{d-1}), \quad X^i \quad i = 1, \ldots, d-2, \tag{3.46}$$

so that the ambient space metric becomes

$$\eta_{+-} = -1 = \eta_{-+}, \quad \eta_{ij} = \delta_{ij}, \tag{3.47}$$

$$X \cdot X = -2X^+X^- + X^iX^i \quad \text{sum over } i \text{ implied.} \tag{3.48}$$

- We finally choose $X^+(\tau, \sigma)$ as the coordinates to be identified with $\tilde{\tau}$:

$$\tilde{\tau} = \frac{\ell}{2\pi\alpha'}\frac{1}{p^+}X^+ - \underbrace{x^+}_{\text{integration constant}}. \tag{3.49}$$

We now relabel $(\tilde{\tau}, \tilde{\sigma}) \to (\tau, \sigma)$ and arrive at

$$X^+(\tau, \sigma) = \frac{2\pi\alpha'}{\ell}p^+\tau + x^+. \tag{3.50}$$

Note that the normalisations were chosen such that $P^+ = T\int_0^l \dot{X}^+ \, d\tilde{\sigma} \equiv p^+$.

Comments:

- We have effectively used the *infinite dimensional* conformal symmetry (3.42) to gauge away an *infinite* number of oscillator degrees of freedom by setting $\alpha_n^+ = 0 \ \forall n \neq 0$.

[2] In addition, one has to show that the specific type of reparametrisation ghosts that arise in the process of the gauge fixing performed here indeed decouple. We thank S. Theisen for discussions on this.

- This procedure is not manifestly Lorentz invariant as it singles out one coordinate, here X^+. We must therefore check for Lorentz invariance at the quantum level to ensure a consistent quantisation scheme.

The advantage of the above spacetime lightcone coordinates is that we can now *explicitly* solve the Virasoro constraints

$$T_{ab} = 0 \quad \Leftrightarrow \quad (\dot{X} \pm X')^2 = 0, \tag{3.51}$$

which in lightcone coordinates become

$$0 \stackrel{!}{=} \quad -2(\dot{X} \pm X')^+(\dot{X} \pm X')^- + (\dot{X} \pm X')^i(\dot{X} \pm X')^i. \tag{3.52}$$

Using $\dot{X}^+ = \frac{2\pi\alpha'}{\ell}p^+$ and $(X^+)' = 0$ this yields

$$(\dot{X} \pm X')^- = \frac{1}{2}\frac{l}{2\pi\alpha'}\frac{1}{p^+}(\dot{X} \pm X')^2_\perp. \tag{3.53}$$

Thus only the transverse oscillator degrees of freedom are independent, those of X^+ and X^- are either vanishing or determined by them. To see this explicitly consider the open string with, say, (NN) boundary conditions $\forall \mu$ and make the familiar ansatz

$$X^- = x^- + \frac{2\pi\alpha'}{\ell}p^-\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-i\frac{\pi}{\ell}n\tau}\cos(\tfrac{n\pi}{\ell}\sigma) \tag{3.54}$$

and similarly for X^i. We now evaluate (3.52), more precisely the sum of the two constraints, to find after some algebra

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}}\frac{1}{p^+}(\frac{1}{2}\sum_{m=-\infty}^{\infty}\alpha^i_{n-m}\alpha^i_m) \qquad \text{classically} \tag{3.55}$$

$\forall n \in \mathbb{Z}$ including the zero mode $\alpha_0^i = \sqrt{2\alpha'}p^i$.
A similar procedure for the closed string yields

$$\tilde{\alpha}_n^- = \frac{1}{\sqrt{2\alpha'}}\frac{1}{p^+}(\sum_{m=-\infty}^{\infty}\tilde{\alpha}^i_{n-m}\tilde{\alpha}^i_m). \tag{3.56}$$

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}}\frac{1}{p^+}(\sum_{m=-\infty}^{\infty}\alpha^i_{n-m}\alpha^i_m). \tag{3.57}$$

3.2.2 Quantisation in LCG

We are now ready to perform the standard quantisation procedure in lightcone coordinates. The canonically conjugate variables following from the action in lightcone gauge

$$S = \frac{1}{4\pi\alpha'}\int d\tau d\sigma((\dot{X}^i)^2 - (X'^i)^2) - \int d\tau\, p^+\dot{q}^- \tag{3.58}$$

with $q^- = \frac{1}{\ell}\int d\sigma X^-$ are

$$X^i, \quad \Pi^i, \qquad i = 1, \ldots, d-2 \tag{3.59}$$

$$q^-, \quad p^+. \tag{3.60}$$

The canonical commutation relations are

$$\boxed{\begin{array}{l}[\alpha^i_m,\alpha^j_n]=m\,\delta_{n+m}\delta^{ij},[x^i,p^j]=i\delta^{ij},\\ [\tilde\alpha^i_m,\tilde\alpha^j_n]=m\,\delta_{n+m}\delta^{ij},\\ [p^+,q^-]=i.\end{array}} \tag{3.61}$$

The classical expression for α^-_n has to be modified by means of a normal ordering constant, e.g. for the open string with (NN) boundary conditions

$$\alpha^-_n=\frac{1}{2\sqrt{2\alpha'}}\frac{1}{p^+}(\sum_{m=-\infty}^{\infty}:\alpha^i_{n-m}\alpha^i_m:-a\delta_{n,0}). \tag{3.62}$$

Mass shell condition

The mass shell condition can now be read off directly from

$$M^2=-p^2=2p^+p^- - p^ip^i, \tag{3.63}$$

where

$$p^\mu=\frac{1}{\sqrt{2\alpha'}}\alpha^\mu_0, \qquad \text{open, NN,} \tag{3.64}$$

$$p^\mu=\sqrt{\frac{2}{\alpha'}}\alpha^\mu_0=\sqrt{\frac{2}{\alpha'}}\tilde\alpha^\mu_0, \qquad \text{closed.} \tag{3.65}$$

a) For the open string with (NN) boundary conditions $\forall\mu$ this yields

$$M^2_{\text{op}}=\frac{1}{2\alpha'}(2\sum_{m=1}^{\infty}:\alpha^i_{-m}\alpha^i_m:+\alpha^i_0\alpha^i_0-2\,a-\alpha^i_0\alpha^i_0), \tag{3.66}$$

$$M^2_{\text{op}}=\frac{1}{\alpha'}(N_\perp-a) \qquad N_\perp=\sum_{m=1}^{\infty}\alpha^i_{-m}\alpha^i_m. \tag{3.67}$$

One can generalise the above expressions for open string with boundary conditions

$$\text{(NN)}:\quad X^+,\,X^-,X^k \tag{3.68}$$

$$\text{(DD)}:\quad X^l \tag{3.69}$$

$$\text{(ND)/(DN)}:\quad X^a \tag{3.70}$$

and finds

$$M^2_{\text{op}} = \frac{1}{\alpha'}(N_\perp+\alpha'(T\Delta x)^2-a), \tag{3.71}$$

$$N_\perp = \sum_{m=1}^{\infty}\left(\alpha^k_{-m}\alpha^k_m+\alpha^l_{-m}\alpha^l_m\right)+\sum_{r\in\mathbb{N}_0+\frac{1}{2}}\alpha^a_{-r}\alpha^a_r \tag{3.72}$$

b) For the closed string one obtains

$$M^2_{\text{cl}}=\frac{2}{\alpha'}(N_\perp+\tilde N_\perp-2a). \tag{3.73}$$

One can show that the difference of the two constraints results in the level matching condition $N_\perp = \tilde{N}_\perp$. Since, apart from this extra constraint, the Virasoro constraints are already implemented explicitly, all string excitations using transverse α^i_{-m}, $\tilde{\alpha}^i_{-m}$ are automatically physical. In particular all excitations are transverse and the spectrum is manifestly free of ghosts. E.g. for the open string, with (NN) boundary conditions $\forall\mu$ the spectrum starts as follows:

- level 0: $|0,p\rangle, \quad M^2 = -\frac{a}{\alpha'}$
- level 1: $\alpha^i_{-1}\,|0,p\rangle, \quad M^2 = \frac{1}{\alpha'}(1-a), \quad i = 1,\ldots,d-2$

We will take a very close look at the open and closed spectrum later.

Criticality

In LCQ the critical values for the normal ordering constant a and for the number d of spacetime dimensions follow by the requirement of Lorentz invariance. Since the definition of the lightcone coordinates singles out a specific coordinate it is not guaranteed that the Lorentz symmetry is non-anomalous in the quantum theory.
For simplicity we carry out this analysis for the open (NN) string. Recall that the generators of Lorentz transformations are given by

$$\begin{aligned} J^{\mu\nu} &= T\int_0^l d\sigma\Big(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\Big) = l^{\mu\nu} + E^{\mu\nu}, && (3.74)\\ l^{\mu\nu} &= x^\mu p^\nu - x^\nu p^\mu, && (3.75)\\ E^{\mu\nu} &= -i\sum_{m=1}^{\infty}\frac{1}{m}\left(\alpha^\mu_{-m}\alpha^\nu_m - \alpha^\nu_{-m}\alpha^\mu_m\right). && (3.76)\end{aligned}$$

These should satisfy the Lorentz algebra

$$\Big[J^{\mu\nu}, J^{\rho\sigma}\Big] = i\eta^{\mu\rho}J^{\nu\sigma} + i\eta^{\nu\sigma}J^{\mu\rho} - i\eta^{\mu\sigma}J^{\nu\rho} - i\eta^{\nu\rho}J^{\mu\sigma}. \qquad (3.77)$$

The transformation J^{i-} mixes $X^\pm$ with X^i. Since $X^\pm$ are singled out in LCQ, a breakdown of Lorentz invariance might show up in an anomaly of the algebra satisfied by J^{i-}.
In absence of an anomaly we should get

$$\Big[J^{-i}, J^{-j}\Big] = i\eta^{--}J^{ij} + i\eta^{ij}J^{--} - i\eta^{-j}J^{i-} - i\eta^{i-}J^{-j} = 0. \qquad (3.78)$$

However, an explicit, long and rather painful calculation sketched below reveals

$$\begin{aligned} \Big[J^{-i}, J^{-j}\Big] &= -\frac{1}{(p^+)^2}\sum_{m=1}^{\infty}\Delta_m\left(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m\right) && (3.79)\\ \text{with } \Delta_m &= m\cdot\frac{26-d}{12} + \frac{1}{m}\Big(\frac{d-26}{12} + 2(1-a)\Big). && (3.80)\end{aligned}$$

$$\boxed{\Rightarrow \qquad \Big[J^{-i}, J^{-j}\Big] = 0 \Leftrightarrow d = 26,\ a = 1.} \qquad (3.81)$$

Remarks:

- A similar computation fixes $d = 26,\ a = 1$ for the closed string.

- The result $d = 26$ also holds in the presence of lower-dimensional branes (i.e. with DD or ND/DN sectors for some excitations). The value of "a" in the presence of more general boundary conditions is derived in the following section.

Extra material (non-examinable):
For completeness we here guide through the main steps in deriving the result (3.79).

- Recall that the oscillators α_n^- are defined, at the quantum level, in the normal ordered manner
$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}}\frac{1}{p^+}\left(\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty} :\alpha_{n-m}^i\alpha_m^i: -a\delta_n\right). \tag{3.82}$$
For simplicity we focus on the open string and set $\alpha' = \frac{1}{2}$. With the help of the expression for α_n^- one sees that $[J^{i-}, J^{j-}]$ involves terms quartic and quadratic in the oscillators α_n^i. One can verify that the commutators involving the quartic terms vanish. This means that one can make the following ansatz for $[J^{i-}, J^{j-}]$:
$$[J^{i-}, J^{j-}] = -\frac{1}{(p^+)^2}C^{ij} \tag{3.83}$$
with
$$C^{ij} = \sum_{m=1}^{\infty}\Delta_m\left(\alpha_{-m}^i\alpha_m^j - \alpha_{-m}^j\alpha_m^i\right), \tag{3.84}$$
where Δ_m are c-numbers which we want to compute. One can furthermore show that
$$\langle 0|\alpha_m^k C^{ij}\alpha_{-m}^l|0\rangle = m^2(\delta^{ik}\delta^{jl} - \delta^{jk}\delta^{il})\Delta_m\,. \tag{3.85}$$
This equation tells us that we can compute the coefficients Δ_m by computing C^{ij} from (3.83), evaluating matrix elements of the above form and comparing the result with (3.85).
- To this end we define $E^j = p^+E^{j-}$ and derive the commutators
$$[x^-, 1/p^+] = i(p^+)^{-2} \tag{3.86}$$
$$[x^i, \alpha_n^-] = \frac{i}{p^+}\alpha_n^i \tag{3.87}$$
$$[x^i, E^j] = -iE^{ij}\,. \tag{3.88}$$
With this we can compute the commutator $[J^{i-}, J^{j-}]$ explicitly, leading to
$$C^{ij} = 2ip^+\alpha_0^-E^{ij} - iE^ip^j + iE^jp^i - [E^i, E^j]\,. \tag{3.89}$$
- To evaluate matrix elements of the form (3.85) we need commutator relations for the α_n^-. By comparing with the Virasoro generators L_m one can argue that the $p^+\alpha_n^-$ satisfy the commutation relations
$$[p^+\alpha_m^-, p^+\alpha_n^-] = (m-n)p^+\alpha_{m+n}^- + \left[\frac{D-2}{12}(m^3 - m) + 2am\right]\delta_{m+n}\,, \tag{3.90}$$
$$[\alpha_m^i, p^+\alpha_n^-] = m\alpha_{m+n}^i\,. \tag{3.91}$$
- Now we use (3.91) as well as the usual commutator $[\alpha_m^i, \alpha_n^j] = m\delta_{m+n}\delta^{ij}$ to compute
$$\begin{aligned}\langle 0|\alpha_m^k C^{ij}\alpha_{-m}^l|0\rangle = \langle 0|&2m(m + p^+\alpha_0^-)\delta^{ik}\delta^{jl} + mp^jp^k\delta^{il} - mp^jp^l\delta^{ik} \\ &+ (m\sum_{n=1}^{m}\frac{1}{n}\alpha_{m-n}^k\alpha_n^i - \delta^{ik}p^+\alpha_m^-) \\ &\times(\delta^{jl}p^+\alpha_{-m}^- - m\sum_{p=1}^{m}\frac{1}{p}\alpha_{-p}^j\alpha_{p-m}^l)|0\rangle - (i\leftrightarrow j)\,.\end{aligned} \tag{3.92}$$

- To evaluate this expression we derive from (3.90)

$$(p^+)^2\langle 0|\alpha^-_m\alpha^-_{-m}|0\rangle = 2m\langle p^+\alpha^-_0\rangle + \frac{D-2}{12}(m^3-m) + 2am \tag{3.93}$$

  and prove the identities

$$p^+\langle 0|\alpha^-_m \sum_{n=1}^{m}\frac{1}{n}\alpha^j_{-n}\alpha^l_{n-m}|0\rangle = p^jp^l + \delta^{jl}m(m-1)/2 \tag{3.94}$$

$$\langle 0|\sum_{n=1}^{m}\frac{1}{n}\alpha^k_{m-n}\alpha^i_n\sum_{p=1}^{m}\frac{1}{p}\alpha^j_{-p}\alpha^l_{p-m}|0\rangle - (i\leftrightarrow j) = (m-1)(\delta^{il}\delta^{jk}-\delta^{jl}\delta^{ik})\,. \tag{3.95}$$

- This allows us to compute (3.92). Comparing the result with (3.85) we finally arrive at

$$\Delta_m = m\frac{26-D}{12} + \frac{1}{m}\left(\frac{D-26}{12} + 2(1-a)\right)\,. \tag{3.96}$$

3.2.3 Normal ordering constant "a" as a Casimir energy

We now provide a deeper interpretation of the normal ordering constant "a".
Recall that the constant "a" is defined as the normal ordering ambiguity in promoting the classical zero-level Virasoro generator to a quantum operator,

$$\text{classical}\quad L_0 = \frac{1}{2}\sum_n \alpha_{-n}\cdot\alpha_n \tag{3.97}$$

$$\downarrow$$

$$\text{quantum}\quad L_0 - a = \frac{1}{2}\sum_n :\alpha_{-n}\cdot\alpha_n: - a. \tag{3.98}$$

Thus a is related to the commutator terms we pick up in performing the normal ordering. For a single open string field X^i with (NN) or (DD) boundary conditions this commutator term is given by

$$\frac{1}{2}\sum_{n\in\mathbb{Z}}\alpha^i_{-n}\alpha^i_n = \frac{1}{2}\sum_n :\alpha^i_{-n}\alpha^i_n: + \frac{1}{2}\sum_{n=1}^{\infty} n \qquad \text{(no sum over } i\text{)}. \tag{3.99}$$

As is stands this sum is divergent. On the other hand, the normal ordering constant of the critical string is finite. To see how the two are related we note that the divergent sum is nothing but the **vacuum energy** of the string. This interpretation follows from the relationship between the quantum Hamiltonian and L_0.
For definiteness we now consider the open string with (NN) boundary conditions $\forall\mu$. The classical and quantum Hamiltonian in LCG are, respectively,

$$\text{classical}\quad H = \frac{\pi}{\ell}\Big(\frac{1}{2}\sum_{n\neq 0}\alpha^i_{-n}\alpha^i_n + \alpha' p^ip^i\Big) \tag{3.100}$$

$$\downarrow$$

$$\text{quantum}\quad H = \frac{\pi}{\ell}\Big(\sum_{n>0}\alpha^i_{-n}\alpha^i_n + \alpha' p^ip^i + \frac{d-2}{2}\sum_{n=1}^{\infty} n\Big). \tag{3.101}$$

The computation of the divergent vacuum energy, given by the last summand, is standard in QFT and proceeds via regularization and renormalisation. In a complete treatment one starts with a regularised theory (e.g. by putting the two-dimensional field theory of the free boson on a lattice), quantises it, extracts the vacuum energy (which includes cutoff-dependent terms), renormalises it by subtracting suitable counterterms such as to remove the cutoff-dependent would-be divergent terms and finally removes the cutoff. This procedure can be carried out in full detail in a quantum field theoretic treatment.[3] With this understanding, we briefly sketch here a somewhat heuristic and quick way to see the result:

- We first **introduce a cutoff Λ to regularise the divergent expression** in such a way that the divergence appears as we remove the cutoff by sending $\Lambda \to \infty$. A convenient cutoff procedure here is e.g. to rewrite the vacuum energy as

$$\frac{\pi}{\ell}\frac{d-2}{2}\sum_{n=1}^{\infty} n \to \frac{\pi}{\ell}\frac{d-2}{2}\sum_{n=1}^{\infty} n e^{-\frac{\pi n}{l\Lambda}} = \frac{\pi}{\ell}\frac{d-2}{2}\sum_{n=1}^{\infty} n (e^{-\frac{\pi}{l\Lambda}})^n. \tag{3.102}$$

From $\sum_{n=1}^{\infty} n q^n = q\frac{d}{dq}\sum_{n=1}^{\infty} q^n = q\frac{d}{dq}\frac{1}{1-q} = \frac{q}{(1-q)^2}$ we find

$$\frac{\pi}{\ell}\frac{d-2}{2}\sum_{n=1}^{\infty} n = \frac{\pi}{\ell}\frac{d-2}{2}\lim_{\Lambda\to\infty}\frac{e^{-\frac{\pi}{l\Lambda}}}{\left(1-e^{-\frac{\pi}{l\Lambda}}\right)^2} \tag{3.103}$$

$$= \frac{d-2}{2}\lim_{\Lambda\to\infty}\Big(\frac{\ell}{\pi}\Lambda^2 - \frac{\pi}{\ell}\frac{1}{12} + \mathcal{O}(\frac{1}{\Lambda})\Big). \tag{3.104}$$

- The expression for the vacuum energy has two non-vanishing contributions: The term proportional to Λ^2 is the divergent piece. It is important that this term scales like ℓ. Therefore, this term can be absorbed by adding a cosmological constant term proportional to $\Lambda^2 \int d^2\sigma\sqrt{-h}$ to the bare Polyakov action via **renormalisation**. This **counterterm** in the bare action then cancels off the divergence arising in the quantum computation of the vacuum energy.

  In addition there is the finite term $-\frac{d-2}{2}\frac{\pi}{\ell}\frac{1}{12}$. This term is present only due to the finite size of the string because it disappears in the limit $\ell \to \infty$. Unlike for the divergent term, there exists no local counterterm that we could add to the action such as to absorb this term. This term is therefore physical and defines the **Casimir energy** of the string.

A priori one might wonder why we *have to* cancel the entire piece of the term scaling like ℓ by adding a suitable cosmological constant - can't we just keep a finite fraction of it and declare it as part of $-a$? The reason why this is not possible in string theory is conformal invariance: The cosmological constant term breaks conformal invariance explicitly in the action - unless the classical counterterm and the quantum term cancel exactly. Therefore $-a$ must be identified with the Casimir energy, i.e.

$$a = \frac{d-2}{24} \qquad \text{for (d}-2\text{) transverse (NN) or (DD) oscillators.} \tag{3.105}$$

For $d = 26$ this gives $a = 1$ as found by requiring Lorentz invariance for the open string with (NN) conditions and also for the closed string. It therefore holds also for the (DD) string.

Remarks:

- Note that $a \neq 0$ still breaks conformal invariance, but merely in form of an acceptable conformal anomaly, i.e. of a quantum anomaly of the conformal symmetry.

[3] I thank A. Hebecker for discussions on this point.

- Very soon we will get to know powerful CFT techniques yielding another derivation of the Casimir energy making the relation to the central term c in in the Virasoro algebra clear.

There is an amusingly quick and efficient manner to re-derive the Casimir energy, i.e. the finite piece of the vacuum energy, by means of **ζ-function regularisation**, which is a formal way to regularize the sum $\sum_{n=1}^{\infty} n$. It makes use of the ζ-function $\zeta(s)$, defined as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{3.106}$$

We need the following two properties of the ζ-function which are proven in standard textbooks on complex analysis:

- $\zeta(s)$ is convergent for $\mathrm{Re}(s) > 1$.
- $\zeta(s)$ allows for an unique analytic continuation to $s = -1$ with

$$\zeta(-1) = -\frac{1}{12}.$$

With two eyes wide shut - or alternatively with the above proper procedure of regularization and renormalisation in mind - we deduce that the contribution to "a" from 1 (NN) or (DD) direction (i.e. one integer moded boson), given precisely by $-\frac{1}{2}\zeta(-1)$, is

$$\boxed{+\frac{1}{24} \quad \text{per (NN) or (DD) direction.}} \tag{3.107}$$

This also gives a quick way to read off the result for a string with (DN) boundary conditions, for which the mode expansion is half-integer. To this end we use

$$\zeta(s, q) = \sum_{n=1}^{\infty} (n+q)^{-s} \tag{3.108}$$

with analytic continuation

$$\zeta(-1, q) = -\frac{1}{12}(6q^2 - 6q + 1).$$

The contribution from one (DN) direction (i.e. one half-integer moded boson) is therefore ($q = \frac{1}{2}$)

$$\boxed{-\frac{1}{48} \quad \text{per (DN) direction.}} \tag{3.109}$$

If you don't believe this result you can derive it in a similar manner as for the (NN)/(DD) string around (3.103).

This allows us to write down the normal ordering constant for the critical string with m (DN) directions and $d - m$ (DD) or (NN) as

$$a = \frac{\#(NN) + \#(DD)}{24} - \frac{\#(DN)}{48} = \frac{\#(NN) + \#(DD) + \#(DN)}{24} - \frac{\#(DN)}{16} = \frac{d-2}{24} - \frac{m}{16}.$$

The mass relation for open string with m (DN) directions is therefore

$$\alpha' M^2 = N + \alpha'(T\Delta x)^2 - a = N + \alpha'(T\Delta x)^2 - \frac{d-2}{24} + \frac{m}{16}. \tag{3.110}$$

3.2.4 Open string spectrum along D-branes

It is time to analyze the critical string spectrum in slightly greater detail.
We make use of the following important **general fact:**

In a Lorentz invariant theory in d dimensions a state forms an irreducible representation under the subgroup of $\mathrm{SO}(1, d-1)$ that leaves its momentum invariant. This group is called "**little group**" or **stabilizer subgroup**.

We must distinguish 3 cases:

- $p^2 = 0$; w.l.o.g. $p = (\omega, \omega, 0) \Rightarrow$ little group = $\mathrm{SO}(d-2)$.
- $p^2 < 0$; w.l.o.g $p = (p, 0) \Rightarrow$ little group = $\mathrm{SO}(d-1)$.
- $p^2 > 0$; w.l.o.g $p = (0, p, 0, \ldots, 0) \Rightarrow$ little group = $\mathrm{SO}(1, d-2)$.

Let us check if this principle is realized in the string spectrum.

All (NN) open strings

We consider first the open string with (NN) b.c. $\forall X^+,\ X^-,\ X^i$, with $d = 26$.

- The tachyonic ground state $|0, p\rangle$ with $M^2 = -\frac{1}{\alpha'}$ is a space-time scalar. In particular it is a scalar of the little group $\mathrm{SO}(1, 24)$.
- The massless transverse vector $\xi_i \alpha^i_{-1} |0, p\rangle$ is a space-time vector in the vector or fundamental representation, denoted by $\square$, of $\mathrm{SO}(24)$.
- All higher states are massive and form complete irreducible representations of $\mathrm{SO}(25)$. To make this manifest we must regroup the polarization degrees of freedom, which in LCQ are a priori representations of $SO(24)$, i.e. purely transverse, into irreducible representations of $SO(25)$.
  Consider e.g. the second excited leve with $M^2 = \frac{1}{\alpha'}$. The different types of states are

$$\xi_i \alpha^i_{-2} |0, p\rangle \quad \& \qquad \xi_{ij} \alpha^i_{-1} \alpha^j_{-1} |0, p\rangle. \tag{3.111}$$

  In terms of $SO(24)$ the polarization tensors form the irreducible representations

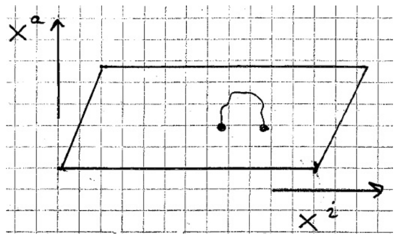
$$\mathrm{SO}(24): \quad \underbrace{\square}_{24} \qquad \underbrace{\square\square}_{\text{symmetric traceless, } \# = \frac{24\cdot 25}{2} - 1} \qquad +1 \tag{3.112}$$

  These 324 degrees of freedom combine into the symmetric traceless $\square\square$ of $\mathrm{SO}(25)$.

A single Dp-brane

Consider now the open string spectrum in the presence of a Dp-brane within $\mathbb{R}^{1,25}$.
Note that the light-cone coordinates $+,\ -$ must be along (NN) directions for a treatment within LCQ so we take the boundary conditions as follows:

(NN) : $i = +, -, 1, ...p{-}1$
(DD) : $a = p + 1, ..., 25$



We distinguish

- excitations along brane in directions X^i, $i = 1, ..., p - 1$ and
- excitations orthogonal to brane in directions X^a, $a = p + 1, ..., 25$.

Let us build the spectrum:

- The ground state is $|0, p\rangle$ with momentum p only in the (NN) directions along the Dp-brane.
- First level:
  i) The parallel excitations are $\alpha^i_{-1}|0, p\rangle$, $\quad i = 1, ..., p - 1$, $\quad M^2 = 0$.
  This forms a **massless vector** from the perspective of the Dp-brane and propagates along the brane (i.e. it transforms as $\square$ of SO($p - 1$)). By general arguments of QFT a massless vector must be a **gauge potential**. More precisely one can explicitly verify by computations of its interactions that this is the case. String interactions will be discussed later in this course. For now we state the key observation:

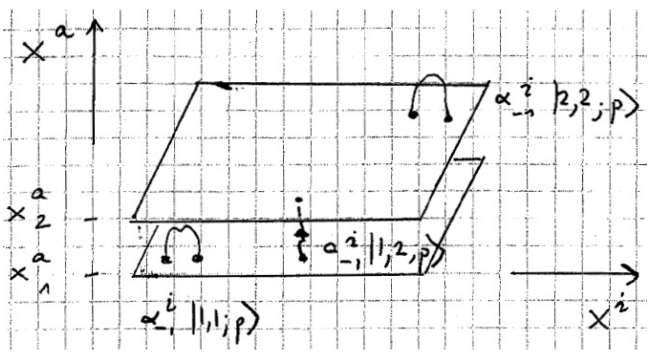
  $$\boxed{\text{A single Dp-brane hosts a } U(1) \text{ gauge theory!}}$$

  ii) The normal excitations are $\alpha^a_{-1}|0, p\rangle$, $\quad a = p + 1, ..., 25$, $\quad M^2 = 0$.
  This forms a collection of 24-p **massless scalar fields** as seen from the perspective of the Dp-brane. I.e. $\forall a$ we have 1 excitation $\alpha^a_{-1}|0, p\rangle$ that does not carry an index "i" along the Dp-brane.
  These are the **Goldstone bosons** associated to the spontaneous breaking of the 26-dimensional Poincaré invariance by the Dp-brane.

Note that (NN) in all directions is the special case of a D25-brane filling all of $\mathbb{R}^{1,25}$.

Parallel Dp-branes

Consider two parallel branes at a distance $x^a_2 - x^a_1$.



We now find 2 sets of massless vectors,

- $\alpha_{-1}^i |11, p\rangle$, i.e. a vector from strings with both ends on brane 1 with $M_{11}^2 = 0$.
- $\alpha_{-1}^i |22, p\rangle$ with both ends on brane 2, $M_{22}^2 = 0$.

Similarly there are two types of massless scalars $\alpha_{-1}^a |11, p\rangle$, $\alpha_{-1}^a |22, p\rangle$.
In addition we find strings stretched between brane 1 and brane 2. These come with both orientations:

- The vectors $\alpha_{-1}^i |12, p\rangle$ and $\alpha_{-1}^i |21, p\rangle$ have mass

$$M_{12}^2 = M_{21}^2 = N - a + \alpha'(T\Delta x)^2 = \frac{1}{2\pi\alpha'} \frac{(x_2^a - x_1^a)^2}{2\pi}. \tag{3.113}$$

  That is, we now also find massive vectors along branes 1 and 2.
- Similarly there are massive scalars $\alpha_{-1}^a |12, p\rangle$, $\alpha_{-1}^a |21, p\rangle$.

Coincident Dp-branes

Now consider the limit $x_2^a \to x_1^a$ by moving the two Dp-branes together. We end up with 4 massless vectors $\alpha_{-1}^i |ij, p\rangle$ and 4 sets of massless scalars $\alpha_{-1}^a |ij, p\rangle$, $\quad i, j = 1, 2$.

In general for N coincident Dp-branes the bosonic open spectrum at level 1 consists of

- N^2 massless vectors $\alpha_{-1}^i |ij, p\rangle$, $\qquad i, j = 1, ..., N$ and
- N^2 sets of massless scalars $\alpha_{-1}^a |ij, p\rangle$.

The labels i, j that keep track of the start- and endpoint of the open string are called **Chan-Paton factors**. It is convenient to introduce a basis λ_{kl}^a, $a = 1, ..., N^2$ of $N \times N$ matrices and rewrite a string state in the form

$$|kl, p\rangle \equiv \lambda_{kl}^a |a, p\rangle. \tag{3.114}$$

The appearance of N^2 massless vectors signals an **enhancement of gauge symmetry to non-abelian gauge symmetry**.
We observe that N^2 is the dimension of the adjoint representation of $U(N)$. Indeed as we will see later the gauge interactions are consistent if $\lambda^a = (\lambda^a)^\dagger$, $a = 1, ..., N^2$, i.e. the λ^a span the Lie algebra of $U(N)$.

$$\boxed{\text{A stack of N coincident Dp-branes "carries" a } U(N) \text{ gauge theory.}}$$

Remarks:

- Note that moving the N D-branes apart breaks $U(N) \to U(1)^N$ as only N massless vectors remain. This process of breaking a non-abelian gauge group to its abelian subgroups is called moving along the Coulomb branch. It is realized here by giving a vacuum expectation value (VEV) to those scalars stretched between the N branes as these indicate the relative position of the branes. Indeed, as these scalars are charged under the non-abelian generators of $U(N)$ a nonzero VEV breaks $U(N) \to U(1)^N$.
- As this simple example shows the physics of D-branes "knows" a lot about gauge theories. This has lead to a modern way to think about gauge theories by thinking of brane dynamics instead and resulted in deep insights into the geometric structure behind Yang-Mills theory.

- Also gauge groups different from $U(N)$ can be realized in D-brane theories. On Assignment 6 we discuss the concept of an **orientifold**: In such theories the possible gauge groups include also $SO(N)$ and the **symplectic group** $Sp(2N)$. In non-perturbative string theories also the exceptional gauge groups E_6, E_7, E_8 and more exotic ones occur.

3.2.5 Closed string spectrum

We now turn to the closed string spectrum. Recall that for the closed string there are independent left- and right-moving oscillators. The tower of closed string excitations is organised by

- the condition level matching $\tilde{N} = N$ and
- the mass-shell condition $M_{\text{cl}}^2 = \frac{4}{\alpha'}(N - a) = \frac{4}{\alpha'}(N - 1)$.

The lower states of the closed string spectrum are therefore as follows:

- The ground state ($N = \tilde{N} = 0$) is a tachyonic spacetime scalar $|0, p\rangle$ with $M^2 = -\frac{4}{\alpha'}$.
- At the first excited level ($N = \tilde{N} = 1$) we encounter a massless state $\xi_{ij}\tilde{\alpha}^i_{-1}\alpha^j_{-1}\,|0, p\rangle, \quad i, j = 1, ..., 24$.
  One can decompose the polarisation two-tensor ξ_{ij} into irreducible representations[4] of the little group SO(24)
$$\xi_{ij} = \underbrace{\xi_{(ij)}}_{\substack{\text{symmetric traceless}\\ \ydiagram{2}}} + \underbrace{\xi_{[ij]}}_{\substack{\text{antisymmetric}\\ \ydiagram{1,1}}} + \underbrace{\xi^{(0)}}_{\text{trace part}} \tag{3.115}$$
  i.e.
$$\begin{aligned}\tilde{\alpha}^i_{-1}\alpha^j_{-1}\,|0, p\rangle \quad &= \quad \left(\frac{1}{2}\left(\tilde{\alpha}^i_{-1}\alpha^j_{-1} + \tilde{\alpha}^j_{-1}\alpha^i_{-1}\right) - \frac{1}{d-2}\delta^{ij}\tilde{\alpha}^k_{-1}\alpha^k_{-1}\right)|0, p\rangle \\ &\quad +\tilde{\alpha}^{[i}_{-1}\alpha^{j]}_{-1}\,|0, p\rangle \\ &\quad +\frac{1}{d-2}\delta^{ij}\tilde{\alpha}^k_{-1}\alpha^k_{-1}\,|0, p\rangle.\end{aligned}$$

Interpretation:
While we will have much more to say about each of these states, we give here a short overview of their physical interpretation.

i) $\xi_{(ij)}$ describes a massless, transversely polarised spin 2 particle.
As we know from the treatment of gravitational waves in General Relativity these are precisely the correct degrees of freedom expected from the **graviton**. In fact an explicit analysis of interactions of this massless spin 2 state confirms this interpretation. Later in this course we will be able to derive General Relativity as an outcome of string theory with the $\xi_{(ij)}$ state playing the role of the graviton.

ii) $\xi_{[ij]} \equiv B_{ij}$ describes the degrees of freedom of an extra antisymmetric tensor field called the **Kalb-Ramond field.**
This antisymmetric 2-tensor B_{ij} plays an important role in that it can be thought of as a generalised (i.e. higher-rank) gauge potential.

[4]The space of symmetric traceless 2-tensors forms a vector space invariant under the action of the Lorentz group in that $\Lambda^i{}_k\Lambda^j{}_l\zeta_{(ij)}$ is again symmetric traceless. Similarly for $\zeta_{[ij]}$.

- As we know from field theory, a gauge potential A_μ describes a 1-form potential $A \equiv A_\mu dx^\mu$. The associated field strength is a 2-form $F = dA = \partial_\mu A_\nu \, dx^\mu \wedge dx^\nu$ which is invariant under the gauge symmetry $A \to A + d\chi$ withχ a 0-form.
- In perfect analogy the higher rank antisymmetric tensor B_{ij} describes a 2-form potential

$$B \equiv \frac{1}{2!} B_{ij} dx^i \wedge dx^j \tag{3.116}$$

One can define a corresponding field strength

$$H = dB = \frac{1}{2!} \partial_i B_{jk} dx^i \wedge dx^j \wedge dx^k$$

invariant under the gauge transformation

$$B \to B + d\Lambda \tag{3.117}$$

with Λ a 1-form.

iii) $\xi^{(0)}$ represents a scalar field, denoted usually by ϕ. It is called the **dilaton** and will be of crucial importance in the context of string interactions.

Final remarks: Two important concepts are covered on Assignment # 6:

i) Orientifold theories

ii) Partition function and Hagedorn temperature

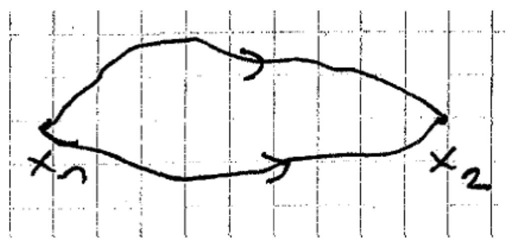
3.3 Modern covariant quantisation

We finally present the modern covariant approach to quantization. Apart from being the standard technique to quantize systems with gauge symmetries, this method is useful to compute string interactions.

3.3.1 The Polyakov Path integral

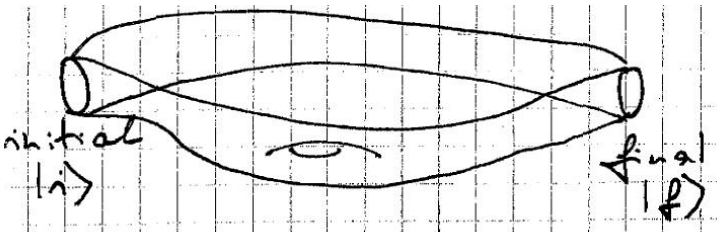
The modern covariant approach to quantization proceeds via the Feynman path integral. We can literally understand the path integral in the form needed as the stringy generalization of the quantum mechanical Feynman path integral for point particles. Recall that for a quantum mechanical point particle Feynman described the amplitude $\langle x_2, \, t_2 | x_1, \, t_1 \rangle$ as a "sum over histories",

$$\langle x_2, \, t_2 | x_1, \, t_1 \rangle = \int_{\substack{x(t_1) = x_1 \\ x(t_2) = x_2}} \mathcal{D}X(t) \, e^{i \, S[X]}. \tag{3.118}$$



Adapting the same logic one can define the amplitude between an initial string state $|i\rangle$ and a final state $|f\rangle$ as the path integral

$$\langle f|i\rangle = \underbrace{\int \mathcal{D}X\,\mathcal{D}h}_{\substack{\text{sum over all WS with}\\ \text{boundaries } |i\rangle,\ |f\rangle}} \quad e^{i\,S_P[X,h]}. \tag{3.119}$$



In principle the initial and final state are accounted for by considering only those worldsheets which correspond the respective states at past and present infinity. The special structure of the worldsheet theory allows for a different procedure in which we consider trivial boundary conditions in the path integral (corresponding to the vacuum). The initial and final states will instead be incorporated by inclusion of so-called vertex operators. Before discussing these vertex operators and the actual computation of scattering amplitudes, however, we take it easy and first compute the unnormalised partition function, corresponding to the path integral with trivial boundary conditions and no operator insertion,

$$\boxed{\begin{aligned} Z &= \int \mathcal{D}X\,\mathcal{D}h\, e^{i\,S_P[X,h]}, \\ S_P &= -\frac{1}{4\pi\alpha'}\int d\tau d\sigma \sqrt{-h} h^{ab}\partial_a X^\mu \partial_b X^\nu\, \eta_{\mu\nu}. \end{aligned}} \tag{3.120}$$

Faddeev-Popov gauge fixing

In its above naive form the path integral Z is not well-defined due to an overcounting of gauge-equivalent configurations of worldsheet metrics h and string fields X which all describe the same physics. Namely, we recall that the action S_P is invariant under a general diffeomorphism $\xi^a \mapsto \xi^a - \epsilon^a(\xi)$ combined with a local Weyl transformation, under which the string field and metric transform as

$$X^\mu \mapsto X^\mu + \epsilon^a \partial_a X^\mu, \tag{3.121}$$

$$h_{ab} \mapsto h_{ab} + (P\cdot\epsilon)_{ab} + 2\,\tilde{\Lambda}\, h_{ab} \tag{3.122}$$

with

$$(P\cdot\epsilon)_{ab} = \nabla_a\epsilon_b + \nabla_b\epsilon_a - h_{ab}(\nabla\cdot\epsilon), \tag{3.123}$$

$$\tilde{\Lambda} = \Lambda + \frac{1}{2}\nabla\cdot\epsilon. \tag{3.124}$$

The operator P maps vectors to symmetric traceless 2-tensors

$$(P\cdot\epsilon)_{ab} = P^c_{ab}\,\epsilon_c \tag{3.125}$$

with

$$P^c_{ab} = \delta^c_{(b}\nabla_{a)} - h_{ab}\nabla^c. \tag{3.126}$$

This overcounting of gauge equivalent configurations is a standard problem in the path integral quantization of gauge theories. The general solution is to isolate the integral over the gauge parameters and divide by the volume of the gauge group. In the case at hand this can be achieved by converting the integral over all worldsheet metrics into an integral over all diffeomorphisms ϵ^a and Weyl rescalings Λ that take us to the gauge transformed $\hat{h}^{(\epsilon^a,\,\Lambda)}$ starting from some fixed $\hat{h}$.

From our earlier discussion of gauge fixing on the worldsheet we know the following: At least locally, if for fixed $\hat{h}_{ab}$ the parameters $\zeta = (\epsilon^a,\,\Lambda)$ run over all diffeomorphisms and Weyl rescalings then $\hat{h}^\zeta = \hat{h} + \delta\hat{h}$ runs over all metrics.

Therefore, given some functional of the metric, $F[h]$, we can rewrite the path integral $\int \mathcal{D}hF[h]$ as follows,

$$\int \mathcal{D}hF[h] = \int \mathcal{D}(P\cdot\epsilon)\,\mathcal{D}\tilde{\Lambda}\,F[\hat{h}^\zeta] = \int \underbrace{d\epsilon\,d\Lambda}_{\equiv d\zeta}\,F[\hat{h}^\zeta]\quad \underbrace{\det\frac{\partial(P\cdot\epsilon,\tilde{\Lambda})}{\partial(\epsilon,\Lambda)}}_{\text{Jacobian = Faddeev-Popov determinant}}\quad . \tag{3.127}$$

In the last step we have introduced the Jacobian for the transition of the integral over $\mathcal{D}(P\cdot\epsilon)\,\mathcal{D}\tilde{\Lambda}$, which is what appears in δh, to an integral over the gauge parameters ζ. This Jacobian is the Faddeev-Popov determinant and can be further massaged as

$$\left|\frac{\partial(P\cdot\epsilon,\tilde{\Lambda})}{\partial(\epsilon,\Lambda)}\right| = \begin{vmatrix} P & 0 \\ * & 1 \end{vmatrix} = \det P. \tag{3.128}$$

Thus the path integral is

$$Z = \int d\zeta\,\mathcal{D}X\,e^{i\,S[X,\hat{h}^\zeta]}\,\det P. \tag{3.129}$$

Note that $S[X,\hat{h}^\zeta] = S[X^{\zeta^{-1}},\hat{h}]$ by invariance under combined diffeomorphisms and Weyl transformations.
If the functional measure is also invariant, then we can perform such a gauge transformation to obtain

$$Z = \int d\zeta\,\mathcal{D}X^{\zeta^{-1}}\,e^{i\,S[X^{\zeta^{-1}},\hat{h}]}\,\det P \tag{3.130}$$

and we can relabel $X^{\zeta^{-1}} \to X$. Then $\int d\zeta$ factors out and yields an overall volume of the group of diffeomorphisms and Weyl rescalings, which was our goal. Omitting this overall factor we arrive at

$$\boxed{\begin{aligned} &Z = \int \mathcal{D}X\,\det P\,e^{i\,S_P[X,\hat{h}]} \\ &\hat{h}: \quad \text{arbitrary reference metric.}\end{aligned}} \tag{3.131}$$

There are a number of subtleties:

- The measure $\int d\zeta\,\mathcal{D}X$ is in general invariant only under diffeomorphisms, not under Weyl rescalings. Later we will find that criticality (i.e. $a = 1,\ d = 26$) is equivalent to the absence of this total Weyl anomaly in the quantum measure. For now we assume that the measure is invariant and proceed as above.

- The above procedure assumes that **every** metric h can be written as $h = \hat{h}^\zeta$ for **precisely one** ζ. There is a double mismatch though: First, the conformal Killing transformations are residual gauge symmetries not fixed above. These extra parametrisations must not be included in the path integral in order to avoid overcounting. Indeed we will be careful to fix this extra residual gauge freedom when computing scattering amplitudes. Second there are caveats related to global properties of the WS in that for worldhseets of complicated topology the metric contains extra parameters, so-called moduli, not accounted for by local gauge transformations ζ. We must therefore sum over these moduli separately. We ignore both these issues now and will come back to them in great detail when discussing the moduli space of Riemann surfaces.

Who's afraid of Faddeev-Popov ghosts?

What remains is a treatment of the functional determinant $\det P$ via the famous Faddeev-Popov trick.
Inspired by the fact that inverse determinants can be expressed as Gaussian integrals as in

$$\int \prod_i dy_i e^{-y_i A_{ij} y_j} \sim \frac{1}{\sqrt{\det A}} \tag{3.132}$$

we try to rewrite $\det P$ in a similar manner as an integral. For this, however, we need to define **Grassmann variables** as **anti-commuting (i.e. fermionic) variables**. E.g. define the variables Θ, ψ with the properties

$$\Theta\,\psi = -\psi\,\Theta, \qquad \Theta^2 = 0 = \psi^2. \tag{3.133}$$

We furthermore define the notion of **Berezin integration** over Grassmann variables by setting

$$\int d\Theta = 0\,, \qquad \int d\Theta\,\Theta = 1. \tag{3.134}$$

Since all higher integrals vanish, $\int d\Theta\,\Theta^2 = 0$ etc., we find the identity

$$\int d\psi\, d\Theta\, e^{\Theta a \psi} = \int d\psi\, d\Theta\,(1 + \Theta a \psi) = a \tag{3.135}$$

for a (bosonic) number a. More generally we define the Grassmann variables

$$\Theta_i : \qquad \Theta_i \Theta_j = -\Theta_j \Theta_i \tag{3.136}$$

and similarly for ψ_i. From

$$\int d\Theta_i\,\Theta_j = \delta_{ij} \tag{3.137}$$

we deduce

$$\int d^n\Theta\,\Theta_{i_1}\ldots\Theta_{i_n} = \epsilon_{i_1\ldots i_n}. \tag{3.138}$$

Then, given a bosonic matrix M_{ij} its determinant can be expressed as a Grassmann integral

$$\int \prod_{i=1}^n d\psi_i\, d\Theta_i\, \exp(\Theta_i M_{ij} \psi_j) \cong \det M. \tag{3.139}$$

As a final technical step we note that we are interested in the determinant of a differential operator P. For this we generalise the above formulae to functional integrals over Grassmann-valued fields by promoting

$$M \to \text{operator}, \qquad \Theta_i \to \Theta(\xi^a)\,, \qquad \psi_i \to \psi(\xi^a). \tag{3.140}$$

Applied to our operator

$$P^c_{ab} \;=\; \delta^c_{(b}\nabla_{a)} - h_{ab}\nabla^c \tag{3.141}$$

this procedure yields[5]

$$\det P \;\cong\; \int \mathcal{D}b_{(ab)}\mathcal{D}c^d \exp\Big(\frac{1}{4\pi}\int d^2\xi\sqrt{-\hat{h}}\, b\cdot(P\cdot c)\Big) \tag{3.142}$$

$$\;\equiv\; \int \mathcal{D}b_{(ab)}\mathcal{D}c^d \exp\Big(\frac{1}{4\pi}\int d^2\xi\sqrt{-\hat{h}}\, b^{ab}\, P^d_{ab}\, c_d\Big). \tag{3.143}$$

Here $b_{(ab)}(\xi^a)$ transforms as a symmetric traceless tensor on the worldsheet and $c^d(\xi^a)$ as a vector. Both are anti-commuting, fermionic fields. c^d and $b_{(ab)}$ are called Faddeev-Popov (FP) ghosts and anti-ghost, respectively. They are fermionic objects with integer spin. A priori they will lead to negative norm states, but we will see how to deal with them.
So as a result we have (after integration by parts in the ghost action)

$$\boxed{\begin{aligned} Z &\cong \int \mathcal{D}X\,\mathcal{D}b\,\mathcal{D}c\, e^{i\,(S_X+S_g)},\\ S_X &= -\frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{-\hat{h}}\,\partial^a X\cdot\partial_a X,\\ S_g &= -\frac{i}{2\pi}\int d^2\xi\sqrt{-\hat{h}}\,\hat{h}^{ab}\,c^d\nabla_a b_{(bd)}.\end{aligned}} \tag{3.144}$$

Note:

- The equation of motion for c^a is given by $P\cdot c = 0$. Therefore **the normalisable solutions for c^a are in one-to-one correspondence with the conformal Killing vectors**, which are the generators of the residual symmetry.
- The equation of motion for $b_{(ab)}$ is

$$\nabla_a b^{ab} = 0. \tag{3.145}$$

  We will understand the geometric meaning of these equations when discussing the moduli space of Riemann surfaces.

3.3.2 Quantisation of the bc-system

The gauge fixed path integral is defined in terms of an arbitrary reference metric $\hat{h}_{ab}$ so let's be cheap pick the simplest by going to flat coordinates with $\hat{h}_{ab} = \eta_{ab}$. The matter and ghost action in flat lightcone coordinates read

$$S_X \;=\; \frac{1}{\pi\,\alpha'}\int d^2\xi\,\partial_+ X\cdot\partial_- X, \tag{3.146}$$

$$S_g[b,c] \;=\; \frac{i}{\pi}\int d^2\xi(c^+\partial_- b_{++} + c^-\partial_+ b_{--}). \tag{3.147}$$

[5]The factor $\frac{1}{4\pi}$ is conventional and merely changes the overall normalization of the partition function.

Tracelessness of b_{ab} translates into $b_{+-} \equiv 0 \equiv b_{-+}$. The equations of motion for the (anti-)ghosts take the simple form

$$\boxed{\begin{aligned}&\partial_+ b_{--} = 0 = \partial_- b_{++},\\ &\partial_+ c^- = 0 = \partial_- c^+\end{aligned}}$$

provided the boundary terms proportional to $\int d\tau (c^+ \delta b_{++} - c^- \delta b_{--})|_{\sigma=0}^{\ell}$ that we pick in the process of varying the action vanish.
In close analogy to the matter fields X we now discuss the boundary conditions and mode expansion:

1) **Closed string**:
The closed string is by definition periodic in σ so we must impose the periodic boundary conditions $b(\sigma+\ell) = b(\sigma)$ and $c(\sigma+\ell) = c(\sigma)$. The most general solution to the e.o.m. is expanded as

$$b_{++}(\tau,\sigma) = (\tfrac{2\pi}{\ell})^2 \sum_{n=-\infty}^{\infty} \tilde{b}_n e^{-\frac{2\pi i}{\ell} n\,(\tau+\sigma)}, \tag{3.148}$$

$$b_{--}(\tau,\sigma) = (\tfrac{2\pi}{\ell})^2 \sum_{n=-\infty}^{\infty} b_n e^{-\frac{2\pi i}{\ell} n\,(\tau-\sigma)} \tag{3.149}$$

and

$$c^{+}(\tau,\sigma) = \frac{\ell}{2\pi} \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{-\frac{2\pi i}{\ell} n\,(\tau+\sigma)}, \tag{3.150}$$

$$c^{-}(\tau,\sigma) = \frac{\ell}{2\pi} \sum_{n=-\infty}^{\infty} c_n e^{-\frac{2\pi i}{\ell} n\,(\tau-\sigma)}. \tag{3.151}$$

As for the X-fields we have an independent set of left- and right-movers. The normalisation is chosen such as to lead to nice expressions for the anti-commutator in the quantum theory.

2) **Open string**:
For the open string the boundary terms must vanish at $\sigma = 0$ and $\sigma = \ell$ separately. Out of the various possible boundary conditions compatible with this we consider the following,

$$c^+(\xi^+)\Big|_{\sigma=0,\,\ell} = c^-(\xi^-)\Big|_{\sigma=0,\,\ell} \qquad \text{and} \qquad b_{++}(\xi^+)\Big|_{\sigma=0,\,\ell} = b_{--}(\xi^-)\Big|_{\sigma=0,\,\ell}. \tag{3.152}$$

This can be rewritten as

$$c^1(\tau,\sigma)\Big|_{\sigma=0,\,\ell} = 0 \qquad \text{and} \qquad b_{01}(\tau,\sigma)\Big|_{\sigma=0,\,\ell} = 0, \tag{3.153}$$

which admits a reasonable geometric interpretation: As noted before the c^a correspond to reparametrisation parameters in that the ghost equation of motion $P \cdot c = 0$ is simply the Killing vector equation. Thus the above ghost boundary condition means that the residual conformal Killing group is restricted to such reparametrisations that do not move the boundary of the string in ξ^1, i.e. σ-direction. Since this is a sensible result, we stick to this boundary condition.

The most general solution of the equations of motion subject to these boundary conditions is

$$c^{\pm} = \frac{\ell}{\pi} \sum_n c_n e^{-\frac{\pi i}{\ell} n\,\xi^{\pm}}, \qquad b^{\pm\pm} = (\tfrac{\pi}{\ell})^2 \sum_n b_n e^{-\frac{\pi i}{\ell} n\,\xi^{\pm}}. \tag{3.154}$$

As expected the boundary conditions identify the left and right-moving oscillators.

Quantisation

The conjugate momentum of the anti-ghost field $b_{\pm\pm}$ follows from the action as

$$\Pi_{b_{\pm\pm}} = \frac{\delta\, S_g[b,c]}{\delta\, \partial_\tau b_{\pm\pm}} = \frac{i}{2\pi} c^{\pm} \tag{3.155}$$

with canonical Poisson-bracket relation $\left\{b_{\pm\pm}(\tau,\sigma), \Pi_{b_{\pm\pm}}(\tau,\sigma')\right\}_{\text{P.B.}} = \delta(\sigma-\sigma')$. To quantise this system we must take into account the fermionic nature of the b and c-fields. As is well-known from quantisation of fermions in QFT, the correct procedure is to replace the Poisson-bracket by $\frac{1}{i}$ times the anti-commutator, defined for operators A, B as $\{A, B\} = AB + BA$. Thus,

$$\left\{b_{++}(\tau,\sigma), c^{+}(\tau,\sigma')\right\} = 2\pi\delta(\sigma-\sigma'), \tag{3.156}$$

$$\left\{b_{--}(\tau,\sigma), c^{-}(\tau,\sigma')\right\} = 2\pi\delta(\sigma-\sigma'). \tag{3.157}$$

One can verfiy that this corresponds to the **anti-commutator relations for the modes**

$$\boxed{\{c_m, b_n\} = \delta_{m+n,0}, \qquad\qquad \{c_m, c_n\} = 0 = \{b_m, b_n\}} \tag{3.158}$$

with the same relations obeyed in addition by $\tilde{c}_{n'}$, $\tilde{b}_n$ for the closed string.

Ghost Virasoro-Algebra

The ghost-energy momentum tensor $T^{(g)}$ follows from the full non-gauge fixed action as

$$T^{(g)}_{ab} = \frac{4\pi}{\sqrt{-h}} \frac{\delta\, S_g[b,c,h]}{\delta\, h^{ab}}. \tag{3.159}$$

In lightcone gauge its non-vanishing components are

$$T^{(g)}_{++} = -i(2b_{++}\partial_+ c^+ + (\partial_+ b_{++})c^+), \tag{3.160}$$

$$T^{(g)}_{--} = -i(2b_{--}\partial_- c^- + (\partial_- b_{--})c^-) \tag{3.161}$$

with corresponding Virasoro generators, for the closed string,

$$L^{(g)}_n \quad = -\tfrac{l}{4\pi^2}\int_0^l d\sigma e^{-\frac{2\pi}{l} i\, n\, \sigma} T_{--}, \tag{3.162}$$

$$\tilde{L}^{(g)}_n \quad = -\tfrac{l}{4\pi^2}\int_0^l d\sigma e^{\frac{2\pi}{l} i\, n\, \sigma} T_{++} \tag{3.163}$$

and similar formulae for the open string. It is not hard to check that in terms of the modes this gives, classically,

$$L^{(g)}_m = \sum_{n=-\infty}^{\infty} (m-n)\, b_{m+n} c_{-n} \qquad \text{(classically)}. \tag{3.164}$$

At the quantum level the Virasoro operators are defined as the normal ordered analgue of this classical expression. Normal ordering is again defined by putting lower-level modes to the left. Due to the anti-commuting nature of the modes, we pick up a minus sign in this process if we have to change the order of the modes, i.e.

$$: b_m b_n := \begin{cases} b_m b_n, & \text{if } m \leq n \\ -b_n b_m, & \text{if } n < m \end{cases} \tag{3.165}$$

and similarly for $: c_m c_n :$ as well as for $: b_m c_n :$. Then,

$$L_m^{(g)} = \sum_{n=-\infty}^{\infty} (m-n) \; : b_{m+n} c_{-n} : . \tag{3.166}$$

Note:

1) For example for the zero-level Virasoro generator application of the normal ordering prescription yields

$$L_0^{(g)} = \sum_{n=-\infty}^{-1} (-n) \underbrace{: b_n c_{-n} :}_{=b_n c_{-n}} + \sum_{n=1}^{\infty} (-n) \underbrace{: b_n c_{-n} :}_{=-c_{-n} b_n}, \tag{3.167}$$

$$L_0^{(g)} = \sum_{n=1}^{\infty} n (b_{-n} c_n + c_{-n} b_n). \tag{3.168}$$

2) Note the slightly different normalization factors compared to the X-field Virasoro generators $L_m^{(X)} = \frac{1}{2} \sum_n : \alpha_{m+n} \alpha_{-n} :$. In particular the zero-modes b_0, c_0 do not appear in L_0.

A computation very similar to the one performed to deduce the Virasoro algebra of the matter-, i.e. X^μ fields, yields the **Ghost Virasoro-algebra**

$$[L_m^{(g)}, L_n^{(g)}] = (m-n) L_{m+n}^{(g)} + \frac{1}{6}(m - 13m^3)\delta_{m+n,0}. \tag{3.169}$$

Note that the generators $L_m^{(g)}$ are bilinears in the fermionic ghost modes and thus bosonic. This is why they indeed satisfy commutator (as opposed to anti-commutator) relations.

Significance of the ghost Virasoro algebra and criticality

The algebra of the conformal transformations of the full action $S = S_X + S_g$ is generated by the combined Virasoro generators

$$L_m^{\text{tot}} = L_m^{(X)} + L_m^{(g)} - a^{\text{tot}} \, \delta_{m,0}, \tag{3.170}$$

where we conventionally include a total normal ordering constant a^{tot} into the definition of L_m^{tot}. It is the sum of the normal ordering constant for the X and for the ghost fields,

$$a^{\text{tot}} = a^{(X)} + a^{(g)}. \tag{3.171}$$

- The first piece is just given by $a^{(X)} = \frac{d-2}{24} + \frac{2}{24}$ corresponding to the $d-2$ transverse X-oscillations familiar from lightcone quantisation together with the contribution from the X^0 and X^{d-1} components. These are not absent here since we are in a covariant gauge.
- To compute $a^{(g)}$ we observe that the ghost system counts as one anti-commuting set of integer moded scalars. By a similar computation as performed in section 3.2.3 its contribution to the Casimir energy is

$$a^{(g)} = -\frac{1}{12}. \tag{3.172}$$

  Since there is no factor of $\frac{1}{2}$ in the definition of $L_0^{(g)}$ this is just minus the contribution of a commuting set of integer-moded scalars.

- Therefore

$$a^{\text{tot}} = \frac{d-2}{24} + \frac{2}{24} - \frac{1}{12} = \frac{d-2}{24} \equiv a. \tag{3.173}$$

The ghost system cancels the contribution from the unphysical non-transverse polarisations, a feature that we will encounter again in the framework of BRST quantisation.

One may verify that the combined Virasoro generators satisfy the commutation relations

$$\boxed{[L_m^{\text{tot}}, L_n^{\text{tot}}] = (m-n)\, L_{m+n}^{\text{tot}} + \delta_{m+n,0}\Big(\frac{c^{\text{tot}}}{12}(m^3-m) + 2m(a-1)\Big)} \tag{3.174}$$

with the central extensions governed by the quantities

$$c^{\text{tot}} = c^{(X)} + c^{(g)}\,, \qquad c^{(X)} = d \quad (\text{for propagation in } \mathbb{R}^{1,d-1}), \qquad c^{(g)} = -26. \tag{3.175}$$

The presence of the central term c^{tot} is equivalent to a Weyl anomaly of the full action $S_X + S_g$, or equivalently to a Weyl anomaly in the path integral

$$\int \mathcal{D}X e^{i\,S_X}(\det P). \tag{3.176}$$

The central term and thus also the Weyl anomaly of the path integral is absent iff

$$d = 26\,, \quad a = 1. \tag{3.177}$$

Thus criticality arises as a self-consistency requirement of the Faddeev-Popov treatment of the path integral.

This gives us a final interpretation of the meaning of criticality: It is the requirement that the X-theory cancels the conformal anomaly of the ghost system,

$$\boxed{0 \overset{!}{=} c^{(X)} + c^{(g)} = c^{(X)} - 26} \tag{3.178}$$

so that the anomaly of the full quantum theory is absent. What is actually fixed is not the number of spacetime dimensions, but the central extension $c^{(X)}$.

3.3.3 BRST Quantisation

The presence of the Faddeev-Popov ghosts, which exhibit the wrong spin-statistics correlation, raises the question of the physical state condition. In path integral quantization of gauge theories the physical state condition is implemented via the important concept of the BRST symmetry. The starting point is the observation that the full action $S_X + S_g$ after gauge fixing $\hat{h} = \eta_{ab}$ enjoys a **global, fermionic, residual symmetry**.
Let ϵ be a constant Grassmann parameter. Then this symmetry is generated by the transformations

$$\boxed{\begin{aligned} \delta_\epsilon X^\mu &= \epsilon(c^+\partial_+ + c^-\partial_-)X^\mu, \\ \delta_\epsilon c^\pm &= \epsilon(c^+\partial_+ + c^-\partial_-)c^\pm, \\ \delta_\epsilon b_{\pm\pm} &= i\,\epsilon(T_{\pm\pm}^{(X)} + T_{\pm\pm}^{(g)}). \end{aligned}} \tag{3.179}$$

Note that the transformations of X^μ are just the conformal Killing transformations with fermonic parameter $\epsilon c^\pm$. This symmetry is named **BRST symmetry** (after Becchi, Rouet, Stora, Tyutin).

Comment:

Here were are considering the gauge fixed action $S_X + S_g$ obtained by explicitly plugging $\hat{h}_{ab} = \eta_{ab}$ into the action. Alternatively one can implement the gauge fixing condition as an extra Gaussian factor in the path integral by introducing a set of auxiliary fields whose equations of motion would classically enforce the gauge fixing condition. In this formulation the transformation of the X^μ-fields is a fermonic version of the full original gauge symmetry (here diffeomorphisms and Weyl rescalings). It is only in the above gauge fixed formulation that the BRST transformation of X^μ reduces merely to the residual symmetry (here the conformal Killing transformations). We compare these two formulations in detail on Assignment 7.

Via Noether's theorem one can define a **BRST charge operator** Q_B as the conserved charge associated with a suitable BRST current.
As always, this charge will then generate the underlying symmetry. Explicit application of the Noether procedure confirms that the BRST charge is fermonic as expected. It generates the BRST symmetry in the sense that

$$\delta_\epsilon X^\mu = \epsilon [Q_B, X^\mu], \tag{3.180}$$

$$\delta_\epsilon c^\pm = \epsilon \left\{ Q_B, c^\pm \right\}, \quad \delta_\epsilon b_{\pm\pm} = \epsilon \left\{ Q_B, b_{\pm\pm} \right\}. \tag{3.181}$$

Note the appearance of the commutator for the action of the fermonic BRST operator on a bosonic field, but of the anti-commutator for its action on the fermonic ghost fields.
One can show explicitly that (for open strings)

$$\boxed{Q_B = \sum_{m=-\infty}^{\infty} : (L_{-m}^{(X)} + \tfrac{1}{2} L_{-m}^{(g)} - a\,\delta_{m,0}) c_m :} \tag{3.182}$$

does the job (and analogously for the left- and right-moving charges in the closed string). In particular

$$Q_B^\dagger = Q_B. \tag{3.183}$$

An important property of the BRST symmetry is that it is **nilpotent**:

$$\delta_\epsilon\, \delta_{\epsilon'} \Phi = 0 \qquad \text{for} \quad \Phi \in \{X^\mu,\, b,\, c\} \tag{3.184}$$

This must translate into the crucial relation

$$Q_B^2 = 0. \tag{3.185}$$

In the quantum theory, evaluation of $Q_B^2 = \frac{1}{2}\{Q_B, Q_B\}$ is complicated by normal ordering subtleties. An explicit computation, which again we are not performing here, yields

$$Q_B^2 = \tfrac{1}{2}\{Q_B, Q_B\} = \tfrac{1}{2} \sum_{m,n=-\infty}^{\infty} ([L_m^{\text{tot}}, L_n^{\text{tot}}] + (m-n) L_{m+n}^{\text{tot}}) c_{-m} c_{-n}. \tag{3.186}$$

This vanishes if and only if the full Virasoro algebra is non-anomalous, which in turn is the case for the critical string with ($d = 26$, $a = 1$):

Consistency of the BRST symmetry is equivalent to absence of the total Weyl anomaly.

The actual significance of the BRST symmetry is that it gives the correct physical state condition. A physical state must be gauge invariant. Given the relation between the gauge transformations and the BRST symmetry it is therefore reasonable to expect that a physical state must be invariant under a BRST transformation. Indeed, on Assignment 7 we will give a formal argument for this assertion in the more general, non-gauge fixed formulation alluded to above. From there we borrow the result that a necessary condition for a state to be physical is that

$$\boxed{Q_B \,|\text{phys}\rangle = 0.} \tag{3.187}$$

Indeed since Q_B acts on X as the (residual) symmetry this **implements** in particular **the constraints** resulting from gauge fixing (here the Virasoro constraints).
This, however, is not enough. Namely there exists a large set of trivially physical states given by

$$|\chi\rangle = Q_B\,|\Psi\rangle, \qquad \text{for } |\Psi\rangle \text{ arbitrary.} \tag{3.188}$$

Indeed since $Q_B^2 = 0$ any such χ satisfies the above criterion

$$Q_B\,|\chi\rangle = Q_B^2\,|\Psi\rangle = 0.$$

These states are **null**, i.e. they are orthogonal to all physical states including themselves,

$$\langle\,\text{physical}|\,Q_B\,|\Psi\rangle = 0, \qquad \text{because } Q_B = Q_B^\dagger, \tag{3.189}$$

$$\langle\chi|\,|\chi\rangle = \langle\Psi|\,Q_B^2\,|\Psi\rangle = 0, \quad \text{(zero norm)}. \tag{3.190}$$

Let us make the following definitions:

- States in the kernel of Q_B, $|\chi\rangle$ s.t. $Q_B\,|\chi\rangle = 0$, are called Q-**closed**.
- States in the image of Q_B, $|\chi\rangle = Q_B\,|\Psi\rangle$, are called Q-**exact**.

To define a positive norm Hilbert space we need to divide the set of Q-closed states by the set of Q-exact states.

The positive norm physical Hilbert space is given by

$$\mathcal{H}_{\text{BRST}} = \frac{\mathcal{H}_{\text{closed}}}{\mathcal{H}_{\text{exact}}} \equiv \textbf{ cohomology } \text{of } Q_B. \tag{3.191}$$

States differing by elements of $\mathcal{H}_{\text{exact}}$ are in the same equivalence class = **cohomology class**:

$$|\Psi\rangle \cong |\Psi\rangle + Q_B\,|\chi\rangle \tag{3.192}$$

Comments:

- We will momentarily find that only a subset of $\mathcal{H}_{\text{BRST}}$ defined by another condition really corresponds to the physical states in the sense of participating in the amplitudes.

- One can formally prove that the space $\mathcal{H}_{\text{BRST}}$ modulo this extra condition is of positive norm. For this we refer to [P], volume 1, section 4.4. Instead of going through this formal proof we demonstrate below that the BRST-cohomology correctly implements the Virasoro constraints.
- The concept of a cohomology as the kernel over the image is defined in mathematics for every nilpotent operator. The probably most famous example is the exterior derivative d that maps a p-form to a $p+1$-form. In this context the p-th cohomology group is defined as

$$H^p = \frac{\text{closed p-forms}}{\text{exact p-forms}}. \tag{3.193}$$

Representation theory
To make all of this explicit we need to define a vacuum $|0^{\text{tot}}\rangle = \left|0^{(X)}\right\rangle \otimes \left|0^{(g)}\right\rangle$ for the full theory defined by $S^{\text{tot}} = S^{(X)} + S^{(g)}$, act with creation operators associated with X, b and c on each factor and then implement the physical state condition. Since the X-sector $\left|0^{(X)}\right\rangle$ and the action of the creators $\alpha^\mu_{-|m|}$ has already been discussed we focus on the ghost sector. A sensible assignment is to declare that

- c_{-n}, b_{-n} , $n > 0$ act as creators,
- c_n, b_n , $n > 0$ act as annihilators.

This is consistent with the normal ordering prescription ("creators to the left") and with the form of the zero-level Virasoro generator

$$L_0^{(g)} = \sum_{n=1}^{\infty} nb_{-n}c_n + nc_{-n}b_n. \tag{3.194}$$

Since the ghost Hamiltonian $H^{(g)} \propto L_0^{(g)}$ we are reassured that b_{-n}, c_{-n} take the role of creators. There is an important difference compared to the X-sector, though: Since the zero modes b_0, c_0 do not appear in $L_0^{(g)}$ they must be treated separately. From the anti-commutation relations we deduce that the zero-modes c_0, b_0 form an algebra defined by

$$c_0^2 = 0 = b_0^2 \ , \qquad \Big\{c_0,\ b_0\Big\} = 1. \tag{3.195}$$

A state in the ghost sector must furnish a representation of this algebra. The smallest representation contains two states $|\uparrow\rangle$, $|\downarrow\rangle$ such that

$$\boxed{c_0\,|\downarrow\rangle = |\uparrow\rangle\ , \quad b_0\,|\uparrow\rangle = |\downarrow\rangle\ , \quad c_0\,|\uparrow\rangle = 0\ , \quad b_0\,|\downarrow\rangle = 0.} \tag{3.196}$$

To see that we need at least two states let us try our luck with only a single state $|\Psi\rangle$. Then

$$c_0\,|\Psi\rangle = \alpha\,|\Psi\rangle, \qquad b_0\,|\Psi\rangle = \beta\,|\Psi\rangle, \qquad \text{for some } \alpha,\ \beta \in \mathbb{C}. \tag{3.197}$$

It is now easy to construct a contradiction because

$$\begin{aligned} |\Psi\rangle = (c_0b_0 + b_0c_0)\,|\Psi\rangle &= 2\alpha\beta\,|\Psi\rangle, \\ \text{but} \quad 0 = c_0^2\,|\Psi\rangle &= \alpha^2\,|\Psi\rangle. \end{aligned} \tag{3.198}$$

We therefore stick to the above simplest non-trivial representation given by $|\uparrow\rangle$, $|\downarrow\rangle$. A priori we could make two inequivalent choices for the full vacuum:

1.) $|0^{\text{tot}}\rangle = |0^{(X)}\rangle \otimes |\uparrow\rangle \equiv |0,\uparrow\rangle$
The vacuum is annihilated by c_0 and by α_n, c_n, b_n with $n > 0$.

2.) $|0^{\text{tot}}\rangle = |0^{(X)}\rangle \otimes |\downarrow\rangle \equiv |0,\downarrow\rangle$
The vacuum is annihilated by b_0 and α_n, c_n, b_n with $n > 0$.

To gain some intuition which is the correct one we evaluate the BRST condition on the special subset $|\Psi^{(X)}\rangle \otimes |\uparrow\rangle$ and, respectively, $|\Psi^{(X)}\rangle \otimes |\downarrow\rangle$ contained in the spectrum that results from the two choices of vacua. This will give us a heuristic argument as to which of the two vacua to choose, which can then be extended to a proper physical state theorem.

Consider first case 2.). The physical state condition implies $Q_B|\chi\rangle = 0$. For $|\chi\rangle = |\Psi^{(X)}\rangle \otimes |\downarrow\rangle$ this gives

$$0 \stackrel{!}{=} Q_B|\chi\rangle = \sum_{m=-\infty}^{\infty} :\left(L^{(X)}_{-m} + \frac{1}{2}L^{(g)}_{-m} - a\delta_{m,0}\right)c_m : |\chi\rangle. \tag{3.199}$$

Using $c_m|\chi\rangle = b_m|\chi\rangle = 0$ for $m > 0$ and setting $a = 1$ this becomes

$$0 \stackrel{!}{=} Q_B|\chi\rangle \quad = \quad \left[(L^{(X)}_0 - 1)c_0 + \sum_{m>0} c_{-m}L^{(X)}_m\right]|\chi\rangle. \tag{3.200}$$

Evaluating the action of the ghost modes on the vacuum yields

$$(L^{(X)}_0 - 1)|\chi\rangle \stackrel{!}{=} 0 \qquad \text{and} \qquad L^{(X)}_m|\chi\rangle \stackrel{!}{=} 0 \ \forall\ m > 0. \tag{3.201}$$

This recovers the correct constraints from OCQ.
By contrast, case 1.) with vacuum $|0,\ \uparrow\rangle$ does not allow us to recover the known constraints in this simple fashion. This suggests that only $|0,\ \downarrow\rangle$ is a meaningful vacuum. Note that the two vacua are distinguished by the defining property $b_0|0,\ \downarrow\rangle = 0$.

Note: For a point particle the physical state condition is simply the mass shell condition $k^2+m^2 = 0$. Q-closed states of the form $|p\rangle\otimes|\downarrow\rangle$ satisfy this and are thus sensible physical states, in the same manner as Q-closed states of the special form $|\Psi^{(X)}\rangle\otimes|\downarrow\rangle$ automatically satisfy the Virasoro constraints. In addition one can convince oneself that Q-closed modulo Q-exact states of the form $|p\rangle \otimes |\uparrow\rangle$, i.e. constructed from the first vacuum, give another copy of physical, i.e. on-shell states; however, these decouple from all scattering amplitudes and so must be discarded. For a derivation of these assertions see [P], volume 1, p.129-131.
For the string theory the same logic goes through even though the details are technically more involved due to the oscillator degrees of freedom. It is still true that a consideration of scattering amplitudes reveals that only BRST closed (modulo exact) states built from $|\Psi^{(X)}\rangle\otimes|\downarrow\rangle$ can contribute to amplitudes (see [P], volume 1, Chapter 9). In addition one must impose the constraint $b_0|\Psi\rangle = 0$, which generalizes the defining property $b_0|0,\ \downarrow\rangle = 0$ of the vacuum.

This can be summarized in form the following

Theorem: (Proof: Polchinski I, Chapter 4.4)
The positive norm physical states are the states $Q_B|\Psi\rangle = 0$ modulo $|\Psi\rangle = Q_B|\chi\rangle$ built on $|0^{\text{tot}}\rangle = |0,\ p\rangle^{(X)} \otimes |\downarrow\rangle^{(g)}$ that satisfy in addition $b_0|\Psi\rangle = 0$.

Example: Level (1) open excitations

For the first excited level we make the ansatz

$$|\Psi\rangle = (\xi \cdot \alpha_{-1} + \beta b_{-1} + \gamma c_{-1}) \left|0^{\text{tot}}\right\rangle. \tag{3.202}$$

This gives us 26 + 2 states to begin with.

On Assignment 7 we will work out the constraints and find the following structure:

i) From $b_0 |\Psi\rangle = 0$ we deduce $0 = \{Q_B,\ b_0\} |\Psi\rangle = L_0^{\text{tot}} |\Psi\rangle$. This yields the mass shell condition $p^2 = 0$.

ii) $Q_B |\Psi\rangle = 0$ leads to

$$\begin{aligned} 0 &\stackrel{!}{=} ((p \cdot \xi)c_{-1} + \beta p \cdot \alpha_{-1}) \left|0^{\text{tot}}\right\rangle \\ \Rightarrow &\quad p \cdot \xi = 0 \ \&\ \beta = 0. \end{aligned} \tag{3.203}$$

Requiring Q-closedness therefore removes the unphysical anti-ghost excitations as well as all polarisations that are not orthogonal to the momentum, thereby eliminating 2 out the 26+2 original states.

iii) To analyse $|\Psi\rangle \cong |\Psi\rangle + Q_B |\chi\rangle$ we observe that for a general state $|\chi\rangle = (\xi' \cdot \alpha_{-1} + \beta' b_{-1} + \gamma' c_{-1}) |0^{\text{tot}}\rangle$ at level 1 we have

$$Q_B |\chi\rangle = ((p \cdot \xi')c_{-1} + \beta' p \cdot \alpha_{-1}) \left|0^{\text{tot}}\right\rangle.$$

This shows that $c_{-1} |0^{\text{tot}}\rangle$ is BRST exact and the polarisation vector is only defined up to the equivalence

$$\xi \cong \xi + \beta' p, \qquad \beta' \in \mathbb{C}. \tag{3.204}$$

(Note that this is consistent with the conditions i) and ii).)

Thus we are left with 24 physical positive norm states, as required.

Chapter 4

Conformal field theory (CFT)

Conformal symmetry is, loosely speaking, invariance under rescaling. Conformally invariant systems possess no intrinsic length, mass or energy scale. In particular there exists no notion of massive particles or massive excitations as these would induce a reference scale.
Conformal invariance plays an important role in many physical systems such as

- string theory, where the worldsheet theory is a two-dimensional conformal field theory;
- at fixed points of the renormalisation group (RG) equations in QFT, at which a theory becomes scale invariant;
- near critical points in condensed matter or statistical physics, at which the correlation length diverges, leaving us again with a scale invariant theory;
- the AdS/CFT correspondence, which relates gravity on AdS space with a CFT on its boundary.

The treatment of such conformally invariant theories necessarily differs from the treatment of non-conformal systems in usual QFT. Two-dimensional conformal field theories are even more special because in two dimensions the group of infinitesimal conformal transformations is infinite. This is large enough as to sometimes allow us to solve the theory exactly and completely. Such theories are called integrable and the search for integrable structures also in higher dimensional theories, e.g. for certain field theories in four dimensions ($\mathcal{N}$=4 Super-Yang-Mills theory), has become an important challenge in theoretical physics. The treatment of two-dimensional CFTs makes use of powerful methods of holomorphic analysis and has developed into an independent field of mathematical physics.

4.1 Conformal invariance

Definition: A **conformal transformation** is a diffeomorphism under which the metric changes only by an overall factor.

In the sequel we consider flat d-dimensional space of arbitrary signature, i.e. $\mathbb{R}^{m,n}$ with $m+n = d$. A conformal transformation is then a diffeomorphism $x \to x'(x)$ under which the metric $\eta_{\alpha\beta}$ transforms like

$$\boxed{\eta_{\mu\nu} \to \eta_{\alpha\beta}\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{x'^\nu} = \Lambda(x)\eta_{\mu\nu}} \tag{4.1}$$

for some conformal factor $\Lambda(x)$. Infinitesimally with $x'^\mu = x^\mu - \epsilon^\mu(x) + \mathcal{O}(\epsilon^2)$ and $\Lambda(x) = e^{\omega(x)} = 1 + \omega(x) + \ldots$ this gives

$$\boxed{\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \omega(x)\eta_{\mu\nu}.} \tag{4.2}$$

Further elementary manipulations carried out on Assignment 7 yield the following set of equations satisfied by $\omega(x)$,

$$\begin{aligned} \omega(x) &= \frac{2}{d}\partial \cdot \epsilon, \\ \left(\eta_{\mu\nu}\partial^2 + (d-2)\partial_\mu\partial_\nu\right)\omega(x) &= 0, \\ (d-1)\partial^2\omega(x) &= 0. \end{aligned} \tag{4.3}$$

Technically this is the origin why we must distinguish the cases $d = 2$ and $d \geq 3$: For $d = 2$ the second equation is vacuous.

4.2 The conformal group in $d \geq 3$

Even though we will mostly be interested in conformal symmetry in two dimensions we briefly consider $\mathbb{R}^{n.m}, d = n + m \geq 3$.
The set of *infinitesimal* conformal transformations can be deduced by finding the most general solution of the constraints (4.3). As discussed on Assignment 7, these are

- translations $x \to x' = x + a$,
- Lorentz transformations $x^\mu \to x'^\mu = x^\mu + m^\mu{}_\nu x^\nu$ with $m_{\mu\nu} = -m_{\nu\mu}$,
- dilatations $x \to x' = (1+\alpha)x$,
- special conformal transformations (SCT) $x \to x' = x + 2(x \cdot b)x - (x \cdot x)b$.
  The *global version* of the special conformal transformations is

  $$\frac{x'}{(x' \cdot x')} = \frac{x}{(x \cdot x)} - b. \tag{4.4}$$

  This can be understood as a successive application of an inversion $x \to \frac{x}{x^2}$, a translation and another inversion.

Note that global special conformal transformations take the form

$$x' = \frac{x - (x \cdot x)b}{1 - 2(b \cdot x) + (b \cdot b)(x \cdot x)}.$$

This is infinite at those points where the denominator vanishes. We conclude that in order for special conformal transformations to be globally defined we must consider the *conformal compactification* $\mathbb{R}^{n,m} \cup \infty$.
This shows that we must carefully distinguish between

a) the group of *globally defined* conformal diffeomorphisms, called *conformal group* and its algebra, the *conformal algebra* and

b) the group of *infinitesimal* conformal transformations.

In particular the conformal group depends on the topology of the space we consider as seen above. E.g. for $\mathbb{R}^{n,m} \cup \infty$ and $d = n + m \geq 3$ the conformal group has $\frac{1}{2}(d+1)(d+2)$ parameters. Indeed one can prove the following theorem:

The conformal group acting on $\mathbb{R}^{n,m} \cup \infty$ is isomorphic to $\mathrm{SO}(m+1, n+1)$.

4.3 The conformal group in $d = 2$

In 2 dimensions the constraints (4.3) reduce to

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = (\partial \cdot \epsilon)\, \eta_{\mu\nu}. \tag{4.5}$$

In the sequel we consider $\mathbb{R}^2$ with Euclidean signature $\eta_{\mu\nu} = \delta_{\mu\nu}$ and complex variables

$$\begin{aligned} z = x^0 + ix^1, &\qquad \epsilon^z = \epsilon^0 + i\epsilon^1 \equiv \epsilon, \\ \bar{z} = x^0 - ix^1, &\qquad \bar{\epsilon}^{\bar{z}} = \epsilon^0 - i\epsilon^1 \equiv \bar{\epsilon}, \\ \partial_z = \frac{1}{2}(\partial_0 - i\partial_1), &\qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1), \\ d^2x = dx^0 dx^1 &= \frac{1}{2} dz d\bar{z}. \end{aligned}$$

4.3.1 Infinitesimal conformal transformations

In complex variables the constraint (4.5) implies

$$\partial_{\bar{z}} \epsilon = 0 = \partial_z \bar{\epsilon} \quad \Rightarrow \quad \epsilon = \epsilon(z) \ \&\ \bar{\epsilon} = \bar{\epsilon}(\bar{z}). \tag{4.6}$$

We have thus established that the group of **infinitesimal conformal transformations** in Euclidean 2 dimensions is generated by all **meromorphic functions** $\epsilon(z)$ and **anti-meromorphic functions** $\bar{\epsilon}(z)$. Note that we say meromorphic, not holomorphic because for infinitesimal conformal transformations we do allow for singularities outside the open set under consideration. The (anti-)meromorphic generators can be expanded in a Laurent series as

$$z \quad \rightarrow \quad z' = z + \epsilon(z) = z + \sum_{n \in \mathbb{Z}} \epsilon_n z^{n+1}, \tag{4.7}$$

$$\bar{z} \quad \rightarrow \quad \bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}) = \bar{z} + \sum_{n \in \mathbb{Z}} \bar{\epsilon}_n \bar{z}^{n+1}. \tag{4.8}$$

A basis of generators for the algebra of infinitesimal conformal transformations is

$$l_n = -z^{n+1}\partial_z\ , \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}\ ,\ \text{for } n \in \mathbb{Z}. \tag{4.9}$$

These generators obey the commutation relations of the **Witt algebra**

$$[l_m, l_n] = (m-n) l_{m+n}, \tag{4.10}$$

$$[\bar{l}_m, \bar{l}_n] = (m-n) \bar{l}_{m+n}, \tag{4.11}$$

$$[\bar{l}_m, l_n] = 0. \tag{4.12}$$

4.3.2 The Möbius group as the group of global conformal transformations

By contrast, the **group of global conformal diffeomorphisms** depends on the topology of the 2-dimensional space. In particular, as for $d \geq 3$, in order for special conformal transformations to be globally defined we need to compactify.
In the sequel we focus on the space $S^2 \cong \mathbb{C} \cup \infty$. The equivalence of S^2 and $\mathbb{C} \cup \infty$ can be seen via the stereographic projection. More complicated topologies, e.g. that of a torus T^2, will be discussed later.

Even if we restrict to S^2 not all generators l_n are well-defined globally, i.e. at all points.

- At $z = 0$ $l_n = -z^{n+1}\partial_z$ is well-defined only for $n \geq -1$.
- To analyse the behaviour at $z = \infty$ we introduce the variable $w = \frac{1}{z}$ and consider the bahaviour at $w = 0$ via
$$\partial_z = -\frac{1}{z^2}\partial_w = -w^2\partial_w \;\Rightarrow\; l_n = w^{-(n+1)}w^2\partial_w = w^{1-n}\partial_w. \tag{4.13}$$
Thus l_n is well-defined at $z = \infty$ for $n \leq 1$.

$$\boxed{\text{The group of finite conformal diffeomorphisms on } S^2 \text{ is generated by } l_{-1},\ l_0,\ l_1 \text{ and } \bar{l}_{-1},\ \bar{l}_0,\ \bar{l}_1.}$$

Geometric interpretation:

- $l_{-1} = -\partial_z$ is the generator of rigid translations $z \mapsto z + b,\ b \in \mathbb{C}$.
- $l_0 = -z\partial_z$ is the generator of complex dilatations $z \mapsto az,\ a \in \mathbb{C}$.
These are best analysed in polar coordinates $z = re^{i\varphi}$, in terms of which $l_0 = -\frac{r}{2}\partial_r + \frac{i}{2}\partial_\varphi$ and $\bar{l}_0 = -\frac{r}{2}\partial_r - \frac{i}{2}\partial_\varphi$. Thus
$$l_0 + \bar{l}_0 = -r\partial_r \qquad \text{generates dilatations,} \tag{4.14}$$
$$i(l_0 - \bar{l}_0) = -\partial_\varphi \qquad \text{generates rotations.} \tag{4.15}$$
- $l_1 = -z^2\partial_z$ generates special conformal transformations because $cl_1 z = -cz^2$ is the infintesimal version of
$$z \mapsto \frac{z}{cz+1}. \tag{4.16}$$

Altogether a globally defined conformal diffeomorphisms can be brought into the form
$$z \mapsto \frac{az+b}{cz+d} \tag{4.17}$$
subject to the following constraints:

- Invertibility requires $ad - bc \neq 0$. Thus we can rescale to $ad - bc = 1$ without changing the transformation. Note that four complex numbers (a, b, c, d) with $ad - bc \neq 0$ can be viewed as the entries of a complex 2×2-matrix with unit determinant. These parameters therefore generate the special linear group $SL(2, \mathbb{C})$.
- Even after rescaling to $ad - bc = 1$ the set of parameters (a, b, c, d) and $(-a, -b, -c, -d)$ give the same transformation. To avoid redundancy we must therefore divide $SL(2, \mathbb{C})$ by the $\mathbb{Z}_2$ action $(a, b, c, d) \to (-a, -b, -c, -d)$.

$\boxed{\text{The group of conformal diffeomorphisms on } S^2 \text{ is the Möbius group } \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2 = \mathrm{PSL}(2,\mathbb{C}).}$

An important property of $\mathrm{PSL}(2,\mathbb{C})$ is that it maps any 3 distinct points to any other 3 distinct points. This will be discussed in the exercises.

4.3.3 Relation between the complex (half-)plane and the worldsheet on the cylinder (or strip)

a.) Closed strings on the cylinder

For closed strings we had defined the theory on the cylinder with coordinates and metric

$$(\tau,\ \sigma),\ \xi^{\pm} = \tau \pm \sigma,\ \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.18}$$

To relate this to the Euclidean theory on S^2 we first perform a Wick rotation $\tau = -i\tau'$ and relabel $\tau' \to \tau$. The new metric in the Euclidean coordinates $(\tau,\ \sigma)$ is

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.19}$$

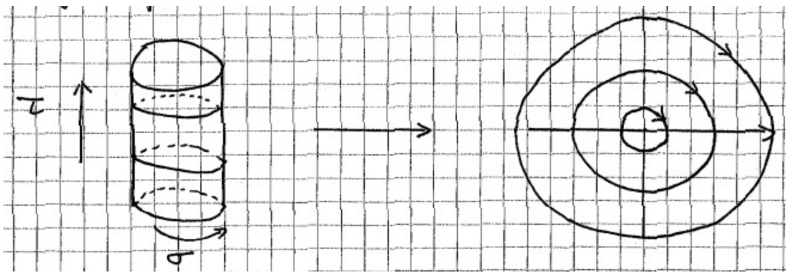
We then define the coordinates

$$\omega = \tau - i\sigma, \qquad \bar{\omega} = \tau + i\sigma. \tag{4.20}$$

The map

$$\omega \mapsto z(\omega) = e^{\frac{2\pi}{\ell}\omega} = e^{\frac{2\pi}{\ell}(\tau - i\sigma)} \tag{4.21}$$

is the conformal mapping from the cylinder to the complex plane $\mathbb{C} \cup \infty$.



In particular it maps

$$\begin{aligned} \tau = -\infty \quad &\longrightarrow \quad z = 0 & (4.22)\\ \tau = +\infty \quad &\longrightarrow \quad z = \infty. & (4.23) \end{aligned}$$

This property will be crucial when considering asymptotically in- or outgoing states in scattering amplitudes and is the basis for the famous operator-state-correspondence in CFTs, to be discussed later.
We can furthermore identify the transformations

$$\begin{aligned} \tau \mapsto \tau + a \quad &\longleftrightarrow \quad z \mapsto e^{\frac{2\pi}{\ell}a} z \text{ (dilatation)}, & (4.24)\\ \sigma \mapsto \sigma + b \quad &\longleftrightarrow \quad z \mapsto e^{-i\frac{2\pi}{\ell}b} z \text{ (rotation)}. & (4.25) \end{aligned}$$

The geometric action of $l_0 + \bar{l}_0$ and $i(l_0 - \bar{l}_0)$ is in perfect agreement with their relation to the Hamiltonian and the momentum operator as the generators of time and spatial translations,

$$l_0 + \bar{l}_0 : \quad \text{dilatation} \quad \leftrightarrow H = \frac{2\pi}{\ell}(L_0 + \bar{L}_0), \tag{4.26}$$

$$i(l_0 - \bar{l}_0) : \quad \text{rotation} \quad \leftrightarrow P = -\frac{2\pi}{\ell} i(L_0 - \bar{L}_0). \tag{4.27}$$

b.) Open strings on the strip

For open strings we have considered the worldsheet in coordinates (τ, σ) with the topology of a strip with boundaries at $\sigma = 0$ and $\sigma = \ell$. By a similar conformal mapping as (4.20), but with adjusted periodicities,

$$(\tau + i\sigma) \mapsto e^{\frac{\pi}{\ell}(\tau + i\sigma)}, \tag{4.28}$$

this is mapped to the **upper half plane**. Note that we are have changed the sign in front of σ to ensure the conventional choice that the open string is defined on the *upper* as opposed to lower half-plane. The boundary of the strip is mapped to the real line $\mathrm{Im}(z) = 0$, with the half-lines $z > 0$ and $z < 0$ corresponding to the two boundaries at $\sigma = 0$ and $\sigma = \ell$, respectively. The boundary conditions now become reflection conditions for the conformal fields on the real line. This gives rise to the concept of **boundary CFT**, which has developed into a rich subfield by itself. Sadly we do not have the time to dig into this topic in more detail and will mostly be working with CFTs on the sphere, corresponding to closed string worldsheets, in what follows.

4.4 Conformal fields and their OPE

A CFT is a physical theory invariant under the group of *infinitesimal* conformal transformations. In particular, as anticipated already, there is no intrinsic notion of length scale or of massive excitations.

Even though the definition of a CFT is much more general, let us first take the lagrangian perspective, whose logic is familiar from the definition of general QFTs:

- Our starting point is a classical theory defined by an action $S[\phi_i(x)]$ with the specific requirement that this action is invariant under infinitesimal conformal transformations.
- The basic objects of this theory are the **fields** $\mathcal{O}_i(x)$. By this we mean, by slight abuse of notation, any *local expression* built out of the $\phi_i(x)$ appearing in the action and derivatives thereof. Local fields are e.g. products or power series such as exponentials of $\phi_i(x)$. These $\mathcal{O}_i(x)$ are also called *local operators.*
- The quantum theory is determined by the correlation functions

$$\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle = \frac{1}{Z} \int \mathcal{D}\phi_i e^{-S[\phi_i]} \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n). \tag{4.29}$$

  Importantly, the expression inside $\langle \ldots \rangle$ is always time-ordered as is familiar from the treatment of correlation functions in usual QFTs.

- In writing equations involving the local $\mathcal{O}_i$ we will always think of operator equations in the full quantum theory, i.e. think of the operators as inserted into a time-ordered path integral as above. For instance an equation of the form

$$\mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2) = f(\mathcal{O}_1(x_1), \mathcal{O}_2(x_2)) \tag{4.30}$$

is shorthand for

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\ldots\rangle = \langle f(\mathcal{O}_1(x_1), \mathcal{O}_2(x_2))\ldots\rangle \tag{4.31}$$

with ... representing arbitrary operator insertions at a distance bigger that $|x_1 - x_2|$.

Now comes an important, though probably unfamiliar point: Conformal symmetry allows for a rather different definition of the theory than the one above starting from a classical action. A generic CFT in d dimensions need not have a description via an action. Rather it is defined by a 'complete set' of local fields $\mathcal{O}_i$ and their correlation functions. If we do have a lagrangian description, these correlation functions are given as above. More generally, however, we can think of the correlators as maps from the space of operators to $\mathbb{C}$ consistent with conformal invariance. We will see that this is very constraining. If really all correlators are known in terms of a finite amount of input data the theory is solved completely and defined through these data.[1]
There are essentially two reasons why this can work: First, because there is a special notion of a 'complete set of operators' available in a CFT which does not exist in a general QFT - the set of quasi-primary fields - and second, because the operator product expansion (OPE) of two such quasi-primaries has remarkable properties. Let us introduce both concepts in turn.

1) Primary and quasi-primary fields

Since in this course we are mainly interested in the applications two-dimensional CFTs to string theory **we restrict the following discussion, unless states otherwise, to a $d = 2$ CFT on $S^2 = \mathbb{C} \cup \infty$, with general fields of the form $\mathcal{O}(z, \bar{z})$.**

1) If a field $\Phi(z, \bar{z})$ transforms under $z \mapsto z' = \lambda z,\ \lambda \in \mathbb{C}$ as

$$\Phi(z, \bar{z}) \mapsto \Phi'(z', \bar{z}') = \lambda^{-h}\bar{\lambda}^{-\bar{h}}\Phi(z, \bar{z}) \tag{4.32}$$

then it has **conformal dimension** $(h, \bar{h})$. Note that in general $\bar{h} \neq h^*$.

2) A **primary field** $\Phi(z, \bar{z})$ transforms as a tensor under conformal tranformations $z \mapsto z' = f(z)$:

$$\Phi(z, \bar{z}) \mapsto \Phi'(z', \bar{z}') = \left(\frac{\partial f}{\partial z}\right)^{-h}\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{-\bar{h}}\Phi(z, \bar{z}). \tag{4.33}$$

Note: In particular, and in the spirit of the discussion around (4.31), we require as part of the defining property of a primary field that any correlation function involving primary fields transforms as

$$\langle \prod_i \Phi(z_i, \bar{z}_i)\rangle \mapsto \prod_i \left(\frac{\partial f}{\partial z}|_{z_i}\right)^{-h_i}\left(\frac{\partial \bar{f}}{\partial \bar{z}}|_{\bar{z}_i}\right)^{-\bar{h}_i}\langle \prod_i \Phi(z_i, \bar{z}_i)\rangle. \tag{4.34}$$

[1] A CFT in $d = 2$ dimensions is indeed exactly solvable in this sense. This is because the two-dimensional Virasoro algebra is infinite-dimensional. In higher dimensional CFTs the computation of all correlators in terms of finite data is possible in principle, but much harder in practice due to the lack of this extra symmetry. We will be more specific at the end of section 4.7.

Expanding $f(z) = z + \epsilon(z) + \dots$ gives the infinitesimal scaling behaviour for primary fields

$$\delta_{\epsilon,\bar{\epsilon}}\Phi(z,\bar{z}) = -(h\partial_z\epsilon + \epsilon\partial_z + \bar{h}\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}})\Phi(z,\bar{z}). \tag{4.35}$$

3) A **quasi-primary field** satisfies (4.33) for $f \in PSL(2,\mathbb{C})$. In particular every primary is a quasi-primary, but not the other way round.

4) A **chiral field** is a field $\Phi(z)$, an anti-chiral field is a field $\Phi(\bar{z})$.

Remarks:

- Quasi-primaries are tensors under the group of globally defined conformal transformations. In a d-dimensional CFT with $d > 2$ (on $\mathbb{R}^{m,n}$ with $m + n = d$), these are the fields with specific transformation behaviour under $SO(m+1, n+2)$.[2] All the statements we make about the quasi-primaries based on their transformations under $PSL(2,\mathbb{C})$ transformations in a two-dimensional CFT carry over analogously to quasi-primaries in higher-dimensional CFTs.
- What has no analogue in higher dimensions is the concept of a primary field, which exploits the infinitesimal structure of the two-dimensional Virasoro algebra.

Mode Expansions

Before we proceed let us define the mode expansion of two-dimensional primaries. Consider the transformation from the worldsheet on the cylinder to $S^2 = \mathbb{C} \cup \infty$.
Suppose on the cylinder a field Φ is purely left-moving and has the mode expansion

$$\Phi_L(\xi^-) = \sum_n \phi_n e^{-i\,n\xi^-}\,(\tfrac{2\pi}{\ell})^h. \tag{4.36}$$

Then it is easy to show that if Φ is primary of weight h, the corresponding expansion on S^2 is

$$\boxed{\Phi_{\text{plane}}(z) = \sum_n z^{-n-h}\phi_n\,.} \tag{4.37}$$

See Ass. 8 for a proof. The generalisation to fields with chiral and anti-chiral pieces is obvious. The modes ϕ_n can be obtained with the help of the residue theorem as

$$\boxed{\phi_n = \frac{1}{2\pi i}\oint dz\,\Phi(z)\,z^{n+h-1}.} \tag{4.38}$$

2) The Operator Product Expansion

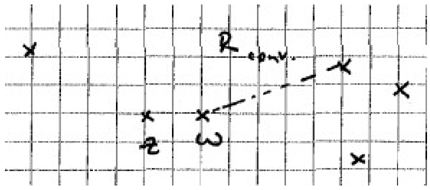
The second important concept in a CFT is that of the **operator product expansion (OPE)**. In a general (Lorentz-invariant) QFT, the OPE is defined as an *approximative expansion* of two operators $\mathcal{O}_i(x_i)$ and $\mathcal{O}_j(x_j)$ *valid in the limit* $x_i - x_j \to 0$,

$$\mathcal{O}_i(x_i)\mathcal{O}_j(x_j) = \sum_k \underbrace{C_{ij}^k\,(|x_i - x_j|)}_{\text{functions in }\mathbb{C}}\,\mathcal{O}_k(x_k). \tag{4.39}$$

[2]Note that in many texts on higher-dimensional CFTs the term primary is used for what we call quasi-primary.

OPEs of the above type are defined generally in QFT as convergent series in a certain open neighbourhood of the operators.
In a d-dimensional CFT the structure of the OPE is much more powerful. This is because the OPE In a d-dimensional CFT satisfies three key properties that rely on conformal invariance, which we state here without proof:

- In a d-dimensional CFT the OPE of two quasi-primaries involves only other quasi-primaries and their derivatives, the so-called descendent fields.
- The functional dependence of the $C_{ij}^k\,(|x_i - x_j|)$ is completely fixed by conformal invariance.
- The OPE is an *exact expression*, i.e. an asymptotic series with radius of convergence the distance to the next field insertion when viewed as an operator equation.



To illustrate this let us specify to a general $d = 2$ CFT. It can be shown that the OPE of two quasi-primaries $\phi_i(z)$ and $\phi_j(w)$ (taken to be chiral for brevity) can be written as

$$\phi_i(z)\phi_j(w) = \sum_{k,n\geq 0} C_{ij}{}^k \frac{a^n_{ijk}}{n!} \frac{1}{(z-w)^{h_i+h_j-h_k-n}} \partial^n \phi_k(w), \tag{4.40}$$

$$a^n_{ijk} = \binom{2h_k + n - 1}{n}^{-1} \binom{h_k + h_i - h_j + n - 1}{n}, \tag{4.41}$$

where the sum over k on the right involves only quasi-primaries. For a proof see e.g. [BP], Chapter 2.6.3. What is important for us is the appearance of the so-called **structure constants** C_{ij}^k. As we will see in chapter 4.7, the structure constants directly determine the 3-point functions of the CFT. Then, by successive application of the OPE we have a chance of reducing all higher n-point functions to lower correlators. This is the idea behind defining the CFT in terms of a finite amount of data.

To summarise, of special importance in a (two-dimensional) CFT is the set of quasi-primaries (primaries) and their OPE. We will now learn how to deduce the OPE for certain fields.

4.5 Conformal Ward-Takahashi identities and energy-momentum tensor

In this section we will demonstrate the importance and power of the operator product expansion. Our aim is to compute the OPE between the energy momentum tensor and a conformal field. This will also shield new light on the nature of the energy momentum tensor and of the conformal anomaly.

4.5.1 General Ward-Takahashi identities

To set the stage we derive here the Ward-Takahashi identities for a general field quantum field theory, with specialisations to CFTs reserved for the next section. Consider therefore a general QFT in d dimensions defined by the path integral

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]}. \tag{4.42}$$

Suppose the theory enjoys a global symmetry

$$\phi \quad \rightarrow \quad \phi' = \phi + \epsilon\delta\phi \qquad \text{with } \epsilon \text{ constant.} \tag{4.43}$$

That is, the classical action transforms as

$$S[\phi] \quad \rightarrow \quad S'[\phi'] = S[\phi]. \tag{4.44}$$

Suppose furthermore that the symmetry is non-anomalous, i.e. also the measure is invariant,

$$\mathcal{D}\phi \rightarrow \mathcal{D}\phi' = \mathcal{D}\phi. \tag{4.45}$$

- **Quantum version of Noether's theorem**:
  We know that a classical continuous symmetry implies the existence of conserved current $\partial_\alpha J^\alpha = 0$. The quantum version of this statement is that $\partial_\alpha J^\alpha(x) = 0$ holds as an operator equation, i.e. together with arbitrary operator insertions away from x under the path integral. To derive this quantum version of Noether's theorem for a non-anomalous global symmetry, we start as in the classical case by promoting $\epsilon \rightarrow \epsilon(x)$. Now the measure $\mathcal{D}\phi$ and the action $S[\phi]$ are no longer separately invariant, but the combined change of the partition function due to the transformation of the measure and of the action can only involve terms proportional to $\partial_\alpha\epsilon(x)$. We can use this observation to define the Noether current by parametrising the change in Z as

$$\begin{aligned} Z \rightarrow Z' = \int \mathcal{D}\phi' e^{-S[\phi']} &= \int \mathcal{D}\phi \, e^{-S[\phi] - \frac{1}{2\pi}\int J^\alpha \partial_\alpha \epsilon(x)} = \\ = \int \mathcal{D}\phi \, e^{-S[\phi]}\Big(1 - \frac{1}{2\pi}\int J^\alpha \partial_\alpha \epsilon(x)\Big) &= \int \mathcal{D}\phi \, e^{-S[\phi]}\Big(1 + \frac{1}{2\pi}\int \partial_\alpha J^\alpha \, \epsilon(x)\Big). \end{aligned} \tag{4.46}$$

  On the other hand

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]} = \int \mathcal{D}\phi' \, e^{-S[\phi']} = Z' \tag{4.47}$$

  because we just changed the integration variable. Note that this is not a contradiction to the statement that the transformation with ϵ replaced by $\epsilon(x)$ is no longer a symmetry of the theory, because $S[\phi']$ and the measure $\mathcal{D}\phi'$ are not invariant independently. Therefore

$$\frac{1}{Z}\int \mathcal{D}\phi \, e^{-S[\phi]}\partial_\alpha J^\alpha(x) = \langle \partial_\alpha J^\alpha(x) \rangle = 0. \tag{4.48}$$

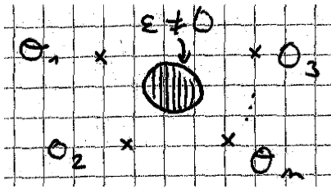
  To show that this indeed holds as an operator equation in the above sense we consider the correlator

$$\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle = \frac{1}{Z}\int \mathcal{D}\phi e^{-S[\phi]}\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n). \tag{4.49}$$

Under the symmetry (4.43) a local operator transforms as

$$\mathcal{O}_i \to \mathcal{O}_i + \epsilon\delta\mathcal{O}_i. \tag{4.50}$$

We now promote $\epsilon \to \epsilon(x)$ such that $\epsilon(x)$ vanishes at the insertion of the operators in the correlator, $\epsilon(x)\Big|_{x=x_i} = 0$.



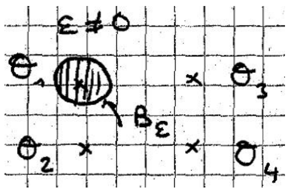
The same steps as before yield $0 = \langle \partial_\alpha J^\alpha(x)\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle$ i.e.

$$\boxed{0 = \partial_\alpha J^\alpha \qquad \text{as an operator equation.}} \tag{4.51}$$

To avoid confusion with what comes next, we hasten to stress that this should really be read as the statement that $0 = \langle \int_B \partial_\alpha J^\alpha(x)\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle$ for an arbitrary region B that does not include any of the operators $\mathcal{O}_i$.

- **Ward-Takahashi identities**:

Now let $\epsilon(x)$ have support only in a region B_ϵ that contains the insertion x_1 of operator $\mathcal{O}_1$, but none of the other operators.



The correlator $\langle\mathcal{O}_1\ldots\mathcal{O}_n\rangle$ transforms as

$$\langle\mathcal{O}_1\ldots\mathcal{O}_n\rangle \to \frac{1}{Z}\int \mathcal{D}\phi\, e^{-S[\phi]}\left(1 + \frac{1}{2\pi}\int_{B_\epsilon}\partial_\alpha J^\alpha \epsilon(x)\right)\left(\mathcal{O}_1(x_1) + \epsilon(x_1)\delta\mathcal{O}_1\right)\mathcal{O}_2\ldots\mathcal{O}_n. \tag{4.52}$$

Let us restrict to ϵ constant inside B_ϵ. For this we deduce

$$-\frac{1}{2\pi}\langle\int_{B_\epsilon}\partial_\alpha J^\alpha(x)\,\mathcal{O}_1(x_1)\ldots\rangle = \langle\delta\mathcal{O}_1(x_1)\ldots\rangle, \tag{4.53}$$

i.e.

$$\boxed{-\frac{1}{2\pi}\int_{B_\epsilon}\partial_\alpha J^\alpha(x)\,\mathcal{O}_1(x_1) = \delta\mathcal{O}_1(x_1)\ldots \qquad \text{as an operator equation.}} \tag{4.54}$$

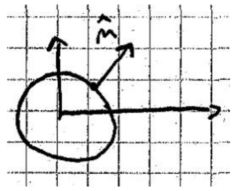
This is the **Ward-Takahashi identity**, which gives a tool to compute the transformation of a local operator by an integral over a certain operator product.

- We finally specialise to a 2d QFT, for which the Ward identities can be rewritten in a particularly neat manner. By Stoke's theorem we can evaluate $\int_{B_\epsilon} \partial_\alpha J^\alpha(x)$ as a line integral over the boundary of B_ϵ,

$$\int_{B_\epsilon} \partial_\alpha J^\alpha(x) = \oint_{\partial B_\epsilon} J_\alpha \hat{n}^\alpha. \tag{4.55}$$

The tangential and normal line element in two dimensions take the form

$$\hat{t}^\alpha = \begin{pmatrix} dx^1 \\ dx^2 \end{pmatrix}, \qquad \hat{n}^\alpha = \begin{pmatrix} dx^2 \\ -dx^1 \end{pmatrix}. \tag{4.56}$$



Therefore we can write

$$\int_{B_\epsilon} \partial_\alpha J^\alpha(x) = \oint_{\partial B_\epsilon} (J_1\, dx^2 - J_2\, dx^1). \tag{4.57}$$

Let us go to complex coordinates: $z = x^1 + i\,x^2$, $\bar{z} = x^1 - i\,x^2$, in which

$$J^z = J^1 + i\,J^2\,, \quad J^{\bar{z}} = J^1 - i\,J^2 \tag{4.58}$$
$$J_{\bar{z}} = g_{\bar{z}z} J^z = \tfrac{1}{2}(J_1 + i\,J_2), \quad J_z = \tfrac{1}{2}(J_1 - i\,J_2) \tag{4.59}$$

and therefore

$$\int_{B_\epsilon} \partial_\alpha J^\alpha(x) = -i \oint_{\partial B_\epsilon} (dz\, J_z - d\bar{z}\, J_{\bar{z}}), \qquad J_z = J_z(z,\bar{z}), J_{\bar{z}} = J_{\bar{z}}(z,\bar{z}). \tag{4.60}$$

Altogether the Ward-Takahashi identities for a 2-dimensional QFT take the form

$$\boxed{\delta\mathcal{O}(w,\bar{w}) = -\frac{1}{2\pi\, i} \oint_{\partial B_\epsilon} (dz\, J_z(z,\bar{z}) - d\bar{z}\, J_{\bar{z}}(z,\bar{z}))\mathcal{O}(w,\bar{w}),} \tag{4.61}$$

where it is important that w lies within the region B_ϵ, i.e. is encircled by the contour integral.

4.5.2 Conformal Ward-Takahashi identities

Let us now apply the Ward-Takahashi identities to the conformal symmetry of a 2-dimensional CFT.

- Recall that in the context of the free boson on the cylinder we have established the general result that Noether currents of conformal transformations are related to the energy-momentum tensor. In lightcone gauge on the cylinder its non-vanishing components are

$$T_{++} = T_{++}(\xi^+)\,, \qquad T_{--} = T_{--}(\xi^+)\,, \qquad T_{-+} = 0. \tag{4.62}$$

- This easily translates into complex coordinates as

$$T_{zz}(z) = T(z), \qquad T_{\bar{z}\bar{z}}(\bar{z}) = \bar{T}(\bar{z}) \tag{4.63}$$

and the Noether current for a conformal transformation $z \to z + \epsilon(z)$ and $\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$ is

$$J_z(z) = \epsilon(z)T(z), \qquad J_{\bar{z}}(\bar{z}) = \bar{\epsilon}(\bar{z})\bar{T}(\bar{z}). \tag{4.64}$$

- Application of eq. (4.61) yields the

Master Formula (**Conformal Ward-Takahashi identity**):

$$\delta_{\epsilon,\bar{\epsilon}}\mathcal{O}(w,\bar{w}) = -\frac{1}{2\pi i}\oint_{\mathcal{C}_w} dz(\epsilon(z)T(z) + \bar{\epsilon}(\bar{z})\bar{T}(\bar{z}))\mathcal{O}(w,\bar{w}), \tag{4.65}$$

with the contour integral counter-clockwise both in z and in $\bar{z}$ (thereby explaining the sign difference of the second term compared to (4.61)).

Note: $T(z)\mathcal{O}(w,\bar{w})$ and $\bar{T}(\bar{z})\mathcal{O}(w,\bar{w})$ are **radially ordered**, i.e. $|z| > |w|$, $|\bar{z}| > |\bar{w}|$. This corresponds to time-ordering inside the path integral $\int \mathcal{D}\phi\, e^{-S[\phi]}$.

- **Important remark on the proof:**
In the derivation of (4.61) we explicitly assumed that the global symmetry under consideration is non-anomalous, see eq. (4.45). At first sight, this seems to clash with the existence of a conformal anomaly in theories with $c \neq 0$. What saves the day is that in order to derive (4.65) it is sufficient to write down the Ward identifies for the subset of conformal transformations given by global translations, by rotations and by scale transformations, from which (4.65) can indeed be derived.[3] Classically, these are given by the $PSL(2,\mathbb{C})$ transformations l_0, l_{-1}, l_1. This group of globally defined conformal transformations is indeed non-anomalous because the corresponding Virasoro operators satisfy $[L_m, L_n] = (m-n)L_{m+n}$ for $m, n \in \{-1, 0, 1\}$.

- **Important conclusion**:
The information about conformal transformations is encoded in the residua of the OPE with the energy-momentum tensor $T(z)$, $\bar{T}(\bar{z})$!

Let us evaluate this for primary fields of dimension $(h, \bar{h})$, thereby deriving their OPE with the energy-momentum tensor.

- From (4.35) we recall the form of the infinitesimal conformal transformation of a primary field,

$$\delta_{\epsilon,\bar{\epsilon}}\phi(w,\bar{w}) = -(h\partial_w\epsilon(w)\phi(w,\bar{w}) + \epsilon(w)\partial_w\phi(w,\bar{w})) - (\bar{h}\partial_{\bar{w}}\bar{\epsilon}(\bar{w})\phi(w,\bar{w}) + \bar{\epsilon}(\bar{w})\partial_{\bar{w}}\phi(w,\bar{w})). \tag{4.66}$$

This stands on the lefthand side of eq. (4.65).

[3]For a proof see Di Francesco et al., Conformal Field Theory, p. 121.

- To compare it to the integral on the righthand side we use the residual formulae (following from $\epsilon(z) = \epsilon(w) + (z-w)\partial_z \epsilon(z)\big|_{z=w} + \ldots$)

$$\epsilon(w)\partial_w \phi(w,\bar{w}) = \oint_{\mathcal{C}_w} dz \frac{1}{2\pi i} \frac{\epsilon(z)}{z-w} \partial_w \phi(w,\bar{w}), \tag{4.67}$$

$$(\partial_w \epsilon(w))\phi(w,\bar{w}) = \oint_{\mathcal{C}_w} dz \frac{1}{2\pi i} \frac{\epsilon(z)}{(z-w)^2} \phi(w,\bar{w}). \tag{4.68}$$

- Comparison with the Ward-Takahashi identities therefore gives the OPE

$$\boxed{\begin{aligned} T(z)\phi(w,\bar{w}) =& \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w \phi(w,\bar{w}) + \text{terms regular at } z=w \\ \bar{T}(\bar{z})\phi(w,\bar{w}) =& \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}} \phi(w,\bar{w}) + \ldots \end{aligned}} \tag{4.69}$$

  Again the lefthand side is radially ordered.

4.5.3 $T(z)$ as a conformal field

Let us first analyse the nature of the energy-momentum tensor as a conformal field. We will argue that $T(z)$ is a quasi-primary field of conformal dimension $h = 2 = \bar{h}$.

- To see that $T(z)$ has scaling dimension $h = 2$ we note that the Hamiltonian is related to the energy-momentum tensor via $H \sim \int d\sigma\, T$. Thus T has mass dimension 2, $[T] = [E^2]$.
- Even we do not know yet if $T(z)$ is a primary field - in fact we will see that it is not - we define its modes as in (4.37) via $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n$, i.e.

$$L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1} T(z). \tag{4.70}$$

- The OPE $T(z)T(w)$ can be computed from the known expression for the commutator $[L_m, L_n]$ as

$$\boxed{T(z)T(w) = \frac{\frac{c}{2}}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \ldots.} \tag{4.71}$$

  From this we conclude that if $c = 0$, then $T(z)$ is a primary field of dimension $h = 2$.

Comment:

The computation leading to (4.71) is important by itself: After plugging the contour integral (4.70) into the commutator one can deform the contour and read off the residue. This is discussed on Assignment 8. This procedure holds more generally:

The information of the OPE is equivalent to the information in the commutators of the modes.

Let us proceed with our investigation of $T(z)$.

- With the help of the same type of computations one can use the OPE for $T(z)\phi(w)$ for a primary $\phi(w)$ to derive the commutators

$$[L_m, \phi_n] = \Big((h-1)m - n\Big)\phi_{m+n}, \qquad [L_m, \phi(z)] = z^m\Big(z\partial_z + (m+1)h\Big)\phi(z). \tag{4.72}$$

If this holds only for $m = -1, 0, 1$, then ϕ is only quasi-primary.

- Therefore if $c \neq 0$, then $T(z)$ is a quasi-primary field. Indeed it was shown in the exercises that there is no central term in the commutators for L_m with $m = -1, 0, 1$, for which $[L_m, L_n] = (m-n)L_{m+n}$. Thus the global conformal group on S^2, $PSL(2,\mathbb{C})$, is always non-anomalous.

Transformation of $T(z)$ under conformal transformations

- With the help of the conformal Ward identities we can now use the OPE of the energy-momentum with itself to compute its transformations under conformal transformations:

$$\begin{aligned}\delta_\epsilon T(w) &= -\mathrm{Res}[\epsilon(z)T(z)T(w)] \\ &= -\mathrm{Res}\left[\epsilon(z)\left(\frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots\right)\right].\end{aligned} \tag{4.73}$$

Expanding $\epsilon(z) = \epsilon(w) + \epsilon'(w)(z-w) + \frac{1}{2}(z-w)^2\epsilon'' + \frac{1}{6}(z-w)^3\epsilon''' + \dots$ we find

$$\delta_\epsilon T(w) = -\epsilon(w)\partial T(w) - 2\epsilon'(w)T(w) - \frac{c}{12}\epsilon'''(w). \tag{4.74}$$

One can verify by a straightforward computation that this is the infinitesimal version of the following transformation under finite $z \mapsto \tilde{z}(z)$:

$$T(z) \mapsto \tilde{T}(\tilde{z}) = \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2}[T(z) - \frac{c}{12}S(\tilde{z}, z)] \tag{4.75}$$

in terms of the Schwarzian derivative

$$S(\tilde{z}, z) = \frac{\partial^3 \tilde{z}}{\partial z^3}\left(\frac{\partial \tilde{z}}{\partial z}\right)^{-1} - \frac{3}{2}\left(\frac{\partial^2 \tilde{z}}{\partial z^2}\right)^2\left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2}. \tag{4.76}$$

With a little more work one can show that indeed the infinitesimal transformation (4.73) uniquely fix $S(\tilde{z}, z)$ to be of this form. For a proof see e.g. [BLT], Chapter 2.1.

- In the exercises we evaluate the Schwarzian derivative for the conformal map from the cylinder to the complex plane, given by $z = e^{\frac{2\pi}{\ell}w} = e^{\frac{2\pi}{\ell}(\tau - i\sigma)}$. This allows us to relate the energy momentum tensor on the cyclinder and on the complex plane as

$$T_{\text{cylinder}}(w) = \left(\frac{2\pi}{\ell}\right)^2\left(z^2 T_{\text{plane}}(z) - \frac{c}{24}\right). \tag{4.77}$$

Relation of c to vacuum energy

The above transformation rule provides a very simple and efficient derivation of the Casimir energy on the cylinder in terms of the central extension c of the Virasoro algebra. The starting point is the intuitive assertion that on the complex plane the Casimir energy vanishes. Thus, in view of the relation between the Hamiltonian and the energy-momentum tensor this implies that the one-point function on the plane is zero, $\langle T_{\text{plane}}\rangle = 0$. Therefore

$$\langle T_{\text{cyl.}}\rangle(w) = -\frac{c}{24}\left(\frac{2\pi}{\ell}\right)^2. \tag{4.78}$$

This beautifully matches with our earlier computation of the vacuum energy if we remember that $w = i\xi^-, \bar{w} = i\xi^+$ in terms of the *Minkowski signature* lightcone coordinates. This gives an extra factor of -1 in relating the one-point function of $T_{\text{cyl.}}(w)$ to the physical value of the vacuum energy on the Minkowski signature cylinder. Altogether one finds with $H = \frac{2\pi}{\ell}(L_0 + \tilde{L}_0)$ and $L_0 = -\frac{\ell}{4\pi^2}\int_0^\ell d\sigma T_{--}(\xi^-)$, $\tilde{L}_0 = -\frac{\ell}{4\pi^2}\int_0^\ell d\sigma T_{++}(\xi^+)$ that the vacuum energy associated with a single string field in Minkowski signature is

$$\langle H_{\text{cyl.}}\rangle = (-1)\left(\frac{2\pi}{\ell}\right)\Big(-\frac{\ell}{4\pi^2}\Big)\int_0^\ell d\sigma(\langle T_{\text{cyl.}}(w)\rangle + \langle\bar{T}_{\text{cyl.}}(\bar{w})\rangle) = -\frac{2\pi}{\ell}\frac{c}{12}. \tag{4.79}$$

Taking into account that a single string field has a conformal anomaly $c = 1$, this agrees with the result $\langle H\rangle = \frac{2\pi}{\ell}(-a - \tilde{a})$ with $a = \frac{1}{24}$ for an integer-moded boson.

Relation of c to conformal anomaly

We have already pointed out several times that $c \neq 0$ signals an anomaly of the conformal symmetry of a CFT. The precise statement is that for a 2-dimensional CFT the conformal anomaly is given by the vacuum expectation value of the trace of the energy-momentum tensor. In lightcone gauge this corresponds to a non-zero expectation value of T_{+-} or, in complex coordinates, of $T_{z\bar{z}}$. In (τ, σ) coordinates the statement is that

$$\langle T^a{}_a\rangle = \frac{c}{12}R^{(2)}. \tag{4.80}$$

Here $R^{(2)}$ the Ricci-scalar of the two-dimensional spacetime on which the CFT is defined. Since classical conformal invariance implies $T^a{}_a = 0$ this indicates a Weyl/conformal anomaly.

While for a precise proof of this important theorem we refer to [P], Chapter 3.4., we can intuitively understand that $\langle T^a{}_a\rangle$ must be proportional to c and to $R^{(2)}$: The proportionality to c follows from our above result that $c \neq 0$ is equivalent to the presence of a vacuum-energy, which breaks scale invariance. Furthermore, non-vanishing curvature implies the notion of a typical length scale in the theory, which again breaks scale invariance.

Note:
By methods similar to those leading to the formal proof of (4.80) one can show that if $c \neq \tilde{c}$ for the left and right-moving Virasoro algebras, the theory suffers from a gravitational anomaly, i.e. an anomaly of diffeomorphism invariance in σ because then $\langle L_0 - \tilde{L}_0\rangle \neq 0$ to shift. This is unacceptable and must be excluded, leading to the level matching condition postulated earlier.

4.6 State-operator correspondence, highest weight states, Verma modules

Let us now investigate the structure of the Hilbert space of a 2-dimensional CFT on S^2. To this end we first need to define the properties of the vacuum. We distinguish between the in-vacuum

$|0\rangle$ corresponding to the vacuum at $\tau = -\infty$ and the out-vacuum $\langle 0|$, the vacuum at $\tau = +\infty$. We now make use of the crucial property of the conformal mapping from the cylinder to the complex plane that $\tau = -\infty$ is mapped to the point $z = 0$ and $\tau = +\infty$ to the point $z = \infty$.

- We postulate that the in-vacuum should have the property that the action of the energy-momentum tensor on it should be well-defined. Regularity of $T(z) = \sum_n z^{-n-2} L_n$ at $\tau = -\infty$, i.e. at $z = 0$, requires
$$L_n |0\rangle = 0 \qquad \forall\, n \geq -1. \tag{4.81}$$
Likewise, regularity of the vacuum at $\tau = +\infty$ implies
$$\langle 0| L_n = 0 \qquad \forall n \leq 1. \tag{4.82}$$
Analogous relations holds for $\bar{L}$. Therefore the only generators that annhilate both $|0\rangle$ and $\langle 0|$ are $\{L_{-1},\, L_0,\, L_1\}$ (and similarly for their anti-meromorphic cousins). This is summarised in the important statement:

$$\boxed{\text{The vacuum of a 2d CFT is invariant only under } PSL(2,\mathbb{C}) \times PSL(2,\mathbb{C}).}$$

- To each primary field $\Phi(z, \bar{z})$ we can associate a state as follows[4]: We again postulate that the action of
$$\Phi(z) = \sum_n \phi_n z^{-n-h} \tag{4.83}$$
on the vacuum must be regular at $\tau = -\infty$ and at $\tau = +\infty$. This requires
$$\phi_n |0\rangle = 0 \qquad \forall n \geq 1 - h, \tag{4.84}$$
$$\langle 0| \phi_n = 0 \qquad \forall n \leq h - 1 \quad (\phi_n^\dagger = \phi_{-n}). \tag{4.85}$$
We now define the in-state and out-state
$$|\phi_{\text{in}}\rangle = \phi_{-h} |0\rangle = \lim_{z\to 0} \phi(z) |0\rangle = \phi(0) |0\rangle, \tag{4.86}$$
$$\langle \phi_{\text{out}}| = \langle 0| \phi_h. \tag{4.87}$$
This gives the **operator-state correspondence**.

- From the relation $[L_m, \phi_n] = (m(h-1) - n)\phi_{m+n}$ for primary fields of dimension h we deduce the following action of the Virasoro generators on the primary state $|\phi\rangle = \phi(0) |0\rangle$:
  i) $|\phi\rangle$ is eigenstate of L_0 with eigenvalue h,
  $$L_0 |\phi\rangle = h |\phi\rangle, \qquad L_n |\phi\rangle = 0 \quad \forall n > 0. \tag{4.88}$$
  ii) $L_{-n}, n > 0$ increases the eigenvalue of L_0,
  $$L_0(L_{-n} |\phi\rangle) = (n + h)(L_{-n} |\phi\rangle) \qquad \forall n \geq 0. \tag{4.89}$$
  Therefore we can identify
  $$\begin{array}{ll} L_n & \text{as anihilators,} \\ L_{-n} & \text{as raising operators w.r.t. eigenstates of } L_0. \end{array} \tag{4.90}$$
  States with the properties (4.88) and (4.89) are called **highest weight states**.

[4] We restrict to chiral fields to avoid writing everything in terms of the meromorphic and the anti-meromorphic sectors. Generalisations are obvious.

This establishes the important isomorphism in a 2-dimensional CFT between

$$\boxed{\text{primary fields} \leftrightarrow \text{highest weight states.}}$$

- The complete Hilbert space is obtained by acting with L_{-n}, $n > 0$ on all highest weight states $|\phi_j\rangle$ where j labels all primary fields.

  Definition 4.1. *The Verma module V_{h_j} is the span of all states of the form*

  $$\left|\phi_j^{k_1 \cdots k_m}\right\rangle = L_{-k_1} \ldots L_{-k_m} |\phi_j\rangle , \quad k_i > 0 \tag{4.91}$$

  of conformal weight $h = h_j + \sum_i^m k_i$.

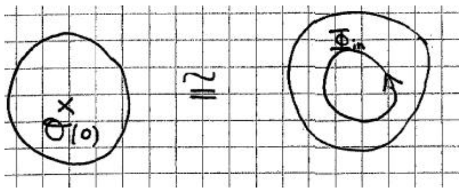
  One can show that the states $\left|\phi_j^{k_1 \cdots k_m}\right\rangle$ with $k_1 \geq k_2 \geq \ldots k_m$ are linearly independent.

  By the operator-state correspondence one can in turn define a conformal field associated with a general state in the Verma module V_{h_j}. These fields are not primary themselves.

  Definition 4.2. *The state $\left|\phi_j^{k_1 \cdots k_m}\right\rangle$ is created by a secondary = descendent field $\phi_j^{k_1 \cdots k_m}(z)$ from the $PSL(2, \mathbb{Z})$ invariant vacuum.*

Important lesson:

- In a general QFT, states and **local** fields are **not** equivalent. While a field is by definition local, a state carries non-local information about a full field configuration. E.g. in the Schrödinger representation the states $\Psi(\phi(x), t)$ correspond to *functionals* of the field configuration.
- In a CFT, on the other hand, we can map the entire spatial slice $\tau = -\infty$ to the **point** $z = 0$ In the path integral information about the state corresponding to a field configuration at $\tau = -\infty$ is thus encoded in the insertion of an arbitrary *local* operator $\mathcal{O}(z)$ at $z = 0$. This is the reason why the operator-state mapping is intimately related to conformal symmetry.



For more information about the Schrödinger representation and a path integral formulation of the state-operator mapping we refer to [P], Chapter 2.8 and 2.9 and especially to [T], Chapter 4.6.

We have seen that the Hilbert space of a 2-dimensional CFT is specified by the spectrum of primary fields ϕ_i and their conformal dimensions $(h_i, \bar{h}_i)$.
If the theory is unitary, the conformal anomaly c and the spectrum is subject to the following constraints:

- $c \geq 0$,
- $h_j \geq 0$,
- $h_\phi = 0 \leftrightarrow \phi = \mathbb{1}$. The only state with associated conformal dimension $h = 0$ is the $PSL(2, \mathbb{C})$ invariant vacuum.

A proof of these assertions will be worked out on Assignment 9.

4.7 Correlation functions in CFT

In a 2-dimensional CFT the correlation functions are extremely constrained by the conformal symmetry. While a thorough derivation of the structure of the correlation functions in a general CFT is beyond the scope of this course, for completeness we here collect the main statements, leaving a more detailed discussion and some of the proofs to Assignment 9.
Let us first restrict ourselves to a CFT on a sphere S^2. For brevity we only consider holomorphic fields; generalizations to more general fields are obvious. First, invariance of the vacuum under globally defined conformal transformations implies the following behaviour of a general n-point function under $PSL(2,\mathbb{C}) \times \overline{PSL}(2,\mathbb{C})$ transformations,

$$\langle \phi_1(z_1) \dots \phi_n(z_n) \rangle = \langle \phi_1'(z_1') \dots \phi_n'(z_n') \rangle |_{z' \to z}. \tag{4.92}$$

This completely fixes the spacetime-dependence of **one-, two- or three-point functions for quasi-primary fields** and constraints the higher correlators as follows:

i) The one-point function must vanish,

$$\langle \phi(z) \rangle = 0, \tag{4.93}$$

unless $h_\phi = 0$, i.e. $\phi = \mathbb{1}$.

ii) The two-point function of quasi-primaries is non-zero only if the conformal dimensions agree,

$$\langle \phi_i(z_i) \phi_j(z_j) \rangle = \delta_{h_i, h_j} \frac{d_{ij}}{(z_i - z_j)^{2h_i}}. \tag{4.94}$$

If d_{ij} is non-degenerate, the fields can be normalised such that $d_{ij} = \delta_{ij}$.

iii) The three-point function is completely fixed up to the appearance of a structure constant C_{ijk},

$$\langle \phi_1(z_1) \phi_2(z_2) \phi_3(z_3) \rangle = \frac{C_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_1+h_3-h_2}}. \tag{4.95}$$

These structure constants are related to the coefficients ${C_{ij}}^k$ in the OPE of two quasi-primaries given in eq. (4.40) as

$$C_{ijk} = {C_{ij}}^l d_{lk}. \tag{4.96}$$

iv) All higher n-point functions are functions of $(n-3)$ $PSL(2,\mathbb{C})$ invariant **cross-ratios**. This is because we can use $PSL(2,\mathbb{C})$ transformations, which map any three points on the sphere to any other three points, to eliminate the spacetime dependence of the correlator to $(n-3)$ combinations of coordinates, which must in addition be invariant. The cross-ratios are discussed on Assignment 8.

In fact, with the help of the OPE and exploiting the associative structure of the correlation functions one can reduce any higher $n-$point correlator to data involving the 3-point correlators and so-called **conformal blocks**, which depend only on the conformal anomaly c and the conformal dimensions $(h_i \bar{h}_i)$ of the quasi-primaries involved (see e.g. [BP], Chapter 2.12 and 2.13 for more information and references). In a 2-dimensional CFT these conformal blocks can be evaluated very explicitly, even though this may be hard in practice. This explicit evaluation makes use of the infinite-dimensional Virasoro algebra. Combined with the results from section (4.6) on the structure of the Hilbert space of a 2-dimensional CFT this is summarised in the remarkable theorem:

> A 2-dimensional CFT is completely specified by its conformal anomaly c, the spectrum of primary fields $\phi_i(z,\bar{z})$ of dimensions $\{h_j, \bar{h}_j\}$ and their OPE coefficients $C_{ij}{}^k$.

In particular, the definition of a 2-dimensional CFT need not involve a Lagrangian. While many examples of 2-dimensional CFTs even without an action have been found, a complete classification of all 2-dimensional CFTs is, to date, not known.

Comment on higher-dimensional CFTs
The above statements about the one, two-, and three-point correlators made use only of the global transformations $PSL(2,\mathbb{C})$. As discussed, these do have an analogue also in higher dimensional CFTs. Therefore, the statements about the one, two-, and three-point correlators involving quasi-primaries continue to hold in higher-dimensional CFTs. The higher correlators also depend only of the cross-ratios and the conformal blocks, which in principle are determined by the spectrum of quasi-primaries and the conformal anomaly. Due to the lack of the Virasoro structure, though, the explicit evaluation of these conformal blocks is much harder and in general these are not known, despite much progress in the recent literature.[5] This is what makes two-dimensional CFTs special.

4.8 Normal ordering and Wick's theorem

For a primary field ϕ the modes $\phi_n, \quad n > -h$ play the role of annihilation operators (and similarly for the anti-meromorphic pieces) in the sense that $\phi_n |0\rangle = 0$ if $n > -h$. Correspondingly the modes $\phi_n, \quad n \leq -h$ are creation operators.

In QFT normal ordering is usually defined as moving all creation operators to the left.
Indeed one can rigorously prove that this notion of normal ordering is equivalent to picking out the non-singular term in the **radially ordered OPE**, i.e.

$$\boxed{\phi(z)\chi(w) = \{ \text{ singular piece } \} + :\phi(z)\chi(w):} \tag{4.97}$$

The proof can be found e.g. in [BP], Chapter 2.7. and will not be presented in this course.

Clearly $\langle :\phi(z)\chi(w): \rangle = 0$. This leads to **Wick's theorem** for two fields:

$$\boxed{\underbrace{\phi(z)\chi(w)}_{\text{time / radially ordered}} = \langle\phi(z)\chi(w)\rangle + \underbrace{:\phi(z)\chi(w):}_{\text{normal ordered}}} \tag{4.98}$$

As in a general QFT one can inductively use this to relate time-ordered and normal-ordered products of more than two fields by replacing any pair of them by their two-point correlator, e.g.

[5] See e.g. Pappadpulo et al., http://arxiv.org/abs/1208.6449, and references therein.

$$\begin{aligned}
:\phi_1(z_1)\phi_2(z_2)\phi_3(z_3): \quad &= \quad \phi_1(z_1)\phi_2(z_2)\phi_3(z_3) \\
&\quad -\phi_1(z_1)\langle\phi_2(z_2)\phi_3(z_3)\rangle - \phi_2(z_2)\langle\phi_1(z_1)\phi_3(z_3)\rangle \\
&\quad -\phi_3(z_3)\langle\phi_1(z_1)\phi_2(z_2)\rangle \\
&\equiv \quad \phi_1(z_1)\phi_2(z_2)\phi_3(z_3) - \sum \text{subtractions},
\end{aligned} \tag{4.99}$$

where the subtractions are obtained by successively taking all pairs of fields and replacing their product by its expectation value.

4.9 Applications to String Theory

We now apply these abstract ideas to the bosonic string, shedding new light on the ontology of the fields we have already got to know in less enlightened language. The theory of the bosonic string freely propagating in $\mathbb{R}^{1,d-1}$ consists of two CFTs on the worldsheet - the collection of d free bosons as well as the ghost system. Let us revisit both in turn.

4.9.1 The free boson on the sphere

The action of a single free boson on the sphere is given by

$$S = \frac{1}{2\pi\alpha'}\int dz\, d\bar{z}\, \partial X(z,\bar{z})\, \bar{\partial} X(z,\bar{z}). \tag{4.100}$$

- Let us first check that the classical equation of motion $\partial\bar{\partial}X(z,\bar{z}) = 0$ holds at the quantum level as an operator equation inside the path integral. Indeed, since $\int \mathcal{D}X \frac{\delta}{\delta X}...$ is the integral over a total derivative, it vanishes and we conclude

$$0 = \frac{1}{Z}\int \mathcal{D}X \frac{\delta}{\delta X} e^{-S} = -\frac{1}{Z}\int \mathcal{D}X e^{-S}\frac{\delta S}{\delta X} = \frac{1}{\pi\alpha'}\left\langle \partial\bar{\partial}X(z,\bar{z})\right\rangle. \tag{4.101}$$

- From the decomposition $X(z,\bar{z}) \equiv X(z) + \bar{X}(\bar{z})$ we find the conserved **chiral and anti-chiral worldsheet currents** $j(z) := i\partial X(z)$ and $\bar{j}(\bar{z}) := i\bar{\partial}X(\bar{z})$. From a CFT perspectives these are well-behaved fields.

- In fact, $X(z,\bar{z})$ **is not a (quasi-) primary field**. This is evident already from the form of its two-point function. To compute this we first show that

$$\partial_z\partial_{\bar{z}}\langle X(z,\bar{z})X(w,\bar{w})\rangle = -\pi\alpha'\delta^{(2)}(z-w). \tag{4.102}$$

  This is because by the same trick as above we find

$$\begin{aligned}
0 \quad &= \quad \int \mathcal{D}X \frac{\delta}{\delta X(z,\bar{z})}\left(e^{-S}X(z',\bar{z}')\right) \\
&= \quad \int \mathcal{D}X e^{-S}\left(\delta^{(2)}(z-z',\bar{z}-\bar{z}') + \frac{1}{\pi\alpha'}\partial_z\partial_{\bar{z}}X(z,\bar{z})X(z',\bar{z}')\right) \\
\Rightarrow 0 \quad &= \quad \left\langle \delta^{(2)}(z-z',\bar{z}-\bar{z}')\right\rangle + \frac{1}{\pi\alpha'}\partial_z\partial_{\bar{z}}\langle X(z,\bar{z})X(z',\bar{z}')\rangle.
\end{aligned} \tag{4.103}$$

Using[6] $\partial_z \frac{1}{\bar z} = 2\pi\delta^{(2)}(z,\bar z) = \partial_{\bar z}\frac{1}{z}$ we can integrate (4.102) to

$$\boxed{\langle X(z,\bar z)X(w,\bar w)\rangle = -\frac{\alpha'}{2}\log(|z-w|^2).} \tag{4.104}$$

This is familiar result that the Green's function in two dimensions is logarithmic. Obviously this two-point correlator is not of the form (4.94) as would be required for a quasi-primary. In other words, $X(z,\bar z)$ does not satisfy the defining property (4.34) for a primary field.

- Correspondingly, the chiral correlators take the form

$$\langle X(z)X(w)\rangle = -\frac{\alpha'}{2}\log(z-w), \qquad \langle \bar X(\bar z)\bar X(\bar w)\rangle = -\frac{\alpha'}{2}\log(\bar z-\bar w). \tag{4.105}$$

- In the quantum theory the energy-momentum tensor is defined as the normal ordered expression

$$\boxed{T(z) := -\frac{1}{\alpha'} :\partial X(z)\partial X(z):} \quad \Longrightarrow \langle T(z)\rangle = 0. \tag{4.106}$$

- The currents $\partial X(z)$, $\bar\partial X$ **are** primary fields of conformal dimension $(1,0)$ and $(0,1)$ respectively.

  i) As a first check, the propagator is indeed of the correct form

$$\langle \partial X(z)\partial X(w)\rangle = -\frac{\alpha'}{2}(z-w)^{-2}, \tag{4.107}$$

  as follows by differentiating $\langle X(z)X(w)\rangle$.

  ii) To prove the primary condition we compute the OPE $T(z)\partial X(w)$ by Wick's theorem,

$$\begin{aligned} T(z)\partial X(w) &= -\frac{1}{\alpha'} :\partial X(z)\partial X(z): \partial X(w) \\ &= -\frac{1}{\alpha'}\left[:\partial X(z)\partial X(z)\partial X(w): + 2\partial X(z)\langle \partial X(z)\partial X(w)\rangle\right]. \end{aligned} \tag{4.108}$$

  Here we used (4.99). Now we insert the (4.107) and use the Taylor expansion

$$\partial X(z)(z-w)^{-2} = \left[\partial X(w) + (z-w)\partial\partial X(w) + \ldots\right](z-w)^{-2} \tag{4.109}$$

  to establish

$$\boxed{T(z)\partial X(w) = \frac{\partial X(w)}{(z-w)^2} + \frac{\partial\partial X(w)}{(z-w)} + \{\text{ non-sing. }\}.} \tag{4.110}$$

  This identifies ∂X as a chiral primary of dimension $h=1$. Correspondingly the mode expansion for the currents takes the form

$$\begin{aligned} i\partial X(z) &= \sum_{n\in\mathbb{Z}} \sqrt{\frac{\alpha'}{2}}\alpha_n z^{-n-1}, \\ i\bar\partial \bar X(\bar z) &= \sum_{n\in\mathbb{Z}} \sqrt{\frac{\alpha'}{2}}\tilde\alpha_n \bar z^{-n-1}. \end{aligned} \tag{4.111, 4.112}$$

[6] To see this we note that $\partial_z \frac{1}{\bar z} = 0$ away from the origin, while at the same time our earlier result (4.60) implies $\int d^2z\,\partial_z\frac{1}{\bar z} = i\oint d\bar z\frac{1}{\bar z} = 2\pi$.

Note that $[L_m, \alpha_n] = -n\alpha_{m+n}$, which had been derived previously, is the commutator version of the statement that $h = 1$.

- The same methods of exploiting Wick's theorem yield the OPE of the energy-momentum tensor with itself, as spelled out on Assignment 9,

$$T(z)T(w) = \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \dots \tag{4.113}$$

This confirms our earlier finding that $c = 1$ for one integer-moded real boson.

- Another important primary field in the X-CFT is the exponential

$$\mathcal{F}_k =: e^{ikX(z,\bar{z})} : = : e^{ikX(z)} e^{ik\bar{X}(\bar{z})} : . \tag{4.114}$$

By Wick's theorem one computes the following OPE by expanding the exponential, as performed again in Assignment 9,

$$\partial X(z) : e^{ikX(w)} : \quad = \quad -i\,\frac{\alpha'}{2}\,k\ : e^{ikX(w)} : \frac{1}{(z-w)} + ..., \tag{4.115}$$

$$T(z) : e^{ikX(w)} : \quad = \quad \left(\frac{\alpha' k^2/4}{(z-w)^2} + \frac{\partial_w}{(z-w)} \right) : e^{ikX(w)} : + ..., \tag{4.116}$$

$$: e^{i\alpha X(z)} :: e^{i\beta X(w)} : \quad = \quad (z-w)^{\alpha'(\alpha\beta)} : e^{i(\alpha X(z) + \beta X(w))} : (1 + \mathcal{O}((z-w))) . \tag{4.117}$$

The OPE with $T(z)$ proves that $: e^{ikX(z,\bar{z})} :$ is primary with $h = \frac{\alpha'}{4} k^2 = \bar{h}$.

4.9.2 The *bc*-ghost-system

The action of the *bc*-ghost-system in complex coordinates takes the form

$$S_{\text{ghost}} = \frac{1}{2\pi} \int d^2 z \left[b_{zz} \bar{\partial}_{\bar{z}} c^z + b_{\bar{z}\bar{z}} \partial_z c^{\bar{z}} \right] . \tag{4.118}$$

The ghost field $c^z(z)$ is a worldsheet vector, i.e. a spin 1 current. Its conformal dimension is $h = -1$ as this corresponds to the mass dimension of a vector.
Correspondingly the anti-ghost $b_{zz}(z)$ is two-tensor current (spin 2) with $h = 2$.
More generally, one considers the so-called (*bc*)-CFT with action

$$S_{bc} = \frac{1}{2\pi} \int d^2 z \left[b(z) \bar{\partial}_{\bar{z}} c(z) + \text{anti-chiral} \right] . \tag{4.119}$$

It is easy to see that this defines a CFT if the fields are of the following nature: $b(z)$ is primary of weight $h_b = \lambda$, $c(z)$ is primary of weight $h_c = 1 - \lambda$, for some $\lambda \in \mathbb{R}$. By the usual methods one finds the energy-momentum tensor

$$T(z) =: (\partial b)c : -\lambda \partial(: bc :). \tag{4.120}$$

The OPE of $T(z)$ with itself shows that the (bc)-central charge is

$$c = -3(2\lambda - 1)^2 + 1 \tag{4.121}$$

(and similarly for the anti-chiral piece). See Assignemnt 10 for more details. The string theoretic ghost-system is therefore the special case $\lambda = 2$, $c = -26$ of the bc-CFT.

4.9.3 String quantisation à la CFT

Having reviewed the two ingredients in the bosonic path integral in CFT language, let us take a fresh look at how the two pieces come about.

- The starting point of the bosonic string quantization is the action $S_{BDH}[X^\mu, h_{ab}]$ with X^μ taking values in $\mathbb{R}^{1,d-1}$. The action enjoys local Weyl and diffeomorphism invariance on the worldsheet.

- Gauge fixing à la Faddeev-Popov leads to a theory defined on $\mathbb{C}\cup\infty$ (for closed strings at tree level). The remnant of the original Weyl and diffeomorphism symmetry is the conformal symmetry on $\mathbb{C}\cup\infty$. The full gauge fixed theory is described by $S^{\mathrm{tot}} = S^{(X^i)} + S^{(bc)}$, where $S^{(X^i)}$ describes d copies of the free-boson-CFT, each with central charge $c^{(X^i)} = 1,\ i = 1,\ldots,d$, and $S^{(bc)}$ describes the $\lambda = 2$-(bc)-CFT with $c^{(bc)} = -26$.

- Self-consistency of the Faddeev-Popov procedure requires absence of the total conformal anomaly,
$$c^{\mathrm{tot}} = \sum_i c^{(X^i)} + c^{(bc)} \stackrel{!}{=} 0. \tag{4.122}$$
Thus the bosonic string in $d = 26$ dimensions is fully consistent as a quantum theory.

- We are now in a position to take a more abstract perspective: The requirement $c^{\mathrm{tot}} = 0$ can also be met by considering more general CFTs and replacing (some of) the 26 copies of the X-CFT therewith. The non-free CFTs are called "internal" sectors. E.g. one can take the tensor product of only 4 copies of the X-CFT (corresponding to propagation of the string in $\mathbb{R}^{1,3}$) and a tensor product of more complicated CFTs with $c = 22$. Later on we will discuss the idea of compactification by considering as target space not flat 26 dimensions, but a space-time of the form $\mathbb{R}^{1,3} \times \mathcal{M}$ with $\mathcal{M}$ some internal compact manifold (here of dimension 22). The "internal" CFT can then be thought of as describing the string propagation on this internal manifold $\mathcal{M}$. Celebrated examples of such CFTs include so-called Gepner models, which are known to describe the string propagation on certain Calabi-Yau manifolds appearing compactifications of the superstring.

  Even though this is not yet apparent from what we have learned, the secret of the worldsheet approach to string theory is the insight that space-time is only a manifestation of an abstract CFT on the worldsheet. What is important is not that space-time is well-defined in the sense of a smooth manifold, but that the CFT that describes the propagation of the string on that space is well-defined. This implies that strings can consistenly propagate even on certain singular spaces, e.g. orbifolds of the form $\mathcal{M}/G$ with G the action of a finite group, as long as the underlying CFT is non-singular. If you want to learn more about this fascinating way to think about the nature of space-time within string theory, you can try already at this stage to read to the introduction to Brian Greene, String Theory on Calabi-Yau manifolds, http://arXiv.org/pdf/hep-th/9702155, even though you might wish to wait until we have introduced the superstring later in this course.

- For the X-CFT the requirement of BRST-invariance gives the physical state condition
$$L_m\,|\phi\rangle = 0, \qquad \tilde{L}_m\,|\phi\rangle = 0, \qquad \forall m > 0 \tag{4.123}$$
$$(L_0 - 1)\,|\phi\rangle = 0, \qquad (\tilde{L}_0 - 1)\,|\phi\rangle = 0. \tag{4.124}$$
This establishes the central insight:

$$\boxed{\text{Physical string states are 1-1 to primary fields of weight } h = 1 = \bar{h}.}$$

It is crucial to appreciate that this restriction to primaries and to $h = 1 = \bar{h}$ does not follow from the CFT itself, but is information "prior to CFT", i.e. it arises as an **extra** consistency condition for the X-CFT to make sense as part of the the gauge fixed version of the original BDH-action. By contrast, in a general CFT the Hilbert space consists of all Verma modules over all primary fields, not just the primaries of $h = \bar{h} = 1$.

- This leads to the concept of a **vertex operator**.

Definition 4.3. *A vertex operator is a primary field of dimension* $(h, \bar{h}) = (1, 1)$.

Its insertion at $z = 0$ creates a physical state from the $\mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C})$-invariant vacuum.

We could also have approached the construction of the string spectrum entirely by constructing the various vertex operators. Let us demonstrate this in two examples:

i) The primary field $: e^{ik_\mu X^\mu(z,\bar{z})} :$ acting on the vacuum creates the state

$$|k\rangle = \lim_{z,\bar{z}\to 0} : e^{ik_\mu X^\mu(z,\bar{z})} : |0\rangle, \quad h = \bar{h} = \frac{\alpha'}{4}k^2. \tag{4.125}$$

The physical state condition is evaluated as follows:

- The field $: e^{ik_\mu X^\mu(z,\bar{z})} :$ is always primary as shown above.
- In order to satisfy $h = 1$ we need $k^2 = -m^2 = \frac{4}{\alpha'}$. This gives the mass shell condition for the lowest-lying state.

Indeed one can rigorously show that the so-defined state $|k\rangle$ is the momentum eigenstate with momentum k,

$$\begin{aligned} \alpha_0^\mu\,|k\rangle &= \sqrt{\frac{2}{\alpha'}}\,i \oint \frac{dz}{2\pi i} \partial X^\mu(z) : e^{ik_\nu X^\nu(0,0)} : |0\rangle \\ &\overset{\text{OPE}}{=} \sqrt{\frac{2}{\alpha'}}\,i \oint \frac{dz}{2\pi i} \left(-\frac{i}{2}\alpha' k^\mu \frac{1}{z}\right) : e^{ik_\nu X^\nu(0,0)} : |0\rangle \\ &= \sqrt{\frac{\alpha'}{2}}\,k^\mu\,|k\rangle. \end{aligned} \tag{4.126}$$

Here we made use of the OPE (4.115).

ii) The fields at the first excited level are created by the following vertex operator:

$$|k,\xi\rangle = \lim_{z,\bar{z}\to 0} \underbrace{\xi_{\mu\nu} : \partial X^\mu(z)\bar{\partial}\bar{X}^\nu(\bar{z}) e^{ik\cdot X(z,\bar{z})} :}_{=:V_1(k,\xi;z,\bar{z})} |0\rangle. \tag{4.127}$$

The physical state conditions are as follows:

- The dimension of V_1 is $h_{V_1} = 1 + \frac{\alpha'}{4}k^2 = \bar{h}_{V_1}$ To achieve $h_{V_1} = 1$ we must set $k^2 \overset{!}{=} 0$.

- Higher constraints arise from demanding that $V_1(k,\xi;z,\bar{z})$ be a primary field. Again with the help of Wick's theorem we compute the OPE

$$\begin{aligned} T(z)V_1(k,\xi;w,\bar{w}) &= \text{const} \times \frac{k^\mu \xi_{\mu\nu}}{(z-w)^3} :\bar{\partial}\bar{X}^\nu(\bar{w})e^{ik\cdot X(w,\bar{w})}: \\ &+ \left[\left(\frac{\alpha'}{4}k^2+1\right)(z-w)^{-2} + \frac{\partial_w}{(z-w)}\right]V_1(k,\xi;w,\bar{w}) \\ &+ \ldots \end{aligned}$$

  This has the form of the OPE of a primary field if $k^\mu \xi_{\mu\nu} \stackrel{!}{=} 0$, thereby reproducing the known transversality constraint of momentum and polarization.

Summary:

Different physical states	$\longleftrightarrow$	Different ways to construct vertex operators

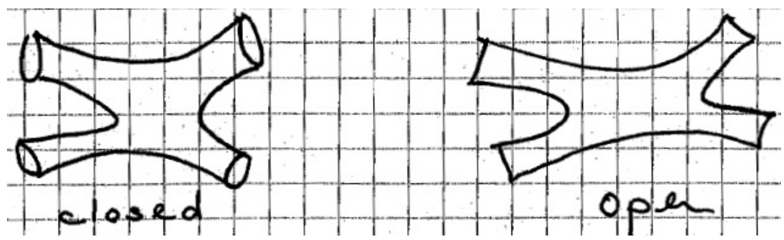
Chapter 5

String Interactions

5.1 Perturbative Expansion

We have finally gathered the technology required to approach the important question of string interactions.

- The basic object to compute in string perturbation theory is the **S-matrix**, defined as the amplitude for the scattering of asymptotic in- and out-states. E.g. scattering of two in-states into two out-states is described by worldsheets of the following form for the closed and open string, respectively:



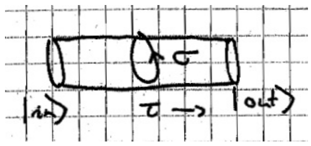
- Just from drawing the worldsheets we note a key property of string scattering - the **absence of definite local interaction vertices**. This fundamentally distinguishes string scattering from the scattering of point particles in QFT as is evident by comparing the above worldsheets with a 2-to-2 Feynman graph, e.g.  in QFT. The string worldsheet always looks **locally** like the worldsheet of a freely propagating string, and **only global properties of the worldsheet capture interactions**.
  Put differently, the string interactions are encoded already in the free two-dimensional CFT **without adding arbitrary further terms in the worldsheet action**.
  This crucial difference compared to point particle QFTs in target space cannot be overestimated - after all the specification of interactions by a QFT Lagrangian adds a degree of arbitrariness into the theory that makes it hard to accept this as a fundamental theory.

- In the following we consider the path integral with a Euclidean metric on the worldsheet. In fact, the only worldsheets for which a globally non-singular *Lorentzian* signature metric exists are the torus T^2, which has no boundaries, or the cylinder. This is because a globally-defined Lorentzian metric requires a globally non-singular Killing vector field that distinguishes the time coordinate from the spatial ones. Such Killing vector fields exist only for the mentioned surfaces. Consider e.g. the surfaces above arising in 2-to-2 scattering: Due the merging and splitting of the in- and out-going pants there is a bifurcation of the "time"-coordinate and thus no globally defined time-like Killing vector field.

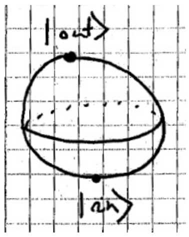
- By means of the operator-state correspondence - the central lesson from the previous CFT chapter - the in- and out-states are encoded in the path integral as by via insertion of the corresponding vertex operators on the worldsheet.
  As the simplest example consider the worldsheet describing 1-to-1 closed string scattering, with Euclidean coordinates and metric

$$w = \tau - i\sigma, \qquad ds^2 = dw\, d\bar{w}. \tag{5.1}$$

It has the topology of a cylinder, with the spatial slices at $\tau = \pm\infty$ corresponding to the in- and out-states:



The conformal map $z = e^{\frac{2\pi}{\ell} w}$ maps this to a sphere, with the in- and out-states inserted at the two poles, or via the stereographic projection to $\mathbb{C} \cup \infty$ with insertions at the origin and at $z = \infty$:



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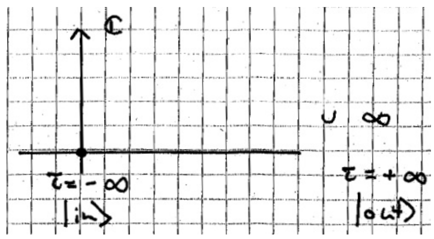
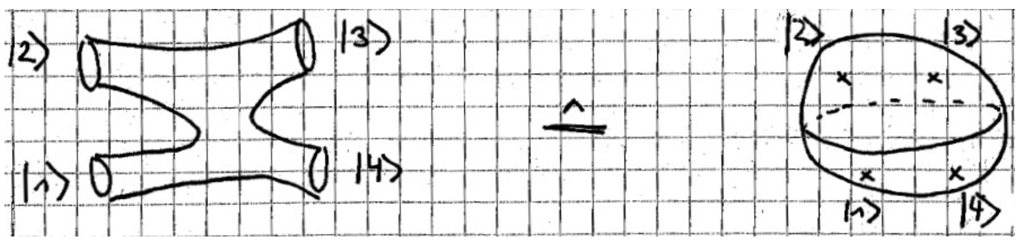
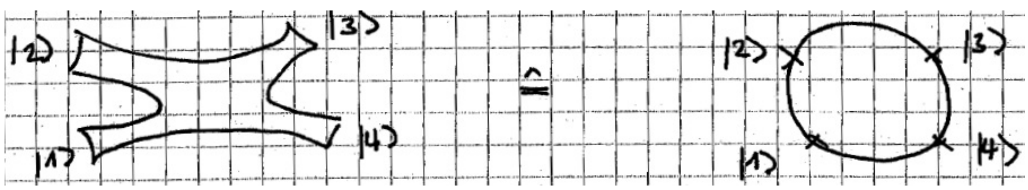


- By similar conformal maps the above closed 2-to-2 scattering process can be mapped to an S^2 with 4 marked points corresponding to the insertion of the 4 vertex operators.



Analogously for open strings the scattering worldsheet is mapped to a disk with insertions on the boundary:



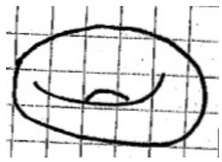
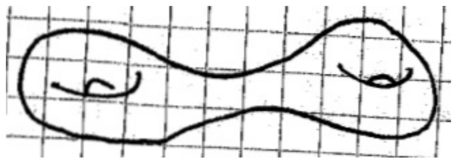
Equivalently we can consider the upper half plane with 4 vertex operators inserted on the real line.

Summary: By conformal symmetry string scattering is described by compact worldsheets with insertions of vertex operators for all in- and out-states.

- Our notation will be to describe by $V_j(k)$ the vertex operator associated with the j-th state with momentum $k^\mu_{\text{in}} = (E, \vec{k})$ or, equivalently, $k^\mu_{\text{out}} = -(E, \vec{k})$.
- The above examples include only the simplest compact worldsheets with 4 insertions, but in the path integral we must **sum over all possible topologies.**
  In the **oriented string** all worldsheets are **orientable.**[1] In characterising the types of worldsheets we need to consider we make use of the following

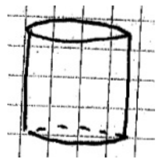
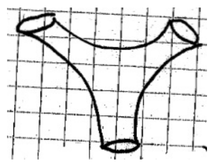
  Theorem 5.1. *Every compact, connected, oriented*[2] *two-dimensional manifold is topologically equivalent to a sphere with g handles and b holes representing the boundaries.*

  As some examples consider the following closed worldsheets

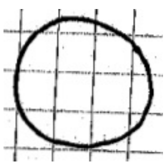
S^2: (g, b) = (0,0) torus T^2: (g, b) = (1,0) double torus (g, b) = (2,0)

or open worldsheets

disk: (g, b) = (0,1) annulus: (g, b) = (0,2) pants: (g, b) = (0,3)

Note that the disk is topologically equivalent to a sphere S^2 with one hole, i.e. with one boundary,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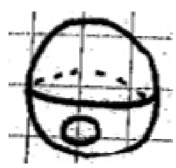


[1] By contrast in orientifold theories we include non-oriented worldsheets. Recall that such unoriented string theories were introduced and discussed in Assignmet 6.

[2] In the unoriented case we need to specify in addition the number c of crosscaps.

and likewise the cylinder can be viewed as a sphere with two holes etc.

- A topological invariant of two-dimensional oriented surfaces[3] is the

$$\boxed{\textbf{Euler number } \chi = 2 - 2g - b.} \tag{5.2}$$

By the famous **Riemann-Roch-theorem** this topological invariant is computed by the expression

$$\boxed{\chi = \frac{1}{4\pi}\int_\Sigma d^2\xi\,\sqrt{h}R^{(2)} + \frac{1}{2\pi}\int_{\partial\Sigma} ds\, k} \tag{5.3}$$

with $R^{(2)}$ the Ricci scalar of the surface and k the geodesic curvature of the boundary.

- Recall that we can add to the string action the term $\lambda\left(\frac{1}{4\pi}\int_\Sigma d^2\xi\,\sqrt{h}R + \frac{1}{2\pi}\int_{\partial\Sigma} ds\, k\right)$ with $\lambda \in \mathbb{R}$ without affecting the dynamics. This term then keeps track of the topology of the worldsheet in the path integral.

Putting everything together we arrive at the following heuristic expression for the S-matrix describing the scattering of n string states (viewed as an expression before gauge fixing):

$$\boxed{S_{j_1\ldots j_n}(k_1,\ldots,k_n) = \sum_{\text{compact topologies}} \frac{\int \mathcal{D}X\mathcal{D}h}{\text{Vol}_{\text{diff.}\times\text{Weyl}}} e^{-S_X - \lambda\chi} \prod_{i=1}^{n} V_{j_i}(k_i).} \tag{5.4}$$

Note that we divide by the volume of the group of diffeomorphisms and Weyl transformations as this group will factor out in the process of the Faddeev-Popov procedure. The particulars of this gauge fixing, however, have to be reconsidered in the presence of vertex operators.

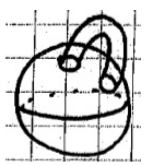
Comments:

1) Let us define the quantity

$$\boxed{g_s = e^\lambda.} \tag{5.5}$$

Then the expansion in terms of worldsheets of different topology is governed by factors of $g_s^{-\chi} = g_s^{-(2-2g-b)}$ (for the oriented case and correspondingly $g_s^{-\chi} = g_s^{-(2-2g-b-c)}$ for the non-oriented theory) in the path integral. For $\lambda << 0$ we observe that $g_s << 1$. Then the sum over topologies defines a perturbative series.

2) Consider a closed string worldsheet and add a handle to it, e.g.



Since this adds two boundaries the Euler number decreases as $\chi \longrightarrow \chi - 2$.

Physically adding a handle describes emission and reabsorption of a closed string. If we think in terms of Feynman diagrams this would corresond to two "vertices", each coming with "coupling constant" g_c. Therefore, we interpret g_s as the

$$\textbf{closed string coupling} \quad \boxed{g_c = e^\lambda = g_s.} \tag{5.6}$$

[3]For non-oriented surfaces we must include the number c of crosscaps, $\chi = 2 - 2g - b - c$.

3) Similarly add a boundary to an open worldsheet, thereby decreasing the Euler number by 1,

The physical interpretation - emission and reabsorption on an open string - indentifies the

$$\textbf{open string coupling} \qquad \boxed{g_o = e^{\lambda/2} = g_c^{1/2}.} \tag{5.7}$$

4) The object $V_{j_i}(k_i)$ is the integrated vertex operator

$$V_{j_i}(k_i) = \text{const} \times \int d^2\xi_i \sqrt{h(\xi_i)} V_{j_i}(k_i;\xi_i) \cong \text{const} \times \int d^2 z_i V_{j_i}(k_i; z_i) \quad \text{in flat gauge.} \tag{5.8}$$

Integration over the insertion of the vertex operator, $\int d^2 z_i$, ensures invariance of the full amplitude under diffeomorphisms on the worldsheets.
The integrated vertex operator is normalised such as to carry one factor of g_o or g_c for open/ closed states.
E.g. the integrated vertex operator for a closed tachyon is

$$V_T(k_i) = g_s \int d^2 z_i \ : e^{ik_i \cdot X(z_i, \bar{z}_i)} : . \tag{5.9}$$

For a general closed state we oftentimes employ the notation

$$V_{j_i}(k_i) = g_s \int d^2 z_i \ V_{j_i}(z_i, \bar{z}_i) : e^{ik_i \cdot X(z_i, \bar{z}_i)} : . \tag{5.10}$$

5.2 Moduli space of Riemann surfaces

It is now time to discuss the degrees of freedom in the metric of two-dimensional surfaces in more detail. As indicated already in the context of the vacuum partition function there is a 2-fold mismatch between the integral over all worldsheet metrics $\int \mathcal{D}h$ and the integral over all diffeomorphisms and Weyl rescalings, $\int d\xi \det P$, where

$$h_{ab} \longrightarrow h_{ab} + (P \cdot \epsilon)_{ab} + 2\tilde{\Lambda} h_{ab}. \tag{5.11}$$

1) First, as stressed several times by now, for the subset of diffeomorphisms and Weyl rescalings given by the conformal Killing vectors the change of the metric under the diffeomorphism can be undone by the Weyl rescaling and thus does not lead to a new metric. These conformal Killing vectors are in the kernel of the differential operator P, i.e. they satisfy

$$P \cdot \epsilon = 0. \tag{5.12}$$

Recall furthermore that these are in 1-to-1 correspondence with the normalisable zero modes of the Faddeev-Popov ghosts.

To avoid overcounting in the path integral one must not integrate over such conformal Killing vectors. In the presence of vertex operator insertions one can impose a further gauge fixing condition by fixing the position of these vertices on the worldsheet. Depending on the number of vertex operators this partially or completely removes the residual symmetry.

2) For 2-dimensional surfaces of non-trivial topology not every metric h_{ab} can be reached from a given reference metric $\hat{h}_{ab}$ by the transformation $\hat{h} \to \hat{h}^{\zeta}$. Consider the operator

$$P^{\dagger}: \quad t_{ab} \mapsto -2\nabla^{b} t_{ab} = (P^{\dagger} t)_{a} \tag{5.13}$$

This is the adjoint of P with respect to the positive definite measure induced by the inner products

$$(\delta h^{(1)}|\delta h^{(2)}) = \int d^{2}\xi \sqrt{-h} h^{ab} h^{cd} \delta h^{(1)}_{ac} \delta h^{(2)}_{bd} \tag{5.14}$$

and

$$\langle \xi^{(1)}, \xi^{(2)} \rangle = \int d^{2}\xi \sqrt{-h} h^{ab} \xi^{(1)}_{a} \xi^{(2)}_{b}. \tag{5.15}$$

Indeed integration by parts straightfowardly confirms that

$$\langle \xi, P^{\dagger} t \rangle = (P \cdot \xi | t). \tag{5.16}$$

Now suppose the exists a symmetric traceless 2-tensor t_0 in the kernel of $P^{\dagger}$, i.e. such that $P^{\dagger} t_0 = 0$. Then for all vectors ξ^{a}

$$\langle \xi, P^{\dagger} t_0 \rangle = (P \cdot \xi | t_0) = 0. \tag{5.17}$$

Since $P \cdot \xi$ is orthogonal to t_0 $\forall \xi$, no ξ^{a} can be found such that $P \cdot \xi = t_0$. Such t_0 are called **metric moduli**.

Definition 5.1. *Deformations of the metric of a differentiable manifold that cannot be absorbed by a diffeomorphism or Weyl rescaling are called metric moduli.*

Comparison with the equations of motion (3.145) of the Faddeev-Popov anti-ghosts reveals that the metric moduli are in 1-to-1 correspondence with the normalisable zero-modes of the anti-ghosts.

In the path integral, the sum over the metric moduli must be carried out extra.

In the sequel we will need the notion of a **Riemann surface**.

Definition 5.2. *A Riemann surface is a 2-dimensional complex manifold.*

Note:

- By a complex manifold we mean a differentiable manifold $\mathcal{M}$ together with an atlas such that in each patch labeled by m we can pick complex coordinates z_m in such a way that the transition functions between the patches are *holomorphic* functions, $z_m = f_{mn}(z_n)$.
- Just as one considers two differential manifolds to be equivalent if there is a differential map between them, two complex manifolds are equivalent if there exists a holomorphic map between them. Now, as discussed in detail in chapter 4.3, on a 2-dimensional manifold the holomorphic maps correspond to the conformal transformations. Therefore, by a Riemann surface we really mean the equivalence class of all complex 2-dimensional manifolds modulo diffeomorphisms (because we consider differentiable manifolds) and Weyl rescalings.
- The degrees of freedom of a Riemann surface are therefore precisely given by the metric moduli.

Of great use is the following

Theorem 5.2. *(Riemann-Roch): Consider an orientable Riemann surface of Euler number χ. Denote by*

$$\begin{aligned} \mu &= \dim(\ker P^\dagger), &(5.18)\\ \kappa &= \dim(\ker P) &(5.19)\end{aligned}$$

the number of metric moduli and conformal Killing vectors, respectively[4]. *Then*[5]

$$\boxed{\mu - \kappa = -3\chi = 6g + 3b - 6.} \tag{5.20}$$

Furthermore,

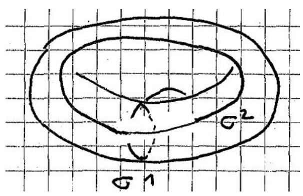
$$\begin{aligned} &\text{if} \quad \chi > 0: \quad \kappa = 3\chi, \quad \mu = 0, &(5.21)\\ &\text{if} \quad \chi < 0: \quad \kappa = 0, \quad \mu = -3\chi. &(5.22)\end{aligned}$$

The proof can be found e.g. in [BLT], Chapter 6.2. or in [P], Chapter 5.3.

Examples:

1) The Riemann surface of maximal Euler number is the sphere S^2 with $\chi = 2$. From the Riemann-Roch theorem we read off that $\mu = 0$, i.e. every S^2 is conformally diffeomorphic to flat space. Furthermore $\kappa = 6 = \dim_{\mathbb{R}}(PSL(2,\mathbb{C}))$, in agreement with the fact that the conformal group of S^2 is $PSL(2,\mathbb{C})$ if we treat the holomorphic and anti-holomorphic coordinates z and $\bar{z}$ as complex conjugates (as we do in the geometry of Riemann surfaces).

2) As the next important example consider a torus T^2 with Euler number $\chi = 0$. There are many representations of the torus. The most intuitive one is as $T^2 = S^1 \times S^1$.



Parametrise both S^1 by periodic coordinates $0 \le \sigma^1 \le 2\pi$, $0 \le \sigma^2 \le 2\pi$ so the torus coordinates are doubly-periodic,

$$(\sigma^1,\, \sigma^2) \cong (\sigma^1,\, \sigma^2) + 2\pi(m,\, n) \qquad m,\, n \in \mathbb{Z}. \tag{5.23}$$

With not too much work one can show that modulo diffeomorphism and Weyl rescalings the most general metric on T^2 can be brought into the form

$$ds^2 = |d\sigma^1 + \tau d\sigma^2|^2 \qquad \tau \in \mathbb{C}. \tag{5.24}$$

For a derivation and much more information on T^2 see Assignment 10.

[4] Here we count the *real* dimension of the moduli space.

[5] For unoriented Riemann surfaces we have $\mu - \kappa = -3\chi = 6g + 3b + 3c - 6$.

The complex number τ is the modulus. Since we count real degrees of freedom we conclude that $\mu = 2$.

Since $\chi = 0$, Riemann-Roch implies that $\kappa = \mu$, i.e. we expect two real conformal Killing vectors.

In fact the globally defined global transformation $\sigma^a \to \sigma^a + v^a$ leaves the metric and periodicity of the coordinates invariant. We have thus established that the conformal group of T^2 is given by $U(1) \times U(1)$.

On Assigment 10 we will see that the shape of a torus does not change under a $PSL(2,\mathbb{Z})$-transformation

$$\tau \to \frac{a\tau + b}{c\tau + d}. \tag{5.25}$$

This restricts the inequivalent values of τ to the so-called fundamental domain. A convenient choice of the fundamental domain is $|\tau| \geq 1, -\frac{1}{2} \leq \Re(\tau) \leq \frac{1}{2}$.

5.3 The gauge fixed S-matrix

Let us discuss in more detail the gauge fixing procedure for the S-matrix, addressing in particular the questions of conformal Killing transformations and the moduli space of metrics. Our starting point is the expression for the S-matrix derived above,

$$S_{j_1 \dots j_n}(k_1, \cdots, k_n) = \sum_{\text{compact topologies}} \int \frac{\mathcal{D}X\mathcal{D}h}{\mathrm{Vol}_{\mathrm{Diff}\times\mathrm{Weyl}}} e^{-S_X - \lambda\chi} \prod_{i=1}^{n} \int d^2\xi_i \sqrt{h(\xi_i)} V_{j_i}(k_i, \xi_i). \tag{5.26}$$

- The worldsheet metric h_{ab} now depends on μ **moduli** t^α, $\alpha = 1, \dots, \mu$. The variation of h with respect to t^α is described by

$$\delta_\alpha h(t^\alpha) = \delta t^\alpha \partial_\alpha h \equiv \delta t^\alpha \partial_{t^\alpha} h. \tag{5.27}$$

  The integral over the degrees of freedom of the metric must therefore include also an integral over the possible values of the moduli, i.e. their fundamental domain F.

  Definition 5.3. *The quantity*

$$\mu_{\alpha,a}{}^b = \frac{1}{2} h^{bd} \partial_{t^\alpha} h_{ad} \tag{5.28}$$

  is called **Beltrami differential**.

- To avoid overcounting we impose an extra gauge fixing condition to fix the residual conformal Killing vector transformations. In the presence of a sufficient number of vertex operator insertions it is convenient to use the κ **Killing vectors** to fix κ positions of vertex operators.

  Remark:
  Alternatively one can divide the non-gauge fixed expression by the volume of the conformal group. In particular if there are not enough vertex operators to fix the conformal Killing transformations completely one divides by the volume of the remaining subgroup of the conformal group. If the volume of the conformal group or a subgroup by which we divide is

infinite, then the corresponding string correlator vanishes. E.g. on the sphere the conformal group is $PSL(2,\mathbb{C})$ and has infinite volume. Thus, the oriented closed string 0-point, 1-point and 2-point function vanishes at tree-level, corresponding to the absence of a vacuum energy, tadpole and, respectively, mass renormalisation at tree-level. For the oriented open string, the conformal group on the disk is $PSL(2,\mathbb{R})$ and the same statements hold for the open 1- and 2-point function at tree-level. On the other hand, the closed string 1-point function on the disk need not vanish as insertion of a closed string vertex operator in the bulk of the disk only leaves a finite volume U(1) subgroup of $PSL(2,\mathbb{R})$ unfixed. Thus there can (and in fact will) be closed string tadpoles in the presence of D-branes.

Naively, the measure becomes

$$\int \mathcal{D}h \prod_{i=1}^{n} \int d^2\xi_i \rightarrow \int \mathcal{D}\zeta \prod_{\alpha=1}^{\mu} \int dt^\alpha \prod_{i=1}^{n-\mu} \int d^2\xi_i \tag{5.29}$$

with κ positions ξ_i fixed at $\hat{\xi}_i$. More precisely one has to compute the modified Faddeev-Popov determinant Δ_{FP}.

Derivation of the measure - non-examinable

As a warm-up we first present, following [P], the following derivation of the Faddeev-Popov determinant for the partition function without any vertex operator insertions and ignoring the subtleties of conformal Killing vectors and metric moduli. It is an alternative to the shortcut presented in section (3.3.1). The partition function is

$$Z \quad = \int \mathcal{D}X\,\mathcal{D}h\, e^{i\,S_P[X,h]} = \int \mathcal{D}X\,\mathcal{D}h\,\mathcal{D}\zeta\, e^{i\,S_P[X,h]}\,\delta(h-\hat{h}^\zeta)\,\Delta_{FP}(h) \tag{5.30}$$

via insertion of

$$1 = \Delta_{FP}(h) \int \mathcal{D}\zeta\,\delta(h-\hat{h}^\zeta). \tag{5.31}$$

Performing the$\int \mathcal{D}h$ integration and exploiting gauge invariance as in section (3.3.1) this becomes

$$Z = \int \mathcal{D}X\,\mathcal{D}\zeta\, e^{i\,S_P[X,\hat{h}]}\,\Delta_{FP}(\hat{h}). \tag{5.32}$$

The *inverse* of the determinant $\Delta_{FP}(\hat{h})$ can be expressed as

$$\Delta_{FP}(\hat{h})^{-1} = \int D\zeta\,\delta(\hat{h}-\hat{h}^\zeta), \qquad \zeta = (\epsilon^a,\Lambda), \quad \hat{h}^\zeta = \hat{h} + P\cdot\epsilon + 2\tilde{\Lambda}\,h. \tag{5.33}$$

With the help of the integral representation of the delta-function and the shorthand notation $(\beta|P\cdot\epsilon) = \int d^2\xi\,\sqrt{\hat{h}}\,\beta_{ab}\,(P\cdot\epsilon)^{ab}$ this is

$$\begin{aligned} \Delta_{FP}(\hat{h})^{-1} &= \int \mathcal{D}\epsilon^a\,\mathcal{D}\Lambda \int \mathcal{D}\beta_{ab}\; e^{2\pi\imath(\beta|-P\cdot\epsilon-2\tilde{\Lambda}\hat{h})} && (5.34)\\ &= \int \mathcal{D}\epsilon^a \int \mathcal{D}\beta'_{ab}\; e^{-2\pi i(\beta'|P\cdot\epsilon)}, && (5.35)\end{aligned}$$

where the integration over Λ restricts β'_{ab} to be symmetric and traceless.
Finally we apply the general rule that if the replace the integration variables in the above expression for $\Delta_{FP}(\hat{h})^{-1}$ by (suitably normalised) Grassmann-valued fields,

$$\epsilon^a \rightarrow c^a, \qquad \beta'_{ab} \rightarrow b_{ab}, \tag{5.36}$$

we obtain $\Delta_{FP}(\hat{h})$ as

$$\Delta_{FP}(\hat{h}) = \int \mathcal{D}b_{(ab)}\mathcal{D}c^d \, \exp\Big(\frac{1}{4\pi}\int d^2\xi\sqrt{-\hat{h}}\, b\cdot(P\cdot c)\Big). \tag{5.37}$$

Now we consider the S-matrix with vertex operator insertions and a moduli dependent metric. According the discussion above the gauge fixing factor to be inserted into (5.26) is

$$1 = \Delta_{FP}(\hat{h},\hat{\xi}_i^a)\int d^\mu t \int \mathcal{D}\zeta\, \delta(\hat{h}-\hat{h}(t)^\zeta) \prod_{(a,i)\in f} \delta(\hat{\xi}_i^a - (\hat{\xi}_i^\zeta)^a). \tag{5.38}$$

The last insertion fixes the position of κ vertex operators to the position $(\hat{\xi}_i^\zeta)^a$, which depends on the particular gauge choice, and

$$\hat{h}(t)^\zeta = \hat{h} + \sum_{\alpha=1}^{\mu}\delta t^\alpha \partial_{t^\alpha}\hat{h} + P\cdot\epsilon + 2\tilde{\Lambda}\, h. \tag{5.39}$$

By the same logic as before we find

$$\Delta_{FP}(\hat{h},\hat{\xi}_i^a)^{-1} = \int d^\mu \delta t \int \mathcal{D}\epsilon^a \int \mathcal{D}\beta'_{ab} \prod_{(a,i)\in f}\int dx_{a,i}\, e^{2\pi i(\beta'|P\cdot\epsilon + \sum_{\alpha=1}^{\mu}\delta t^\alpha \partial_{t^\alpha}\hat{h})} \tag{5.40}$$

$$\times e^{2\pi i\sum_{(a,i)\in f} x_{a,i}\epsilon^a(\hat{\xi}_i^a)}. \tag{5.41}$$

Again we replace the integration variables by suitably normalised Grassmann fields,

$$\epsilon^a \to c^a, \qquad \beta'_{ab} \to b_{ab}, \qquad x_{a,i} \to \eta_{a,i}, \qquad \delta t^\alpha \to \gamma^\alpha \tag{5.42}$$

and deduce

$$\Delta_{FP}(\hat{h},\hat{\xi}_i^a) = \int \mathcal{D}b\,\mathcal{D}c \int \mathcal{D}^\mu\gamma\,\mathcal{D}^\kappa\eta\, e^{-\frac{1}{4\pi}(b|P\cdot c - \gamma^\alpha\partial_{t^\alpha}\hat{h}) + \sum_{(a,i)}\eta_{a,i}\, c^a(\xi_i^a)}. \tag{5.43}$$

All that is left is to perform the Grassmann integral $\int \mathcal{D}^\mu\gamma\,\mathcal{D}^\kappa\eta$.

The final **result** is

$$\Delta_{\mathrm{FP}} = \int \mathcal{D}b\,\mathcal{D}c\; e^{-S_{gh}} \prod_{\alpha=1}^{\mu}\frac{1}{4\pi}(b|\partial_\alpha\hat{h}) \prod_{(a,i)\in f} c^a(\hat{\xi}_i), \tag{5.44}$$

where

- $(b|\partial_\alpha\hat{h}) = \int d^2\xi\,\sqrt{\hat{h}}\, b_{ab}\,(\partial_\alpha\hat{h})^{ab}$ in terms of the gauge fixed metric and $\partial_\alpha\,\hat{h} := \partial_{t^\alpha}\hat{h}$ and
- $(a,i)\in f$ refers to the set of fixed vertex operator positions.

In total the gauge fixed S-matrix takes the form

$$\boxed{\begin{aligned} S_{j_1\ldots j_n}(k_1,\cdots,k_n) = \quad & \sum_{\text{top.}}\int\prod_{\alpha=1}^{\mu}dt^\alpha\int\mathcal{D}X\,\mathcal{D}b\,\mathcal{D}c\, e^{-S_X - S_{gh} - \lambda\chi}\,\times \\ & \prod_{(a,i)\notin f}\int d\xi_i^a \prod_{\alpha=1}^{\mu}\frac{1}{4\pi}(b|\partial_\alpha\hat{h})\prod_{(b,j)\in f}c^b(\hat{\xi}_j)\prod_{i=1}^{n}\sqrt{\hat{h}(\xi_i)}V_{j_i}(k_i,\xi_i).\end{aligned}} \tag{5.45}$$

This looks more scary than it is - after all, what we have done can simply be summarised as the following **take-home message**:

- For each conformal Killing vector we place one vertex at $\hat{\xi}_i$: $V_{j_i}(k_i, \hat{\xi}_i)$ and replace the integral over the corresponding vertex operator position $\int d\xi_i^a$ by one insertion of $c^a(\hat{\xi}_i)$.

- For each modulus we insert $\frac{1}{4\pi}(b|\partial_\alpha \hat{h})$ and integrate over the fundamental domain $\int_F dt^\alpha$.

  Note that the insertion $\frac{1}{4\pi}(b|\partial_\alpha \hat{h})$ can be written in terms of the Beltrami differentials introduced above as

$$\frac{1}{4\pi}(b|\partial_\alpha \hat{h}) = \frac{1}{4\pi}\int d^2 z b_{zz}\, \mu_{\alpha,z}{}^{\bar{z}} + b_{\bar{z}\bar{z}}\, \mu_{\alpha,\bar{z}}{}^{z}. \tag{5.46}$$

  More on the use of Beltrami differentials and the philosophy behind the gauge fixing can found e.g. in [P], Chapter 5.4.

5.4 Tree-level amplitudes

We are finally in a position to compute the first non-trivial S-matrix, beginning with processes at tree-level. This corresponds to amplitudes on Riemann surfaces with positive Euler characteristic. There are three such surfaces:

- Correlators on **the sphere** $S^2 = \mathbb{CP}^1$ $(g = b = c = 0)$ yield oriented closed string amplitudes.

- Correlators on **the disk or upper half-plane** D_2 $(b = 1, g = c = 0)$ with open string vertex operators inserted on the boundary of D_2 or, equivalently, the real line, correspond to oriented open string amplitudes. Insertion of closed string vertex operators in the bulk of the D_2 describe the scattering of oriented closed strings off D-branes.

- Correlators on the **real projective plane** RP^1 $(c = 1, b = c = 0)$ describe processes in unoriented string theory.

5.4.1 Correlators on the sphere - the Virasoro-Shapiro amplitude

We start with a process on the sphere, corresponding to oriented closed string scattering at tree-level in perturbation theory. In view of the technology acquainted in the previous section we note:

- The sphere S^2 has no moduli, $\mu = 0$. Thus no insertion of factors $\frac{1}{4\pi}(b|\partial_\alpha \hat{h})$ is required.

- The conformal group $PSL(2, \mathbb{C})$ has 3 complex parameters. Since its volume is infinite, the first non-vanishing correlator on the sphere is the 3-point correlator - see the remark after (5.28). For an n-point correlator with $n \geq 3$, we fix the $PSL(2, \mathbb{C})$ invariance by fixing the position of three vertex operators. Thus we must include three Faddev-Popov ghost c-modes as specified above.

Example: Scattering of four closed string tachyons

The tachyon vertex operator is

$$V_{j_i}(k_i) = g_c \int d^2 z_i :e^{i\, k_i X(z_i, \bar{z}_i)}: . \tag{5.47}$$

Thus the S-matrix describing the scattering four closed string tachyons at tree-level is

$$S(k_1,k_2,k_3,k_4) = g_c^4\, e^{-2\lambda} \int_{\mathbb{C}^2} d^2 z_4 \,\langle \prod_{i=1}^{3} :\tilde{c}\, c\, e^{i\,k_i X} : (z_i,\bar{z}_i) : e^{i\,k_4 X} : (z_4,\bar{z}_4)\rangle_{S^2}, \tag{5.48}$$

where as always the z_i inside $\langle\rangle$ are radially ordered. Since the X-CFT and the (bc)-CFT are independent the problem factorises in the computation of the following two correlators,

i) $\langle : e^{i\,k_1 X(z_i,\bar{z}_i)} : \ldots : e^{i\,k_4 X(z_4,\bar{z}_4)} :\rangle_{S^2}$ and

ii) $\langle : \tilde{c}\, c(z_i,\bar{z}_i) : \ldots : \tilde{c}\, c(z_3,\bar{z}_3) :\rangle_{S^2}$.

ad i) The first correlator reduces to a Gaussian path integral with the result

$$\langle \prod_{i=1}^{m} e^{i\,k_i X(z_i.\bar{z}_i)}\rangle = \text{const.} \times \delta(\sum_{i=1}^{m} k_i) \prod_{j<l} |z_j - z_l|^{\alpha'\, k_j \cdot k_l}. \tag{5.49}$$

While a more through derivation is reserved to Assignment 10, we can understand the structure by bringing the correlator into the form

$$\begin{aligned} \langle \prod_{i=1}^{m} e^{i\,k_i X(z_i,\bar{z}_i)}\rangle &= \int \mathcal{D}X \exp\Big(\frac{1}{2\pi\alpha'}\int d^2 z\, X(z,\bar{z})\, \partial\bar{\partial}\, X(z,\bar{z}) + iJ(z,\bar{z})\, X(z,\bar{z})\Big), \\ J(z,\bar{z}) &= \sum_i k_i\, \delta^{(2)}(z - z_i, \bar{z} - \bar{z}_i). \end{aligned} \tag{5.50}$$

The Gaussian can be performed explicitly. Up to an irrelevant overall factor given by a functional determinant and some more subtleties, which are treated with care in Assignment 10, it yields

$$\langle \prod_{i=1}^{m} e^{i\,k_i X(z_i.\bar{z}_i)}\rangle \cong \exp\Big(\frac{\pi\alpha'}{2}\int d^2 z\, d^2 z'\, J(z,\bar{z}) G(z,\bar{z},z'\bar{z}') J(z',\bar{z}')\Big) \tag{5.51}$$

with the Green's function

$$\partial\bar{\partial}\, G(z,\bar{z},z'\bar{z}') = \delta^{(2)}(z - z', \bar{z} - \bar{z}') \Longrightarrow G(z,\bar{z},z'\bar{z}') = \frac{1}{2\pi}\log|z - z'|^2. \tag{5.52}$$

The form of $G(z,\bar{z},z'\bar{z}')$ follows from the discussion around (4.102) and (4.104).

Among the subtleties we ignored here are the zero modes of the Green's function, which give us the factor $\delta(\sum_{i=1}^{m} k_i)$, and the normal ordering of the vertex operators. Taking both into account leads us to (5.50).

ad ii) The ghost-sector 3-point function is

$$\langle \prod_{i=1}^{3} : \tilde{c}\, c(z_i,\bar{z}_i) :\rangle_{S^2} = \text{const.} \times\ z_{12} z_{13} z_{23}\, \bar{z}_{12} \bar{z}_{13} \bar{z}_{23}. \tag{5.53}$$

This follows from general expression for 3-point correlator in CFT together with $h = -1$ for c,

$$\langle c(z_1)\, c(z_2)\, c(z_3)\rangle = \frac{C_{123}}{z_{12}^{-1-1+1} z_{13}^{-1-1+1} z_{23}^{-1-1+1}}. \tag{5.54}$$

In combination these two results give the following z_i dependence of the integrand appearing in $S(k_1, k_2, k_3, k_4)$,

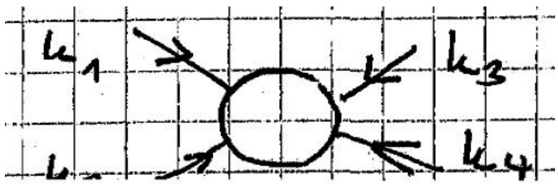
$$|z_{12}|^{2+\alpha' k_1\cdot k_2}|z_{13}|^{2+\alpha' k_1\cdot k_3}|z_{23}|^{2+\alpha' k_2\cdot k_3}|z_{34}|^{\alpha' k_3\cdot k_4}|z_{14}|^{\alpha' k_1\cdot k_4}|z_{24}|^{\alpha' k_2\cdot k_4}. \tag{5.55}$$

By $PSL(2,\mathbb{C})$-invariance of the final amplitude we can fix z_1, z_2, z_3 to convenient positions, e.g. $z_1 = 0$, $z_1 = 1$, $z_1 = \infty$.
In particular in the limit $z_3 \to \infty$ all terms involving z_3 cancel with the help of the kinematic relations $k_1 + k_2 k_3 + k_4 = 0$ and $k_i^2 = \frac{4}{\alpha'}$.
What remains is therefore the non-trivial integral

$$I = \int d^2 z_4 |z_4|^{\alpha' k_1\cdot k_4}|1 - z_4|^{\alpha' k_2\cdot k_4}. \tag{5.56}$$

Let us introduce the famous **Mandelstam variables**

$$s = -(k_1 + k_2)^2\,, \qquad t = -(k_1 + k_3)^2\,, \qquad u = -(k_1 + k_4)^2. \tag{5.57}$$



These satisfy

$$s + t + u = -\sum_{i=1}^{4} k_i^2 = \sum_{i=1}^{4} M_i^2 \overset{\text{for 4 tachyons}}{\equiv} -\frac{16}{\alpha'}. \tag{5.58}$$

Therefore the integral can be expressed as

$$I = \int d^2 z |z|^{-\alpha' \frac{u}{2} - 4}|1 - z|^{-\alpha' \frac{t}{2} - 4} \equiv J(s, t, u) \tag{5.59}$$

and we find

$$\boxed{S(k_1, k_2, k_3, k_4) = i\, g_c^4\, C_{S_2} (2\pi)^{26} \delta^{(26)}(\sum_i k_i) J(s, t, u).} \tag{5.60}$$

The constant prefactor C_{S_2} comprises all constants which we have swept under the rug in the computation and will be fixed later by unitarity.
The function $J(s, t, u)$ appearing in the integrand can be represented via **Euler Γ-functions**

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}, \qquad\qquad z \in \mathbb{C}. \tag{5.61}$$

From standard texts on complex analysis we quote the following **properties** of the Γ-function:

- $\Gamma(z)$ is convergent for $\Re(z) > 0$ and has a unique analytic continuation to $\mathbb{C}$.
- From $\Gamma(.z+1) = z\,\Gamma(z)$, $\Gamma(1) = 1$, which is easily derived via integration by parts, it follows that the Γ-function coinicides with the factorial for natural numbers, $\Gamma(n) = (n-1)!$ for $n \in \mathbf{N}$.

- $\Gamma(z)$ has poles at $z = -n$, $n = 0, 1, 2, \ldots$, in whose vicinity it enjoys the expansion

$$\Gamma|_{z\to -n} = \frac{1}{z+n}\frac{(-1)^n}{n!}. \tag{5.62}$$

Without proof[6] we now quote the following integral

$$\int d^2z |z|^{2a-2}|1-z|^{2b-2} = 2\pi \frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}{\Gamma(a+b)\,\Gamma(a+c)\,\Gamma(b+c)}, \qquad a+b+c = 1. \tag{5.63}$$

To apply this to (5.59) we identify

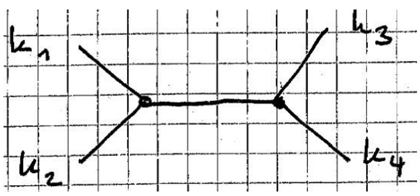
$$a = -1 - \alpha'\frac{u}{4}, \qquad b = -1 - \alpha'\frac{t}{4} \tag{5.64}$$

and arrive at the **Virasoro-Shapiro amplitude**

$$\boxed{S(k_1, \ldots, k_4) = i\, g_c^4 C_{S_2} (2\pi)^{26}\delta(\sum_i k_i)\, 2\pi\, \frac{\Gamma(-1-\frac{\alpha'}{4}s)\,\Gamma(-1-\frac{\alpha'}{4}t)\,\Gamma(-1-\frac{\alpha'}{4}u)}{\Gamma(2+\frac{\alpha'}{4}s)\,\Gamma(2+\frac{\alpha'}{4}t)\,\Gamma(2+\frac{\alpha'}{4}u)}.} \tag{5.65}$$

Properties of the amplitude:

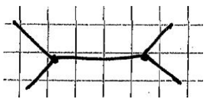
- Consider a process in the "s-channel", i.e. scattering at fixed Mandelstam variable t with s varying,



Of physical importance are the poles of $\Gamma(-1-\frac{\alpha'}{4}s)$, located at

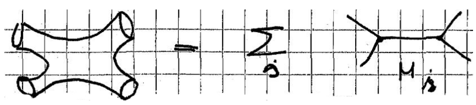
$$-1 - \frac{\alpha'}{4}s = -n, \quad \text{i.e.} \quad s = \frac{4}{\alpha'}(n-1) \qquad \forall n \in \mathbb{N}_0. \tag{5.66}$$

As is well-familiar from scattering theory in QFT, these are the resonances due to exchange of string states of mass $m^2 = \frac{4}{\alpha'}(n-1)$, with the poles resulting from the propagator of the intermediate states,



$$\sim \frac{1}{s-m^2}. \tag{5.67}$$

That the full tower of string excitations appears in the S-matrix, c.f. (5.66), just means that we sum over infinitely many exchanged particles,

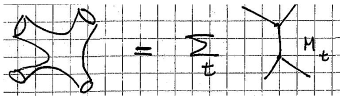


$$\cdot\, \delta(s - M_s^2) + \text{terms analytic at } M_s^2. \tag{5.68}$$

[6]This can be derived via the trick $|z|^{2a-2} = \frac{1}{\Gamma(1-a)}\int_0^\infty dt\, t^{-a} e^{-|z|^2 t}$. Together with the analogous expression for $|1-z|^{2b-2}$ the integral can be written as a Gaussian integral in the coordinates x, y for $z = x+iy$. Evalutation of the Gaussian then leads to the quoted result if we recognize on the way the Euler Beta-function $B(a,b) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, which in turn has the representation $B(x,y) = \int_0^1 dt t^{x-1}(1-t)^{y-1}$.

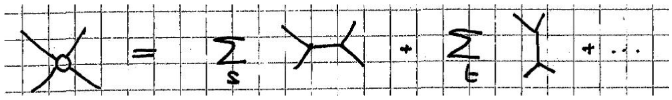
In other words, the amplitude knows about the entire spectrum of string excitations, a fact which is quite remarkable and a highly non-trivial consistency check of the framework.

- By inspection the Virasoro-Shapiro amplitude is symmetric in t and s (and u). So it also allows an expansion for fixed s in t, corresponding to the t-channel:



$$\cdot\, \delta(t - M_s^2) + \text{terms analytic at } M_t^2. \tag{5.69}$$

- The Virasoro-Shapiro amplitude can therefore be written as a sum over infinitely many s-channel poles **or** over infinitely many t-channel poles. This property is called **duality**. It holds more generally for string amplitudes and distinguishes string amplitudes from point particle QFT amplitudes in the following sense. In QFT one has to sum over finitely many s- **and** t-channels,



(5.70)

In string theory, one single string diagram at genus g corresponds to what would be described by several Feynman graphs at a given perturbative order of different topologies.

Determining the normalisation

What remains is to determine the constant C_{S^2} in the amplitude (5.65). This factor contains the product of the constant $C^X_{S^2}$ due to functional determinants in the X-sector and $C^{bc}_{S^2}$ from the ghost 3-point function,

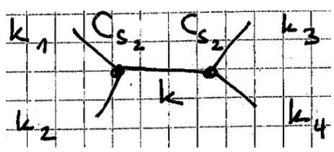
$$C_{S^2} \simeq g_c^{-2}\, C^X_{S^2}\, C^{bc}_{S^2}. \tag{5.71}$$

A closer look in particular at the functional determinants in the X-sector, see Assignment 10, reveals that C_{S^2} is a universal constant valid for **all** $(3+n)$-point functions on the sphere. In particular it is the same constant that appears also in the 3-point function for scattering of 3 tachyons. This amplitude is easily computed with our methods,

$$S_{S_2}(k_1, k_2, k_3) = i\, g_c^3 C_{S_2} (2\pi)^{26} \delta^{(26)}(\sum_i k_i) \tag{5.72}$$

Unitarity now allows us to fix C_{S^2} by demanding that each pole in the s-channel of the 4-point function give the same amplitude as two 3-point amplitudes. Consider the first pole at $s = -\frac{4}{\alpha'}$. Interpretating the 4-point function at $s = -\frac{4}{\alpha'}$ as a sequence of two 3-point functions is equivalent to the ansatz

$$S(k_1, k_2, k_3, k_4) = i \int \frac{d^{26}k}{(2\pi)^{26}} \frac{S_{S_2}(k_1, k_2, k)\, S_{S_2}(-k, k_3, k_4)}{-k^2 + \frac{4}{\alpha'} + i\,\epsilon} + \text{terms analytic at } \frac{4}{\alpha'} \tag{5.73}$$



Near the pole $s = -\frac{4}{\alpha'}$ the 4-point function takes the form

$$S(k_1, k_2, k_3, k_4) = -\frac{8\pi\, i\, C_{S_2} g_c^4}{\alpha'\, s + 4}(2\pi)^{26}. \tag{5.74}$$

This is most easily seen directly from the representation (5.56): The pole in s comes from the bahaviour of $z \to \infty$. There the integral can be evaluated with the help of the residue theorem. Plugging (5.74) into our factorisation ansatz (5.73) we deduce

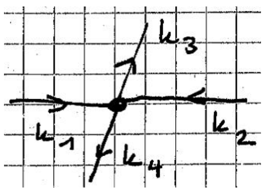
$$\boxed{C_{S_2} = \frac{8\pi}{\alpha'} \frac{1}{g_c^2}.} \tag{5.75}$$

Ultraviolet behaviour

Of special interest is the ultraviolet behaviour of the string scattering amplitude as this is the regime that probes the stringy nature of the excitations. Kinematically this is the limit

$$s \to \infty, \qquad t \to \infty, \qquad \frac{s}{t} \ \text{ fixed,} \tag{5.76}$$

corresponding to scattering at high energies and at fixed angle θ.



E.g. for massless particles scattering at fixed angels would be described by the following choice of momenta:

$k_1 = \frac{\sqrt{s}}{2}(1, 1, 0, \dots, 0)$,
$k_2 = \frac{\sqrt{s}}{2}(1, -1, 0, \dots, 0)$,
$k_3 = \frac{\sqrt{s}}{2}(1, \cos(\theta), \sin(\theta), 0, \dots, 0)$,
$k_4 = \frac{\sqrt{s}}{2}(1, -\cos(\theta), -\sin(\theta), 0, \dots, 0)$.

This limit is called **"hard-scattering" limit**.

To describe the behaviour of the amplitude in this regime we make us of the asymptotic behaviour of the Γ-function,

$$\Gamma \approxeq \exp(x \ln x) \quad \text{for} \quad \Re(x) \to \infty, \tag{5.77}$$

and deduce

$$\to S(k_1, k_2, k_3, k_4) \approxeq \exp(-\frac{\alpha'}{2}(s \ln s + t \ln t + u \ln u)) \quad \text{as } |s|,\ |t| \to \infty \text{ and } s/t \text{ fixed.} \tag{5.78}$$

This establishes an **exponential fall-off of the amplitude**.

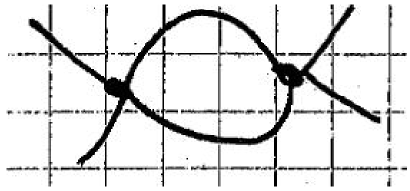
Lesson:

String amplitudes fall off much faster than point-particle amplitudes in QFT.

For comparison, the amplitude for exchange of a spin-J particle in QFT falls off like a power-law,

$$\mathcal{A} \sim \frac{t^J}{s - M^2}. \tag{5.79}$$

- The exponential fall-off of string amplitude is consistent with the s-channel picture (i.e. the power-law behaviour near the zeroes of the propagator) because we sum over *infinitely* many states.
- It is (partially) responsible for UV finiteness when considering strings running in the loop.



- Strings behave differently because high-energy processes probe the string length $\sqrt{\alpha'}$. In this regime the string is non-local due to its extended structure.

One can now perform a systematic analysis of the closed string scattering amplitudes at tree-level, in particular of the massless graviton $g_{\mu\nu}$, the B-field $B_{\mu\nu}$ and the dilaton ϕ. The interactions can be compared with the vertices from a **low-energy effective action** in the 26-dimensional ambient spacetime. This effective action describes the low-energy point particle regime of string theory. In order to describe the interactions well below the string scale M_s we include only the massless excitations. In the bosonic theory we could also include the tachyon, but in the full superstring theory this excitation is absent anyway so including the tachyon is only for reasons of demonstration. In this fashion one can explicitly confirm at the level of interactions the claim that the spin-two excitation $g_{\mu\nu}$ describes the graviton of Einstein gravity. We will soon find an alternative method to arrive at the same conclusion.

5.4.2 Correlators on the disk

The computation of open string correlators proceeds in a manner very similar to the closed string amplitudes, with the difference that due to the boundary conditions the vertex operators are effectively chiral. More precisely:

- Oriented open string amplitudes at tree-level are described by correlators on the disk D_2 with $(g, b) = (0, 1)$, which is conformally equivalent to the upper half plane. The conformal group is $PSL(2, \mathbb{R})$.
- The open string vertex operators are inserted on the boundary of D_2 or, respectively, on the real line, which forms the boundary of the upper half plane. They depend on the real variable y parametrizing, say, the real line.
- An important difference to the closed string sector is that the boundary of D_2 admits the notion of ordering of the vertex operator insertions. We must therefore sum over all possible orderings of the vertex operators (up to one overall permutation symmetry).
- For open strings on a stack of N coincident D-branes the full vertex operator carries the Chan-Paton labels. Recall that an open string at oscillation level n with both ends on a stack of N coincident D-branes is characterised by

$$|k; n; a\rangle = \sum_{i,j=1}^{N} |k; n; ij\rangle \lambda^a_{ij}, \qquad \lambda^a_{ij} = (\lambda^a_{ij})^\dagger, \qquad a = 1, \dots, N^2. \tag{5.80}$$

The N^2 hermitian $N \times N$ matrices are in the adjoint representation of $U(N)$. In computing the scattering amplitude of r open strings we must include, for each summand with a given ordering of the vertices, the trace over the Chan-Paton matrices,

$$\mathrm{Tr}(\lambda_{a_1}\,\lambda_{a_2}\,\ldots\lambda_{a_r}). \tag{5.81}$$

Following this procedure the 4-tachyon amplitude amplitude can be computed, mutatis mutandis, in a manner similar to its closed string counterpart. The resulting **Veneziano amplitude** takes the form

$$\boxed{S(k_1,k_2,k_3.k_4) = \frac{2ig_o^2}{\alpha'}(2\pi)^{26}\delta^{(26)}(\sum_i k_i)\Big(\frac{\Gamma(-\alpha's-1)\Gamma(-\alpha't-1)}{\Gamma(-\alpha's-\alpha't-2)} + (t\leftrightarrow u) + (s\leftrightarrow t)\Big).} \tag{5.82}$$

This will be derived as an exercise.

The open string effective action

One can now systematically compute the scattering amplitudes involving the level-one states, i.e. the gauge bosons along the D-brane, and the tachyonic ground state as well the massive string excitations. Taking into account the ordering prescription for the vertex operators and the Chan-Paton structure one finds explicitly that the interactions agree with the structure of a $U(N)$ gauge theory with extra matter in suitable representations of the gauge group. In particular, the tachyonic and positive-mass excitations from open strings with both ends on the stack of D-branes give rise to states in the adjoint representation of $U(N)$.
E.g. the scattering amplitude involving one gauge boson with momentum k_1 and polarisation vector ζ_1 and two open tachyons with momenta k_2 and k_3 takes the form

$$S(k_1,\zeta_1;k_2,k_3) \simeq (2\pi)^{(26)}\delta^{(26)}(k_1+k_2+k_3)\,\mathrm{Tr}(\lambda_1[\lambda_2,\lambda_3]). \tag{5.83}$$

This corresponds to the cubic interaction vertex derived from an effective action in the 26-dimensional ambient spacetime of the form

$$-\int_{\mathbb{R}^{1,25}}\frac{1}{2}\mathrm{Tr}(D_\mu\varphi\,D^\mu\varphi) \tag{5.84}$$

where

$$\varphi \equiv \varphi_a\lambda^a \tag{5.85}$$

describes the tachyon in the adjoint of $U(N)$ with covariant derivative

$$D_\mu\varphi = \partial_\mu - i[A_\mu,\varphi], \qquad A_\mu = (A_\mu)_a\lambda^a. \tag{5.86}$$

Similarly one can reconstruct the other interaction terms. The spacetime effective action for the gauge boson and the tachyonic ground state at string tree-level and to first order in the momenta takes the form

$$\frac{2\alpha'}{g_o^2}\int_{\mathbb{R}^{1,25}} -\frac{1}{2}\mathrm{Tr}(D_\mu\varphi\,D^\mu\varphi) + \frac{1}{2\alpha'}\mathrm{Tr}(\varphi^2) + \frac{1}{3}\sqrt{\frac{2}{\alpha'}}\mathrm{Tr}(\varphi^3) - \frac{1}{4}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}). \tag{5.87}$$

5.5 1-loop amplitudes

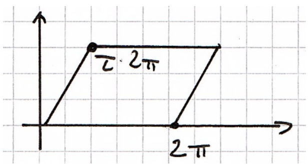
5.5.1 Oriented closed theory

Consider the closed oriented n-point function at one-loop in perturbation theory. It is described by an amplitude on $\Sigma = T^2$. Recall that the torus is topologically $T^2 = S^1 \times S^1$ and can be parametrised by the doubly-periodic flat real coordinates

$$(\sigma^1,\, \sigma^2) \cong (\sigma^1,\, \sigma^2) + 2\pi(m,\, n) \qquad m,\, n \in \mathbb{Z}. \tag{5.88}$$

We now need to revisit the moduli space and conformal group of T^2 in order to arrive at the correct measure for the gauge fixed S-matrix.

1) **Moduli space**
We have already pointed out that the metric $ds^2 = |d\sigma^1 + \tau d\sigma^2|^2$ can be expressed in terms of one complex modulus $\tau = \tau_1 + i\tau_2$. A useful representation of T^2 is a as a lattice in the complex plane.



The flat complex coordinates on the complex lattice are $w = \sigma^1 + \tau\sigma^2, \bar{w} = \sigma^1 + \bar{\tau}\sigma^2$.

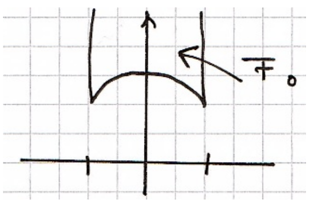
From its representation as a lattice one can deduce that a T^2 is shape-invariant under a PSL$(2,\mathbb{Z})$ transformation

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1, \quad (a,b,c,d) \cong (-a,-b,-c,-d) \in \mathbb{Z}. \tag{5.89}$$

The set of PSL$(2,\mathbb{Z})$ transformations define the modular group of the torus. In fact these are generated by the transformations

$$T : \tau \to \tau + 1, \qquad S : \tau \to -\frac{1}{\tau}. \tag{5.90}$$

The fundamental domain F_0, i.e. the domain of inequivalent values of τ after modding out by the action of the modular group, is:



$$F_0 = \{\tau| \qquad |\tau| \geq 1, -\tfrac{1}{2} \leq \Re(\tau) \leq \tfrac{1}{2}, \Im(\tau) > 0\} \,.$$

Furthermore the torus enjoys the additional discrete symmetry $w \to -w$. For further details of these assertions see Assignment 10.

Consequences for path integral: Following the general discussion of the gauge-fixed S-matrix we insert a factor of

$$\mathcal{B}\bar{\mathcal{B}} = \frac{1}{4\pi}\,(b|\partial_\tau \hat{g}) \cdot \frac{1}{4\pi}\,\left(\bar{b}|\partial_{\bar{\tau}}\hat{g}\right) \tag{5.91}$$

and integrate over the amplitude over $\frac{1}{2}\int_{F_0} d^2\tau$. The factor $\frac{1}{2}$ is due to the extra $\mathbb{Z}_2$ symmetry $w \to -w$.

2) **Conformal Killing vectors:**
The conformal group of T^2 is given by $U(1) \times U(1)$. This allows us to fix one closed vertex operator at, say, w_1. In addition we must insert a factor of $c, \bar{c}$ at w_1. Altogether the amplitude therefore contains a factor of

$$: c\bar{c}V_1(w_1, \bar{w_1}) : \tag{5.92}$$

Alternatively we can perform the integral over the position w_1 of the vertex operator and divide by the volume of $U(1) \times U(1)$, which is finite,

$$\mathrm{Vol}(U(1) \times U(1)) = \mathrm{Vol}(T^2) = (2\pi)^2 \tau_2.$$

If we follow this prescription the amplitude contains the integral

$$\int_{F_0} \frac{dw_1 d\bar{w}_1}{2(2\pi)^2 \tau_2} : c\bar{c}V_1(w_1, \bar{w_1}) : \; . \tag{5.93}$$

The factor of $\frac{1}{2}$ is due to the conversion from complex to real coordinates.
Altogether the n-point amplitude takes the form

$$\boxed{S^{(n)}_{T^2} = \int_{F_0} \frac{d\tau d\bar{\tau}}{4\tau_2} \left\langle \frac{1}{4\pi^2} \mathcal{B}\bar{\mathcal{B}} \int d^2 w_1 : c\bar{c}V_1(w_1, \bar{w_1}) : \prod_{i=2}^{n} \frac{1}{2} \int dw_i d\bar{w}_i V_i(w_i, \bar{w}_i) \right\rangle_{T^2} .} \tag{5.94}$$

Computation of the vacuum amplitude

To keep things as simple as possible we now compute the 1-loop vacuum amplitude

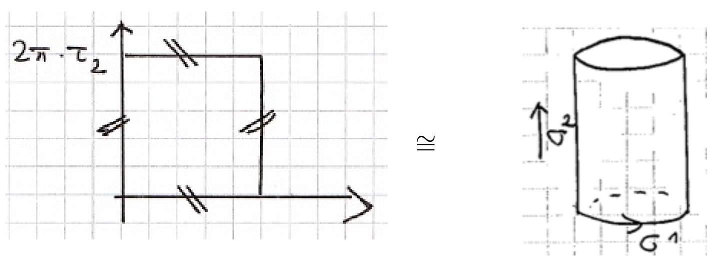
$$Z_{T^2} := \int_{F_0} \frac{d\tau d\bar{\tau}}{4\tau_2} \langle 1 \rangle_{\tau, \otimes X^i} \cdot \langle \text{ ghost insertions } \rangle_{\tau,(bc)} . \tag{5.95}$$

As we will see, this amplitude has a very important physical interpretation. It is furthermore sufficient to demonstrate the key properties of string-loop amplitudes.
The vacuum amplitude splits into a correlator in the X- and the bc-CFT and an integral over the moduli space. Consider first the X-CFT piece,

$$Z(\tau) \equiv \langle 1 \rangle_{\tau, \otimes_{i=1}^{d} X^i} , \tag{5.96}$$

where we work in d-dimensional target space.

i) First suppose that $\tau = i\tau_2$, corresponding to a rectangular torus, which can be viewed as a cylinder of length $(2\pi\tau_2)$ with both ends identified.



The path integral $\int \mathcal{D}X e^{-S_X}$ on a worldsheet given by the above cylinder with both ends identified admits the following interpretation:
Start from the vacuum, create a state, evolve it by Euclidean time $2\pi\tau_2$ and identify in- and out-state. In canonical formalism, this amounts to evaluating

$$
\begin{aligned}
Z(\tau = i\tau_2) &= \sum_{\text{all internal } n} \langle 0| \underbrace{|0\rangle\langle n|}_{\substack{\text{identify w.}\\ |0\rangle}} \underbrace{e^{-2\pi\tau_2 H}}_{\text{evolution}} \underbrace{|n\rangle\langle 0|}_{\substack{\text{create } |n\rangle \\ \text{from } |0\rangle}} |0\rangle \\
&= \sum_n \langle n| e^{-2\pi\tau_2 H} |n\rangle \equiv \mathrm{Tr} e^{-2\pi\tau_2 H}. \qquad (5.97)
\end{aligned}
$$

This is an example of a general principle in quantum field theory:

The path integral in compactified Euclidean time yields the partition function at temperature $T = \frac{1}{2\pi\tau_2}$.

ii) To generalise this to a torus with modulus $\tau = \tau_1 + i\tau_2$ we must in addition translate the fields by $2\pi\tau_1$ in spatial direction,

$$Z(\tau) = \langle 1 \rangle_{\tau, \otimes X^i} = \sum_n \langle n| e^{2\pi i \tau_1 P - 2\pi\tau_2 H} |n\rangle$$

or more compactly

$$\boxed{Z(\tau) = \mathrm{Tr} e^{2\pi i\tau_1 P - 2\pi\tau_2 H}.} \qquad (5.98)$$

With the relations

$$P = \frac{2\pi}{\ell}(L_0 - \bar{L}_0), \qquad H = \frac{2\pi}{\ell}(L_0 + \bar{L}_0 - \frac{c+\bar{c}}{24}) \qquad \text{with} \qquad \frac{2\pi}{\ell} \equiv 1 \qquad \text{from now on,}$$

this can written as

$$Z(\tau) = (q\,\bar{q})^{-\frac{d}{24}}\ \mathrm{Tr} q^{L_0}\,\bar{q}^{\bar{L}_0}, \qquad q = e^{2\pi i\tau}. \qquad (5.99)$$

We now express L_0 in terms of the momentum operator and the number operator,

$$L_0 = \frac{\alpha'}{4}k^2 + N, \qquad \bar{L}_0 = \frac{\alpha'}{4}k^2 + \bar{N}, \qquad (5.100)$$

and express the trace as an integral over the momentum modes times the trace over the oscillator part of a string state,

$$\mathrm{Tr}\, q^{L_0}\,\bar{q}^{\bar{L}_0} = \int \frac{d^d k}{(2\pi)^d} \underbrace{\mathrm{Tr}'}_{\substack{\text{trace over}\\ \text{oscillators}}} \langle k; N, \bar{N}|\, (q\bar{q})^{\frac{\alpha'}{4}k^2} q^N \bar{q}^{\bar{N}}\, |k; N, \bar{N}\rangle. \qquad (5.101)$$

The inner product over the momentum state gives a factor of the spacetime volme $\langle k|k\rangle = \delta^{(d)}(k-k) \equiv V_d$, and with $(q\,\bar{q})^{\frac{\alpha'}{4}k^2} = (\exp(4\pi\tau_2))^{\frac{\alpha'}{4}k^2}$ we find

$$Z(\tau) = V_d\,(q\,\bar{q})^{-\frac{d}{24}} \int \frac{d^d k}{(2\pi)^d}\, e^{-\pi\tau_2\alpha' k^2} \cdot \mathrm{Tr}' q^N\,\bar{q}^{\bar{N}}. \qquad (5.102)$$

The computation of the trace was performed in detail on Assignment 6, to which we refer for details. The result is

$$\mathrm{Tr}' q^N \bar{q}^{\bar{N}} = \prod_{n=1}^{\infty} (1-q^n)^{-d} (1-\bar{q}^n)^{-d}. \tag{5.103}$$

We finally perform a Wick rotation $k^0 \to ik^0$ to render the integral $\int d^dk$ finite and perform the Gaussian integration. This yields the final expression for the partition function $Z(\tau)$,

$$\boxed{\begin{aligned} Z(\tau) &= iV_d\,(Z_X(\tau))^d, && (5.104)\\ Z_X(\tau) &= (4\pi^2\alpha'\tau_2)^{-\frac{1}{2}} \cdot |\eta(\tau)|^{-2} \qquad \text{with} && (5.105)\\ \eta(\tau) &= q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \qquad \text{the Dedekind } \eta \text{ function.} && (5.106) \end{aligned}}$$

The next task is to compute the one-loop integral of the ghost insertions. For reasons of time we do not present this computation here. It can be found e.g. in [P], Chapter 7, p. 212. We merely quote the final result: The ghost sector yields a factor of $|\eta(\tau)^2|^2$. Setting now $d = 26$ the one-loop amplitude is

$$\boxed{Z(\tau) = iV_{26} \int_{F_0} \frac{d\tau d\bar{\tau}}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-13} \cdot |\eta(\tau)|^{-48}.} \tag{5.107}$$

Note:

- The ghost contribution cancels the contribution from the 2 non-transverse polarisations

$$|\eta(\tau)|^{-48} = |\underbrace{\eta(\tau)^{-26}}_{X} \cdot \underbrace{\eta(\tau)^2}_{bc}|^2. \tag{5.108}$$

  This demonstrates nicely how the ghost sector restores unitarity by cancelling the non-physical excitations. The underlying reason for this cancellation is the anti-commuting nature of the ghost fields. Commuting and and anti-commuting fields contribute with opposite sign in the loop, as is familiar from QFT with bosons and fermions.

- An important property of Z_{T^2} is **modular invariance**, i.e. invariance under $PSL(2,\mathbb{Z})$ transformations $\tau \to \frac{a\tau+b}{c\tau+d}$ of the torus. The underlying reason is that the Dedekind-function is an example of a **modular form**, i.e. a function of τ with definite transformation under $PSL(2,\mathbb{Z})$. Without proof we state that under S and T-transformations, which generate the modular group, the Dedekind function transforms as

$$\eta(-\frac{1}{\tau}) = (-i\tau)^{\frac{1}{2}}\,\eta(\tau)\ , \qquad \eta(\tau+1) = e^{i\frac{\pi}{12}}\,\eta(\tau). \tag{5.109}$$

  Modular invariance of Z_{T^2} follows because $\frac{d\tau d\bar{\tau}}{\tau_2^2}$ and $\tau_2|\eta(\tau)|^4$ are individually invariant, as can be checked with the above and the $PSL(2,\mathbb{Z})$ transformation

$$\tau_2 \to \frac{\tau_2}{|c\tau+d|^2}, \qquad d^2\tau \to \frac{d^2\tau}{|c\tau+d|^4}. \tag{5.110}$$

  This is an important consistency check. Note that modular invariance holds already for $Z_X(\tau)$ defined above.

- For a general CFT the torus amplitude can be written as

$$Z_{T^2} = iV_d \int_{F_0} \frac{d\tau d\bar{\tau}}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-\frac{d}{2}} \underbrace{\sum_{i\in\mathcal{H}^\perp} q^{N_i-1}\bar{q}^{\bar{N}_i-1}}_{\substack{\text{yields more general modular}\\ \text{forms } (\Theta-\text{functions})}} \quad . \tag{5.111}$$

Modular invariance is an important consistency condition for the defining data of a CFT.

Ultraviolet finiteness

To appreciate the properties of the stringy 1-loop amplitude we compare Z_{T^2} to its point particle analogue. The partition function of a field theory describing a particle of mass m is given by the sum over all particle paths with the topology of a circle,

$$Z_{S^1}(m^2) = V_d \int \frac{d^dk}{(2\pi)^d} \int_0^\infty \frac{dl}{2l} e^{-\frac{1}{2}l\cdot(k^2+m^2)}. \tag{5.112}$$

This is easily understood from our derivation of the stringy partition function once we take into account that

$$\begin{aligned} \frac{1}{2}(k^2+m^2) \qquad & \text{corresponds to the Hamiltonian,}\\ l \qquad & \text{describes the circumference of } S^1 \equiv \text{ the compact Euclidean time,}\\ \int_0^\infty \frac{dl}{2l} \qquad & \text{takes into account division by the volume of Diff}(S^1) \text{ and of the}\\ & \mathbb{Z}_2 \text{ symmetry} \quad x\to -x. \end{aligned}$$

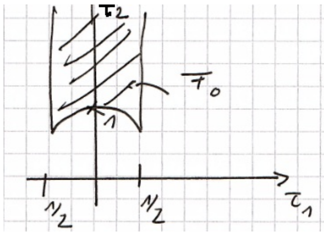
In fact we can bring this into a form very similar to the stringy expression Z_{T^2}. String theory contains an infinite tower of states of mass

$$m_i^2 = \frac{2}{\alpha'}(N_i + \bar{N}_i - 2), \qquad N_i = \bar{N}_i. \tag{5.113}$$

The field theory analogue of Z_{T^2} therefore corresponds to summing $Z_{S^1}(m_i^2)$ over the string tower, where we include the level-mathcing condition via $\delta_{N\bar{N}} = \int_{-\pi}^{\pi} \frac{d\Theta}{2\pi} e^{i(N-\bar{N})\Theta}$. In this spirit we find

$$\begin{aligned} \sum_{i\in\mathcal{H}^\perp} Z_{S^1}(m_i^2) &= iV_d \int_0^\infty \frac{dl}{2l} \int_{-\pi}^{\pi} \frac{d\Theta}{2\pi} (2\pi l)^{-\frac{d}{2}} \cdot \sum_i e^{-\frac{l}{\alpha'}(N_i+\bar{N}_i-2)+i(N_i-\bar{N}_i)\Theta},\\ &= iV_d \int_R \frac{d\tau d\bar{\tau}}{4\tau_2} (4\pi^2\alpha'\tau_2)^{-\frac{d}{2}} \cdot \sum_i q^{N_i-1}\bar{q}^{\bar{N}_i-1} \qquad (5.114)\\ \text{with } 2\pi\tau &= \Theta + i\frac{l}{\alpha'}, \qquad q = e^{2\pi i\tau}, \qquad \text{and integration region } R: \ \tau_2 \geq 0, \ |\tau_1| \leq \frac{1}{2}. \end{aligned}$$

We notice a crucial difference of this field theory approach to the string theoretic result:
In Z_{T^2} we integrate over the fundamental domain $F_0: \ |\tau| \geq 1, \ |\tau_1| \leq \frac{1}{2}$, which is a subregion of the particle integration domain R.



The importance of this is the following: The limit $\tau_2 \to 0$ describes the ultraviolet (UV) regime as in this limit the Euclidean time becomes very small, corresponding to processes at high energies. Indeed the integral over τ is divergent in this regime. The point is now that this UV divergent region is absent in string theory due to modular invariance of the torus, but present in field theory. Thus we have established the crucial result:

Z_{T^2} is UV-finite in string theory. Modular invariance acts as an intrinsic UV cutoff and removes the UV divergence of analogous point particle theories.

Comments:

- Modular invariance is therefore the secret behind UV finiteness of the 1-loop amplitude. While demonstrated only for the simplest example of a vacuum amplitude, the mechanism of cutting out the UV divergent regent continues to work for n-point functions.
- To date, UV finiteness has been proven rigorously at 2-loop level in superstring theory.[7] Beyond that, technicalities concerning the (super)moduli space of higher genus Riemann surfaces hinder a general proof, but there are no indications that the situation changes at higher loop order. In fact, using the pure spinor formalism examples of UV finite 5-loop amplitudes have been computed. This serves as evidence for the wildly accepted conjecture of UV finiteness of string theory.
- It is important to appreciate that superstring theory is to date the only quantum theory of gravity *and* Yang-Mills-theory that is believed to be UV-finite.[8]

Infrared behaviour

The IR regime corresponds to the limit $\tau_2 \to \infty$, in which the torus (or cylinder with both ends identified) becomes very long. In this regime we can expand the inegral over the η-function as

$$\begin{aligned}
\int^\infty d^2\tau |\eta(\tau)|^{-48} &= \int^\infty d^2\tau\, (q\bar{q})^{-1} |\prod_{n=1}^\infty (1-q^n)|^{-48} \\
&\cong \int_{-\frac{1}{2}}^{\frac{1}{2}} d\tau_1 \int^\infty d\tau_2\, (q\bar{q})^{-1} (1+24q+\ldots)(1+24\bar{q}+\ldots) \\
&\cong \int^\infty d\tau_2 \left((q\bar{q})^{-1} + 24^2 + \ldots \right),
\end{aligned}$$

[7] For the state of the art in this direction consult e.g. the review by D'Hoker and Phong, http://arXiv.org/pdf/hep-th/0211111.

[8] There are certainly nontrivial UV finite Yang-Mills theories such as $\mathcal{N}=4$ Super-Yang-Mills, and there are speculations about the possible UV-finiteness of $\mathcal{N}=8$ supergravity, but none of these describes both particle interactions and gravity at the same time.

where terms of the type $q + \bar{q}$ vanish upon performing the integral over the τ_1-coordinate. Thus we deduce the following IR-behaviour

$$iV_{26} \int^{\infty} \frac{d\tau_2}{2\tau_2} (4\pi^2 \alpha' \tau_2)^{-13} \left[\underbrace{e^{4\pi\tau_2}}_{\text{tachyon}} + \underbrace{24^2}_{\substack{\text{massless}\\ \text{modes}}} + \underbrace{\ldots}_{\substack{\text{massive}\\ \text{modes}}} \right] \tag{5.115}$$

- The first term is divergent. This divergence, however, is an artefact due to the appearance of the tachyon. Since the tachyon will be removed in the final superstring theory we can safely ignore this nuisance.
- The next term is due to the massless states. The long-distance behaviour is therefore governed by **massless** states. Their contribution to Z_{T^2} is finite.

Relation to vacuum energy:

Finally let us give the physical intepretation of the partition function. Consider again a particle theory with mass m. As we have seen $Z_{S^1}(m^2)$ computes the amplitude of a single particle running in one loop. The total vacuum amplitude, on the other hand, contains all disconnected vacuum loops, weighted with combinatorial factors due to permutation symmetry,

$$Z_{S^1}(m^2) \qquad + \frac{1}{2} \;(Z_{S^1}(m^2))^2 \qquad + \frac{1}{3!}\; (Z_{S^1}(m^2))^3 \qquad + \ldots$$

The total vacuum amplitude is therefore

$$\Rightarrow Z_{vac}(m^2) = e^{Z_{S^1}(m^2)}.$$

On the other hand, in canonical formalism

$$Z_{vac}(m^2) = \langle 0| \, e^{-iHT} \, |0\rangle = e^{-i\rho_0 V_d},$$

where T is the time in spacetime, and ρ_0 the vacuum energy density. Thus

$$\boxed{\rho_0 = \frac{i}{V_d} Z_{S^1}(m^2).} \tag{5.116}$$

This analogy between $Z_{S^1}(m^2)$ and Z_{T^2} suggests therefore that also in string theory

$$\boxed{\rho_0 = \frac{i}{V_d} Z_{T^2}.} \tag{5.117}$$

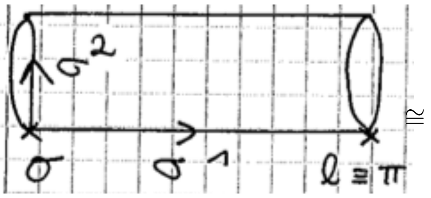
Thus Z_{T^2} computes the 1-loop correction to the vacuum energy, i.e. the 1-loop correction of the spacetime cosmological constant. As we have seen, in the bosonic string this vacuum energy is finite (if we ignore the artifical tachyon) and of order the string scale. On the other hand, in the supersymmetric superstring theory one finds

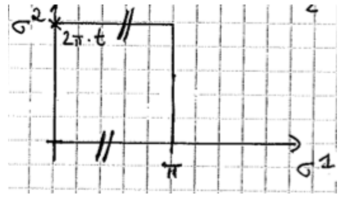
$$Z_{T^2} \equiv 0$$

due to an exact cancellation between fermions and bosons running in the loop.

5.5.2 Oriented open theory

The open string oriented one-loop amplitudes are defined on the cylinder C_2, which can be represented as a strip of length $2\pi t$ with two boundaries at $\sigma^1 = 0, \ell$ and the line $\sigma^2 = 0$ and $\sigma^2 = 2\pi t$ identified. We will set $\ell \equiv \pi$ in the sequel.





- The cylinder C_2 is described by one real modulus t. Unlike the torus, there is no analogue of the modular group action $PSL(2,\mathbb{Z})$. Thus the modulus can take values in the full regime $0 \leq t \leq \infty$.
- The group of conformal Killing transformations consists of translations parallel to the σ^2-axis, i.e. such that they preserve the boundary of the strip at $\sigma^1 = 0, \pi$. Its volume is $2\pi t$.

The vacuum amplitude

The computation is very similar to the computation on the torus.

- For simplicity we consider the situation of a stack of N D-branes filling the entire 26-dimensional space, corresponding to (NN) boundary conditions in all dimensions. Generalisations to lower-dimensional branes are simple.
- As in the closed sector, the ghost contribution turns out to cancel the oscillator trace of precisely two non-transverse directions. With this in mind the amplitude is

$$\begin{aligned} Z_{C_2} &= \int_0^\infty \frac{dt}{2t} \mathrm{Tr}\, e^{-2\pi t (L_0 - \frac{c}{24})} && (5.118) \\ &= i V_{26} \int_0^\infty \frac{dt}{2t} (8\pi^2 \alpha' t)^{-13} \mathrm{Tr}'_{\otimes_{i=1}^{24} X^i} q^{L_0 - \frac{1}{24}}. && (5.119) \end{aligned}$$

For a stack of N concindent D-branes filling all of spacetime this is

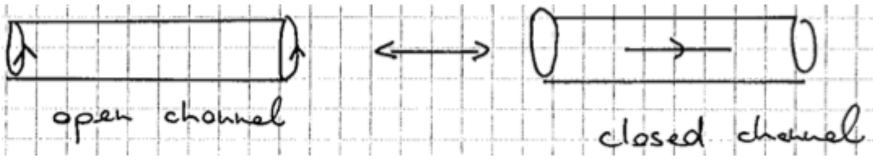
$$Z_{C_2} = i V_{26}\, N^2 \int_0^\infty \frac{dt}{2t} (8\pi^2 \alpha' t)^{-13} \eta(it)^{-24}. \qquad (5.120)$$

- The IR-limit, $t \to \infty$, works out like for the closed string: The only IR-divergent term is due to the open tachyon, which is absent in the eventual superstring theory.
- The UV-limit $t \to 0$, on the other hand, is different from the closed string sector: Unlike on the torus, the UV-divergent region is not absent from the integral because there is no analogous modular group action serving as an intrinsic regulator.

Thus, it might seem that we do face a UV divergence, contrary to our previous claim that string theory is UV finite. To see why the divergence as $t \to 0$ is not in contradiction with UV finiteness we need to discuss the **worldsheet duality between the open and closed string channel**.

Open versus closed channel

- In the UV-divergent limit $t \to 0$, the cylinder is infinitely long.
- The remarkable insight is the following: We can either view the long cylinder as describing an open string stretching between the boundaries at $\sigma^1 = 0, \pi$ and running in the loop described by the Euclidean time σ^2. Or, alternatively, we may interpret the annulus as a closed string propagating at tree-level from the left to the right. The two interpretations of the cylinder are referred to as open and closed string channel.



- Technically, the two viewpoints are related by interchanging the role of the Euclidean time and the spatial coordinate on the worldsheet. From our analysis of $PSL(2,\mathbb{Z})$ transformations of the torus on Assignment 10 we recall that an S-duality transformation $\tau \to -\frac{1}{\tau}$ exchanges the coordinates σ^1 and σ^2. The same applies to the cylinder with it taking the role of τ. Including a conventional rescaling of the spatial coordinate the transition from the open to the closed string channel is accomplished by

$$t \longrightarrow s = \frac{\pi}{t}. \tag{5.121}$$

With the help of the transformation of the Dedekind function

$$\eta(it) = t^{-\frac{1}{2}}\eta\Big(\frac{i}{t}\Big) = \Big(\frac{s}{\pi}\Big)^{\frac{1}{2}}\eta\Big(\frac{is}{\pi}\Big) \tag{5.122}$$

the annulus amplitude in closed string channel is

$$Z_{C_2} = iV_{26}\, N^2 \,\frac{1}{2\pi(8\pi^2\alpha')^{\frac{1}{2}}}\int_0^\infty ds\,\eta\Big(\frac{is}{\pi}\Big)^{-24}. \tag{5.123}$$

- The UV limit $t \to 0$ in the open channel has translated in the IR limit $s \to \infty$ of the closed channel. This describes a closed string tree-level process with the string propagating over long Euclidean time. Thus we have reinterpreted the UV-divergence as an IR-divergence. This is in fact a general feature of string amplitudes:

All UV divergencies in string amplitudes can be reinterpreted as IR divergencies of dual diagrams.

- In fact, we can make the propagation of the closed strings visible in the limit $s \to \infty$ by expanding

$$\eta\Big(\frac{is}{\pi}\Big)^{-24} = \underbrace{e^{2s}}_{\text{tachyon}} + \underbrace{24}_{\text{massless}} + \mathcal{O}(e^{-2s}). \tag{5.124}$$

The tachyonic term is again an artifact of the bosonic theory. Of importance is the second term. It shows that **the IR divergence is due to the exchange of massless closed string states at zero momentum.**

Tadpoles in string theory and field theory

A diagram where a state - here a closed string state - is created from the vacuum is called a tadpole.

- The IR divergence is the $\frac{1}{k^2}$ divergence from joining two tadpoles by the propagator of a massless state with $k^2 = 0$.

- The tadpole diagram as such is computed by the 1-point function of a single closed string operator inserted in the interior of the disk. Note that such a diagram is in general non-zero even though the 1-point function on the sphere and the open 1-point function on the disk vanish - see the remark after (5.28).

- In QFT a tadpole diagram results from a term linear in the field in the lagrangian as this is what gives rise to a single field vertex. **Tadpoles therefore signal an instability of the vacuum**, which is defined as the locus in field space that satisfies $V'(\phi) = 0$. For example in the presence of a tadpole for a bosonic field,

$$S = \int -\frac{1}{2}(\partial\phi)^2 + \Lambda\phi \tag{5.125}$$

the locus $\phi = 0$ does not correspond to the true vacuum. If we set out at $\phi = 0$ the field configuration will change. In the presence of higher terms there may be a new vacuum at $\phi \neq 0$ and the theory will flow to that correct vacuum. In the above action, by contrast, there is no such vacuum and the theory is entirely unstable.

- To see how to deal with the tadpole in string theory we need to include also unoriented worldsheets.

5.5.3 Non-oriented vacuum amplitudes and tadpole cancellation

So far we have been analysing mainly the oriented closed and open string. The unoriented theory, aka orientifold theory, was introduced on Assignment 6 as the quotient of the oriented theory by worldhseet parity $\Omega : \sigma \to -\sigma$. In the orientifold theory, we must include in addition amplitudes on non-oriented Riemann surfaces.

- For the closed and open theory at $\chi = 0$, respectively, these are the Klein bottle and the Möbius strip. For brevity we can only mention these amplitudes here and refer the interested reader e.g. to [P], Chapter 7 for details on the geometry of these un-oriented Riemann surfaces and the computation of the vacuum amplitudes on them.

- Suffice it here to note the following important result: Both the Klein bottle and the Möbius strip vacuum amplitude suffer from UV/IR-divergencies in form of tadpoles. Summing over

Klein bottle, annulus and Möbius amplitude the total IR divergence in the closed channel can be written as a perfect square and is proportional to

$$(2^{13} \pm N)^2 \int_0^\infty ds, \tag{5.126}$$

where N is the number of spacetime-filling coincident D-branes and the two signs depend on the details of the orientifold projection.

- This means that the theory is tadpole free if we choose the projection such that the lower sign arises and take $N = 2^{13} = 8192$. With this orientifold projection the gauge group on the spacetime-filling branes is

$$SO(8192). \tag{5.127}$$

Tadpole cancellation as a stringy consistency condition

One can read this result as follows:

- A priori, any number N of spacetime filling D-branes seems to be allowed. However, precisely if $N = 8192$ the theory is tadpole free, meaning that the perturbative string vacuum is indeed stable (up to the annoying tachyon which, as we reiterate, doesn't bother us much for the reasons stated.)
- While for the bosonic theory is merely a toy model, the same logic will fix, in the superstring theory, the gauge group of the 10-dimensional open plus closed theory to be $SO(32)$ (more precisely $\mathrm{Spin}(32)/\mathbb{Z}_2$).
- Upon compactification of some of the 26 or, respectively, 10 dimensions more general gauge groups are consistent with the requirement of tadpole cancellation, but not every anomaly-free gauge configuration that is allowed in field theory is fully consistent within string theory. Tadpole cancellation implies absence of gauge anomalies in the effective theory, but it is stronger. It is only one example of such stringy consistency conditions with no analogue in field theory. This is because string theory is indeed a theory of quantum gravity, and the consistent coupling of a gauge sector to gravity entails further constraints.

5.6 Strings on curved backgrounds

5.6.1 The non-linear σ-model

So far we have focused on string propagation in flat ambient spacetime by coupling the string fields in the Polyakov action to the flat metric $\eta_{\mu\nu}$ of $\mathbb{R}^{1,25}$,

$$S_P = \frac{1}{4\pi\alpha'} \int d^2\xi \sqrt{h}\, h^{ab}\, \partial_a X^\mu\, \partial_b X^\nu\, \eta_{\mu\nu}. \tag{5.128}$$

To extend the theory to string propagation in a curved target spacetime with metric $G_{\mu\nu}(X)$ we generalise the Polyakov action to

$$S_\sigma = \frac{1}{4\pi\alpha'} \int d^2\xi\, \sqrt{h}\, h^{ab}\, \partial_a X^\mu(\xi)\, \partial_b X^\nu(\xi)\, G_{\mu\nu}(X(\xi))\,. \tag{5.129}$$

The curved metric $G_{\mu\nu}(X(\xi))$ can be understood as a coherent state of gravitons describing the fluctuations of the metric around $\eta_{\mu\nu}$. To appreaciate this let us consider as an analogy a laser field in quantum optics. We set out in perturbative QED to quantize the electromagnetic vacuum and describing its fluctuations by photons. A coherent state of these vacuum fluctuations represents a non-trivial field configuration, in this case a laser field. Likewise, the gravitational field as encoded in $G_{\mu\nu}(X(\xi))$ can be viewed as a coherent excitation of gravitons. This statement can be made more precise as follows:
Suppose $G_{\mu\nu}(X)$ describes a close-to-flat metric so that we can expand

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + \chi_{\mu\nu}(X). \tag{5.130}$$

Inserted into the string theoretic path-integral this becomes

$$\exp(-S_\sigma) = \exp(-S_P)\Big(1 - \frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X^\nu \chi_{\mu\nu}(X(\xi)) + \dots\Big). \tag{5.131}$$

Working for simplicity on a flat worldsheet, the effect of the curved metric is captured by insertion and exponentiation of a graviton vertex operator

$$V = \frac{1}{2\pi\alpha'}\int d^2z\, \partial X^\mu \bar{\partial} X^\nu \chi_{\mu\nu}(X(\xi)). \tag{5.132}$$

For comparison, previously we have only considered the special case

$$\chi_{\mu\nu} = g_c 2\pi\alpha' \zeta_{\mu\nu}\, e^{ikX} \tag{5.133}$$

corresponding to a plane graviton wave.
A general deviation $\chi_{\mu\nu}$ from the flat metric describes a superposition of such graviton waves. This indeed justifies the general lesson:

> The string propagates in a background described by a coherent state of its own massless fluctuations!

Once we have realised that propagation in curved spacetime corresponds to coupling the string fields to the graviton sector of its massless excitations, we are naturally lead to including also the other massless fields. In the closed sector these are

$$\begin{aligned} B_{\mu\nu} = B_{[\mu\nu]}, &\qquad \text{the Kalb-Ramond B-field, and} \\ \Phi, &\qquad \text{the dilaton.} \end{aligned}$$

A natural generalisation of the closed Polyakov action is therefore the **closed σ-model action**

$$\boxed{S_\sigma = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi\sqrt{h}\Big\{\big(h^{ab}G_{\mu\nu}(X) + i\epsilon^{ab}B_{\mu\nu}\big)\,\partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)}\Phi(X)\Big\}}. \tag{5.134}$$

- The factor ϵ^{ab} is the antisymmetric worldsheet two-tensor. It is required by antisymmetry of $B_{\mu\nu}$.
- The factor of i follows by Wick rotation to Euclidean worldsheet signature because ϵ^{ab} always involves one time derivative.

- $R^{(2)}$ is the Ricci scalar on the worldsheet.
  The last term is thus a generalization of the topological term $\frac{1}{4\pi}\int d^2\xi\sqrt{h}R^{(2)}\lambda = \lambda\chi$ with $\lambda = \Phi(X)$. Most importantly, the strings coupling is therefore really given by
  $$g_s = e^{\Phi} = e^{\Phi(X)}, \text{ i.e. the coupling "constant" is dynamical.} \tag{5.135}$$

Lesson:

In string theory there are no dimensionless coupling constants. All couplings are determined by VEVs (vacuum expectation values) of dynamical fields.

Symmetries of the σ-model action:

- Poincaré invariance in spacetime, which holds for S_P, is replaced by general covariance.
- In addition a new symmetry arises: A transformation of the Kalb-Ramond 2-form $B = \frac{1}{2}B_{\mu\nu}dx^\mu dx^\nu$ of the type
  $$B \to B + d\Lambda, \qquad \text{i.e.} \qquad B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\Lambda_\nu(X) - \partial_\nu\Lambda_\mu(X), \tag{5.136}$$
  leaves the action S_σ invariant. Under this symmetry the **3-form field strength** $H = dB$ with components
  $$H_{\alpha\beta\gamma} = \partial_\alpha B_{\beta\gamma} + \partial_\beta B_{\gamma\alpha} + \partial_\gamma B_{\alpha\beta} \tag{5.137}$$
  is invariant. As anticipated already around eq. (3.116) this generalises the concept of an abelian gauge symmetry to higher-rank gauge potentials.[9]

  The string therefore carries "charge" under this symmetry. Indeed the coupling
  $$\int d^2\xi\sqrt{h}\epsilon^{ab}B_{\mu\nu}\partial_a X^\mu\partial_b X^\nu \tag{5.138}$$
  generalises the coupling $\int d\tau\partial_\tau X^\mu A_\mu$ of a point particle to a gauge field.

Comment on open string σ-model:

For open strings the transformation (5.136) leads to boundary terms. These vanish if we include a coupling of the background gauge field $A_\mu(X)$ to the string endpoints along with a suitable transformation of A_μ. This is discussed in detail in Assignment 11, to which we refer for more details.

5.6.2 α'-Expansion and Conformal Invariance

The string action for propagation in flat space leads to a free theory on the worldsheet, which, as we have seen, can be solved exactly. By contrast, the non-linear σ-model is an interacting theory.
To analyze the worldsheet interaction terms we ignore for the time being the terms describing the worldsheet coupling to $B_{\mu\nu}$ and Φ and focus on
$$S = \frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{h}h^{ab}G_{\mu\nu}(X)\partial_a X^\mu\partial_b X^\nu. \tag{5.139}$$

[9] In mathematical terms, B is the connection of a *gerbe*, much in the same sense in which a gauge potential A_μ is the connection of a vector bundle.

It is instructive to expand

$$X^\mu(\xi) \equiv X_0^\mu + \sqrt{\alpha'} Y^\mu(\xi) \tag{5.140}$$

into a ξ-independent background value X_0^μ and the fluctuations $Y^\mu(\xi)$. The factor of $\sqrt{\alpha'}$ renders these dimensionless. A Taylor expansion of $G_{\mu\nu}(X)$ in the action,

$$\begin{aligned} G_{\mu\nu}(X)\partial X^\mu \partial X^\nu &= \alpha' \Big(G_{\mu\nu}(X_0) + \sqrt{\alpha'} G_{\mu\nu,\rho}(X_0) Y^\rho(\xi) \\ &\quad + \frac{\alpha'}{2} G_{\mu\nu,\rho\tau} Y^\rho(\xi) Y^\tau(\xi) + \ldots \Big) \partial Y^\mu \partial Y^\nu, \end{aligned} \tag{5.141}$$

yields an infinite number of couplings in the worldsheet theory with the following properties:

- If the target space has typical radius R_c then $\frac{\partial G}{\partial X} \propto \frac{1}{R_c}$, and the effective dimensionless coupling constant is of the order $\frac{\sqrt{\alpha'}}{R_c}$. Thus the worldsheet theory is weakly coupled if $\frac{\sqrt{\alpha'}}{R_c} << 1$. Indeed the extended nature of the string becomes important for $R_c \cong l_s$, i.e. when the σ-model is strongly coupled.

- This establishes the following important concept: String perturbation theory really involves a **double expansion**:

$g^2 = e^\Phi \Leftrightarrow$ loop expansion from spacetime perspective

$\frac{\sqrt{\alpha'}}{R_c} \Leftrightarrow$ loop expansion on the worldsheet

- From the spacetime perspective, the expansion in $\frac{\sqrt{\alpha'}}{R_c}$ will be interpreted as an expansion in higher derivatives of the (background) metric.

- For $\frac{\sqrt{\alpha'}}{R_c} \to 1$ there exists no accurate point particle description of the effective theory because the variation of the metric over distances of the string is big. This leads to the concept of **"stringy geometry"**: A string probes the target space geometry very differently from a point particle. E.g. string theory can be perfectly-well defined on backgrounds which contain certain singularities. While point particle theory would be inconsistent in such a background, string theory can effectively resolve these geometries.

Conformal invariance

The central condition in order to define a well-defined quantum string theory is, as we have seen, conformal invariance on the worldsheet, i.e. the constraint $c^{(\mathrm{tot})} = c^{(X)} + c^{(\mathrm{ghost})} = 0$.

- For the free theory of strings propagating in flat spacetime $\mathbb{R}^{1,d-1}$, where the string coordinates are simply described by a copy of d X-CFTs with $c = 1$, this translates into the constraint $d = 26$.

- The interacting worldsheet theory corresponding to string propagating in the most *general* curved background cannot be solved *exactly* at the quantum level, i.e. it is in general not known how to find the exact spectrum of string states etc.[10] However, for backgrounds with $\frac{\sqrt{\alpha'}}{R_c} << 1$ we can treat the worldsheet interactions perturbatively. As in conventional QFT we start from the free theory and compute deviations order by order in the coupling, here given by $\frac{\sqrt{\alpha'}}{R_c}$.
- From the logic of the path integral conformal invariance continues to be a consistency condition for the so-defined quantum theory. However, we need to check if **extra** constraints arise from requiring conformal invariance in addition to the constraint that $d = 26$. Put differently we need to analyze if the interactions on the worldsheet induce **new** contributions to the Weyl anomaly of S_X.
- For a general QFT, conformal invariance implies scale invariance. It is not known whether the converse is also true in general. However, for a 2-dimensional unitary QFT with compact spatial dimensions (a requirement shared by our worldsheet theory), scale invariance does imply full conformal invariance[11]. It is therefore sufficient to analyze possible deviations from scale invariance due to the worldsheet interactions. In QFT such a deviation is captured by the β-function. We thus seek to compute the β-function of the space-time metric
$$\beta_{\nu\rho}(G) = \mu \frac{\partial}{\partial \mu} G_{\nu\rho}(X;\mu), \qquad \text{with } \mu \equiv \text{ the energy scale.} \tag{5.142}$$
Absence of a Weyl anomaly is equivalent to
$$\beta_{\nu\rho}(G) = 0. \tag{5.143}$$
This must be enforced order by order in $\frac{\sqrt{\alpha'}}{R_c}$ in σ-model perturbation theory.

Computation of 1-loop β-function:

As a first approximation we evaluate $\beta_{\nu\rho}(G)$ at 1-loop level in $\frac{\sqrt{\alpha'}}{R_c}$. From standard QFT we recall the following procedure to compute the 1-loop β-function of a coupling g (e.g. the gauge coupling in QCD):

- Compute the 1-loop correction to the coupling g.
- Extract the divergent piece, e.g. as the $\frac{1}{\epsilon}$ term in dimensional regularisation.
- The 1-loop β-function is $\beta(g) = g\times$ coefficient of $\frac{1}{\epsilon}$-term.

For the β-function of $G_{\nu\rho}(X;\mu)$ in the case at hand we must compute the loop correction to the propagator (i.e. the self-energy of the scalar field $Y^\mu(\xi)$). This computation proceeds as follows, where are not careful about numerical prefactors:

[10] Nonetheless, for special background manifolds the corresponding interacting CFT can be solved exactly; e.g. propagation of the superstring on a certain class of Calabi-Yau manifolds can be described by Gepner models, which are modular invariant tensor products of minimal models for which an exact solution is known.

[11] See the seminal paper by J. Polchinski, "Scale and Conformal Invariance In Quantum Field Theory", Nucl. Phys. B 303, 226 (1988).

- Locally around X_0 we choose **Riemann normal coordinates** familiar from General Relativity. In these the Levi-Civita metric takes the form

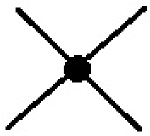
$$G_{\mu\nu}(X) = \delta_{\mu\nu} - \frac{\alpha'}{3} R_{\mu\lambda\nu\kappa}(X_0) Y^\lambda Y^\kappa + \mathcal{O}(Y^3) \tag{5.144}$$

$$\tag{5.145}$$

so that

$$S = \frac{1}{4\pi} \int d^2\xi \sqrt{h} \left[\delta_{\mu\nu} \partial^a Y^\mu \partial_a Y^\nu - \frac{\alpha'}{3} R_{\mu\lambda\nu\kappa}(X_0) Y^\lambda Y^\kappa \partial^a Y^\mu \partial_a Y^\nu \right] + \ldots \tag{5.146}$$

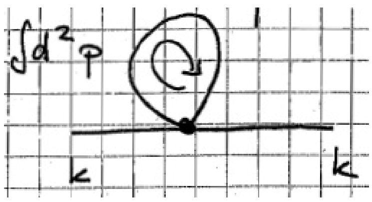
- This leads to a quartic interaction vertex with the following Feynman rule:



$\simeq R_{\mu\lambda\nu\kappa} k^\mu_a (k^a)^\nu$

Here k^μ_a denotes the the 2-dimensional worldsheet momentum of the scalar field Y^μ and appears due to the two derivatives in (5.146).

- There is thus a one-loop correction to the kinetic term of the scalar fields from the diagram



In momentum space this correction is proportional to

$$\int \frac{d^2p}{(2\pi)^2} R_{\mu\lambda\nu\kappa} (k^\mu \cdot k^\nu) \qquad \underbrace{\left\langle Y^\lambda(p) Y^\kappa(p) \right\rangle}_{\text{propagator in momentum space}} \qquad . \tag{5.147}$$

- This integral is divergent from the $p \to \infty$ region as can be anticipated already from the logarithmic divergence $\xi \to \xi'$ of the propagator in position space,

$$\left\langle Y^\lambda(\xi) Y^\kappa(\xi') \right\rangle = -\frac{1}{2} \delta^{\lambda\kappa} \log(|\xi - \xi'|^2). \tag{5.148}$$

- We can easily extract the divergence by dimensional regularisation. Setting $d = 2 + \epsilon$ the divergent piece of the integral is isolated as

$$\int \frac{d^{2+\epsilon}p}{(2\pi)^2} \left\langle Y^\lambda(p) Y^\kappa(p) \right\rangle = \int \frac{d^{2+\epsilon}p}{(2\pi)^2} \frac{\delta^{\lambda\kappa}}{p^2} = \frac{1}{2\pi} \int dp\, p^{\epsilon - 1} \simeq \frac{1}{2\pi} \frac{\delta^{\lambda\kappa}}{\epsilon}. \tag{5.149}$$

The counterterm for this divergence is proportional to

$$-\frac{1}{\epsilon} R_{\mu\nu} \partial Y^\mu \partial Y^\nu \qquad \text{in } S_\sigma. \tag{5.150}$$

- The β-function at 1-loop is therefore given by $\alpha' R_{\mu\nu}$. We conclude that absence of an (extra) Weyl anomaly at 1-loop requires

$$\beta_{\mu\nu}(G) = \alpha' R_{\mu\nu} \stackrel{!}{=} 0. \tag{5.151}$$

This is a spectacular result: The above equation is nothing other than the Einstein equation for the vacuum. Indeed since we have set all other fields, such as Φ and B to zero in our computation, the target space is precisely described by the vacuum, i.e. pure geometry with no matter.

To first order in $\frac{\sqrt{\alpha'}}{R_c}$ consistency of the σ-model yields the Einstein equation for background metric.

We have just **derived** the dynamical laws of Einstein gravity as a corollary from worldsheet consistency of a string propagating in the corresponding target space. This confirms that string theory indeed yields a quantum theory of gravity.
What is more, we can next compute the 2-loop β-function, with the result

$$\beta^{(2)}_{\mu\nu} = \alpha' R_{\mu\nu} + \frac{1}{2}\alpha'^2 R_{\mu\lambda\rho\sigma} R_{\nu}^{\lambda\rho\sigma} \stackrel{!}{=} 0. \tag{5.152}$$

This way **one can systematically find stringy higher curvature corrections to the Einstein equations**.
This logic generalises to the full σ-model including the coupling to $B_{\mu\nu}$ and to Φ. Here we quote the 1-loop β-functions:

$$\begin{aligned}
\beta_{\mu\nu}(G) &= \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\kappa} H_{\nu}^{\lambda\kappa} + \mathcal{O}(\alpha'^2) \stackrel{!}{=} 0, \\
\beta_{\mu\nu}(B) &= -\frac{\alpha'}{2} \nabla^\gamma H_{\gamma\mu\nu} + \alpha' \nabla^\gamma \Phi H_{\gamma\mu\nu} + \mathcal{O}(\alpha'^2) \stackrel{!}{=} 0, \\
\beta_{\mu\nu}(\Phi) &= \frac{d-26}{6} - \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_\gamma \Phi \nabla^\gamma \Phi - \frac{\alpha'}{24} H_{\mu\nu\gamma} H^{\mu\nu\gamma} + \mathcal{O}(\alpha'^2) \stackrel{!}{=} 0.
\end{aligned} \tag{5.153}$$

The first term in $\beta_{\mu\nu}(\Phi)$ can be viewed as the tree-level contribution in σ-model perturbation theory.
The β-functions are solved e.g. for $\Phi = \Phi_0 \equiv$ const., $d = 26$ and $B_{\mu\nu} = 0 = R_{\mu\nu}$.
One can now deduce the **low energy effective action** for the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ as the target space action whose equations of motion reproduce the β-function equations order by order in perturbation theory. The result can be found e.g. in [P], eq. (3.7.20), along with many more illuminating explanations. See also Assignment 12 for a derivation of some of these facts.

Chapter 6

Superstring theory

As stressed several times the bosonic string theory, which we have analysed up to now, is merely a toy model because of the following two major shortcomings:

- The tachyonic ground state signals a vacuum instability.
- The string spectrum contains only bosonic excitations. This lack of fermionic states is in contrast to observations and makes the bosonic string unrealistic.

Both of these challenges are remedied in superstring theory. Concerning the tachyonic groundstate, the basic idea is to realise that its appearance is due to the negative zero-point energy of the bosonic string theory on the worldsheet. This suggests that one should modify the theory by adding fermions on the worldsheet, hoping that their opposite statistics might cancel this vacuum energy.[1]
In fact, we will see in this chapter that solving both abovementioned problems replaces the bosonic string theory in $d = 26$ dimensions by a supersymmetric theory in $d = 10$ dimensions. Supersymmetry is a symmetry that exchanges bosons and fermions. The worldsheet superstring theory consists of a bosonic and a fermionic sector. The bosonic sector is identical to the worldsheet theory of the bosonic string. We can therefore view our efforts up to now as a preliminary study of one half of the superstring theory.
There exist two major formulations of this superstring theory. Both theories enjoy supersymmetry on the worldsheet and in spacetime, but they differ in the following respect:

- In the **Ramond-Neveu-Schwarz (RNS)** formulation, supersymmetry is *manifest* on the worldsheet, but not in spacetime.
- In the **Green-Schwarz (GS)** formulation, supersymmetry is *manifest* in spacetime, but not on the worldsheet.

More recently, the **pure-spinor** or Berkovitz formulation has been developed as yet another approach to the superstring.
In this course, we will only discuss the RNS formalism.

[1] Indeed this will remove the tachyon at least in the so-called Ramond subsector of the superstring theory; in the Neveu-Schwarz sector things turn out more complicated.

6.1 The classical RNS action

The superstring theory is obtained by adding to the bosonic string, whose action in flat gauge is

$$S_B = -\frac{1}{8\pi}\int d^2\xi \frac{2}{\alpha'}\partial_\alpha X^\mu \partial^\alpha X_\mu \tag{6.1}$$

with X^μ a worldsheet scalar, a sector describing 2-dimensional worldsheet spinors.

- Recall that a spinor is by definition a representation of the Clifford algebra. Applied to the two-dimennsional worldsheet with flat metric $\eta_{\alpha\beta}$ the Clifford algebra is generated by the two-dimensional γ-matrices with anti-commutation relations

$$\{\gamma^\alpha, \gamma^\beta\}_{AB} = 2\,\eta^{\alpha\beta}\,\mathbb{1}_{AB} \qquad \text{where} \qquad \gamma^\alpha = \gamma^\alpha_{AB}. \tag{6.2}$$

  Here A, B are spinor indices on the worldsheet and α, β are vector indices on the worldsheet, α, $\beta = 0$, 1. For simplicity we will take all spinor indices as indices downstairs.

- A spinor ψ_A transforms under Lorentz transformations as

$$\psi_A \to S_{AB}\,\psi_B, \qquad S_{AB} = [\exp(i\omega_{\alpha\beta}\tfrac{i}{4}[\gamma^\alpha,\gamma^\beta])]_{AB}, \tag{6.3}$$

  with $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ an infinitesimal Lorentz transformation. We understand that all repeated indices, here the spinor index B, are summed over.

- We will discuss the systematic construction of spinor representations in arbitrary dimensions later. For the time being we simply observe that in two dimensions we can pick the γ-matrices as real 2×2-matrices, e.g. as

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \{A, B\} = \{1, 2\}. \tag{6.4}$$

  In view of (6.3) also ψ_A can be taken to be real

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \qquad \psi^* = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}^* = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \psi. \tag{6.5}$$

  This reality condition on the spinors is called **Majorana** condition, the corresponding spinor is a **Majorana spinor**.

- The labelling $\psi_\pm$ refers to the chirality, i.e. to the eigenvalues under $\gamma \equiv \gamma^0\gamma^1$,

$$\gamma\begin{pmatrix} \psi_+ \\ 0 \end{pmatrix} = \begin{pmatrix} \psi_+ \\ 0 \end{pmatrix}, \qquad \gamma\begin{pmatrix} 0 \\ \psi_- \end{pmatrix} = -\begin{pmatrix} 0 \\ \psi_- \end{pmatrix}. \tag{6.6}$$

  Spinors of definite chirality are called **Weyl spinors**.

- The objects $\psi_\pm$ are thus **Majorana-Weyl spinors**, i.e.they are both real and chiral. Such Majorana-Weyl spinors exist in 2 mod 8 dimensions.

Let us now define the RNS action in flat gauge as the action obtained by adding the canonical kinetic terms for free bosons and Majorana fermions on the worldsheet,

$$\boxed{S = -\frac{1}{8\pi}\int d^2\xi\,\tfrac{2}{\alpha'}\partial_\alpha X^\mu \partial^\alpha X_\mu + 2\,i\,\bar{\psi}^\mu_A \gamma^\alpha_{AB}\partial_\alpha \psi_{\mu\,B},} \tag{6.7}$$

where

- $\psi_A^\mu = \begin{pmatrix} \psi_+^\mu \\ \psi_-^\mu \end{pmatrix}$ with $\psi_\pm^\mu$ representing Grassmann valued spacetime vectors, and
- $\bar{\psi} = \psi^\dagger \gamma^0 = \psi^T \gamma^0 = (-\psi_-, \psi_+)$.

In light-cone coordinates

$$\boxed{\begin{aligned} S =& S_B + S_F, && (6.8)\\ S_B =& \frac{1}{2\pi} \int d^2\xi \, \tfrac{2}{\alpha'} \partial_+ X \cdot \partial_- X, && (6.9)\\ S_F =& \frac{1}{2\pi} \int d^2\xi \, i \, (\psi_+ \cdot \partial_- \psi_+ + \psi_- \cdot \partial_+ \psi_-). && (6.10) \end{aligned}}$$

Note that the mass dimensions of the fields are $[X] = -1$, $[\psi] = \frac{1}{2}$, explaining the relative factor of $\frac{1}{\alpha'}$.

Ignoring potential boundary terms, to be discussed later, the equation of motion for ψ_A^μ is the **Dirac equation**

$$\gamma^\alpha \partial_\alpha \psi = 0 \qquad \text{or in components} \qquad \partial_+ \psi_- = 0\,, \quad \partial_- \psi_+ = 0. \tag{6.11}$$

Thus the Weyl spinors $\psi_\pm$ are also chiral in the sense that $\psi_\pm = \psi_\pm(\xi_\pm)$.

As can be verified by brute force computation, the action $S_B + S_F$ is invariant under the **fermionic symmetry**

$$\boxed{\begin{aligned} \sqrt{\tfrac{2}{\alpha'}} \delta X^\mu &= i \, \bar{\epsilon} \psi^\mu, \\ \delta \psi^\mu &= \sqrt{\tfrac{2}{\alpha'}} \tfrac{1}{2} \gamma^\alpha \partial_\alpha X^\mu \cdot \epsilon. \end{aligned}} \tag{6.12}$$

Here $\epsilon \equiv \epsilon_A$ denotes an infinitesimal Majorana spinor,

$$\epsilon_A = \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix} \qquad \text{with} \quad \epsilon_+, \, \epsilon_- \quad \text{Grassmann fields.} \tag{6.13}$$

In order for (6.12) to be a symmetry of the full action, $\epsilon_A(\xi)$ must obey

$$\gamma^\beta \, \gamma_\alpha \, \partial_\beta \, \epsilon(\xi) = 0. \tag{6.14}$$

In components the symmetry (6.12) reads

$$\boxed{\begin{aligned} \sqrt{\tfrac{2}{\alpha'}} \delta X^\mu &= i \, (-\epsilon_- \psi_+^\mu + \epsilon_+ \psi_-^\mu), \\ \delta \psi_\pm^\mu &= \sqrt{\tfrac{2}{\alpha'}} \epsilon_\mp \partial_\pm X^\mu. \end{aligned}} \tag{6.15}$$

- The symmetry (6.12) relates the bosonic and fermionic degrees of freedom. This is the characteristic property of a **supersymmetry (SUSY)**.

- The symmetry is "chiral" in the sense that (6.14) becomes

$$\partial_+\epsilon_+ = 0, \qquad \partial_-\epsilon_- = 0. \tag{6.16}$$

- In the above formulation the symmetry closes only upon use of the e.o.m., i.e. it holdes on-shell.

Remarks:

- SUSY is a deep concept that extends (in some sense uniquely) the Poincaré symmetry. While found for the first time in the context of the two-dimensional RNS theory, it has become an important principle of more general physical systems.

- The supersymmetry generators Q_A are spinorial and can be shown to obey the typical anti-commutation relations

$$\boxed{\{Q_A, \bar{Q}_B\} \cong 2\,\gamma^\alpha_{AB}\,P_\alpha,} \tag{6.17}$$

  where the momentum operator P_α generates translations.

- Just like translations are related to position and thus to the conventional spacetime coordinates, the existence of a fermionic symmetry Q_A implies the notion of fermionic, i.e. Grassmann-valued coordinates. Together with the conventional, bosonic coordinates of spacetime these span what is called **superspace**. We do not discuss the formulation of RNS string in superspace notation in this course, referring the interested reader instead to [P], Chapter 12.3. In such a superspace formulation the supersymmetry holds off-shell, i.e. without use of the equations of motion.

6.2 Super-conformal invariance

In the previous section we have presented the RNS action formulated on a flat worldsheet with metric $\eta_{\alpha\beta}$. As in the bosonic theory, this action is really the result of gauge fixing an action formulated on a worldsheet with dynamical worldsheet metric $h_{\alpha\beta}$. This action describes 2-dimensional gravity coupled to a supersymmetric theory of worldsheet scalars and fermions. As a consequence of the fundamental anti-commutation relation $\{Q, \bar{Q}\} = 2\gamma \cdot P$ for the supersymmetry generator Q_A, local diffeomorphism invariance together with supersymmetry implies **local supersymmetry**. Such a theory is called two-dimensional **supergravity (SUGRA)**. In the same way as the worldsheet spinors ψ^μ_A are the superpartners of the bosons X^μ (in that they are exchanged by supersymmetry, cf (6.12)), the metric $h_{\alpha\beta}$ has a superpartner whose degrees of freedom give rise to the gravitino.

For brevity we do not discuss this supergravity in this course. The details can be found in [BLT], Chapter 7.2. and 7.3. Suffice it here to state that the logic leading from the full action to the gauge fixed action is as for the bosonic string and proceeds along the following lines:

- The full action enjoys **local super-Weyl and diffeomorphism invariance**. In particular, the supersymmetry parameter $\epsilon_A = \epsilon_A(\tau, \sigma)$ is unconstrained.

- After fixing the gauge to flat gauge we are left with a **remnant super-conformal symmetry**, where SUSY is only "chiral" ($\epsilon_- = \epsilon_-(\xi^+)$, $\epsilon_+ = \epsilon_+(\xi^-)$) and diffeomorphism invariance reduces to conformal symmetry.

- The generators of the **super-conformal symmetry**, i.e. their conserved currents are

– **the energy-momentum-tensor**,

$$T_{++} = -\frac{1}{\alpha'}\partial_+ X \cdot \partial_+ X - \frac{i}{2}\psi_+^\mu \partial_+ \psi_{+\,\mu}, \tag{6.18}$$
$$T_{--} = -\frac{1}{\alpha'}\partial_- X \cdot \partial_- X - \frac{i}{2}\psi_-^\mu \partial_- \psi_{-\,\mu}. \tag{6.19}$$

Conformal symmetry implies $T_{+-} = 0 = T_{-+}$, conservation of the energy-momentum tensor means $\partial_+ T_{--} = 0 = \partial_- T_{++}$.

– **the supercurrent**, which can be found via Noether's method as

$$\delta_{\mathrm{SUSY}} S = \int d^2\xi\, (\partial_\alpha \bar{\epsilon}) J^\alpha, \qquad J_\pm = -\frac{1}{2}\sqrt{\frac{2}{\alpha'}}\psi_\pm^\mu \partial_\pm X_\mu. \tag{6.20}$$

To derive this form of $J_\pm$ certain conditions from local supersymmetry must be used, see [BLT], Chapter 7.2 and 7.3. From this form of the supercurrent we see that

$$\partial_- J_+ = 0, \qquad \partial_+ J_- = 0. \tag{6.21}$$

- As we fix the gauge by integrating out the worldsheet metric and its superpartner, the gravitino, one must keep their equations of motion in the form of generalised constraint equations. These are the **super-conformal Virasoro constraints**

$$T_{\pm\pm} \stackrel{!}{=} 0, \qquad J_\pm \stackrel{!}{=} 0, \tag{6.22}$$

which must be imposed on solutions of the equations of motion if we use flat gauge RNS action.

6.3 Mode expansions: Ramond vs. Neveu-Schwarz

We continue with our analysis of the flat gauge RNS action and proceed to the boundary conditions and mode expansion of the worldsheet fields.

- The bosonic mode expansion and boundary conditions are just as in the bosonic string.
- The fermionic equations of motion follow from variation of

$$S_F = \frac{i}{2\pi}\int d^2\xi (\psi_+ \partial_- \psi_+ + \psi_- \partial_+ \psi_-), \tag{6.23}$$

which yields

$$\delta S_F \cong \int d\tau [\psi_+ \delta\psi_+ - \psi_- \delta\psi_-]_{\sigma=0}^{\sigma=\ell} + \text{terms vanishing upon use of the e.o.m. (6.11)}. \tag{6.24}$$

a) **Closed sector**:
The boundary terms at $\sigma = 0$ and $\sigma = \ell$ cancel each other,

$$\psi_+ \delta\psi_+ - \psi_- \delta\psi_- \Big|_{\sigma=0} \stackrel{!}{=} \psi_+ \delta\psi_+ - \psi_- \delta\psi_- \Big|_{\sigma=\ell}. \tag{6.25}$$

The most general boundary conditions that do not mix ψ_+ and ψ_- and respect spacetime Poincaré symmetry are

$$\psi_+^\mu(\sigma) = \pm\psi_+^\mu(\sigma+\ell), \tag{6.26}$$
$$\psi_-^\mu(\sigma) = \pm\psi_-^\mu(\sigma+\ell). \tag{6.27}$$

Since ψ is a worldsheet spinor, the minus sign is possible as we go around the worldsheet once, taking $\sigma \to \sigma+\ell$. This was not possible for worldsheet scalars X^μ.
Thus there are 4 independent sectors since for $\psi_\pm$ we can independently choose either sign, In short the boundary conditions can be written as

$$\psi_\pm(\sigma+\ell) = e^{2\pi\, i\, \phi_\pm}\psi_\pm(\sigma), \qquad \text{where} \tag{6.28}$$

$\phi = 0$ denotes the **Ramond sector**, and
$\phi = \frac{1}{2}$ denotes the **Neveu-Schwarz sector**.

i) The Ramond sector (R) corresponds to periodic boundary conditions with **integer mode expansion**

$$\psi_-^\mu(\tau,\sigma) = \sum_{n\in\mathbb{Z}} \sqrt{\frac{2\pi}{\ell}}\, b_n^\mu e^{-\frac{2\pi}{\ell} i\, n(\tau-\sigma)}, \tag{6.29}$$

$$\psi_+^\mu(\tau,\sigma) = \sum_{n\in\mathbb{Z}} \sqrt{\frac{2\pi}{\ell}}\, \tilde{b}_n^\mu e^{-\frac{2\pi}{\ell} i\, n(\tau+\sigma)}. \tag{6.30}$$

ii) The Neveu-Schwarz sector (NS) corresponds to anti-periodic boundary conditions with **half-integer mode expansion**

$$\psi_-^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \sqrt{\frac{2\pi}{\ell}}\, b_r^\mu e^{-\frac{2\pi}{\ell} i\, r(\tau-\sigma)}, \tag{6.31}$$

$$\psi_+^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \sqrt{\frac{2\pi}{\ell}}\, \tilde{b}_r^\mu e^{-\frac{2\pi}{\ell} i\, r(\tau+\sigma)}. \tag{6.32}$$

The four different sectors are therefore

$$(\phi_+,\, \phi_-) = (0,\, 0) \leftrightarrow \text{RR}, \tag{6.33}$$
$$(\phi_+,\, \phi_-) = (\frac{1}{2},\, \frac{1}{2}) \leftrightarrow \text{NS-NS}, \tag{6.34}$$
$$(\phi_+,\, \phi_-) = (\frac{1}{2},\, 0) \leftrightarrow \text{NS-R}, \tag{6.35}$$
$$(\phi_+,\, \phi_-) = (0,\, \frac{1}{2}) \leftrightarrow \text{R-NS}. \tag{6.36}$$

b) **Open sector**:
The boundary terms at $\sigma = 0$ and $\sigma = \ell$ vanish individually. This relates ψ_+ and ψ_- as

$$\psi_+^\mu(\sigma)\Big|_{\text{boundary}} = \pm\psi_-^\mu(\sigma)\Big|_{\text{boundary}}. \tag{6.37}$$

We are left with $\underbrace{2}_{\pm} \times \underbrace{2}_{\sigma=0,\,\ell}$ sectors, of which two differ only by an insignificant overall sign.

Consider a direction μ with Neuman-Neumann (NN) boundary conditions for X^μ. By convention we choose the overall sign such that

$$\psi_+^\mu(\sigma)\Big|_{\sigma=0} = \psi_-^\mu(\sigma)\Big|_{\sigma=0}. \tag{6.38}$$

There are now two inequivalent cases

$$\psi_+^\mu(\sigma)\Big|_{\sigma=\ell} = \eta\, \psi_-^\mu(\sigma)\Big|_{\sigma=\ell}, \qquad \eta = \pm 1. \tag{6.39}$$

i) The Ramond sector corresponds to periodic boundary conditions with integer modes:

$$\psi_+^\mu(\sigma)\Big|_{\sigma=\ell} = \psi_-^\mu(\sigma)\Big|_{\sigma=\ell} \Rightarrow \qquad \psi_\pm^\mu(\tau,\sigma) = \sqrt{\frac{\pi}{\ell}} \sum_{n\in\mathbb{Z}} b_n^\mu e^{-\frac{\pi}{\ell} i\, n(\tau\pm\sigma)}. \tag{6.40}$$

ii) The Neveu-Schwarz sector corresponds to anti-periodic boundary conditions and half-integer modes:

$$\psi_+^\mu(\sigma)\Big|_{\sigma=\ell} = -\psi_-^\mu(\sigma)\Big|_{\sigma=\ell} \Rightarrow \qquad \psi_\pm^\mu(\tau,\sigma) = \sqrt{\frac{\pi}{\ell}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-\frac{\pi}{l} i\, r(\tau\pm\sigma)}. \tag{6.41}$$

In both cases reality implies $b_r^\mu = (b_{-r}^\mu)^\dagger$.

If we consider (DD) boundary conditions for X^μ instead of (DD) we find the corresponding solution for the superpartners by noting that a change of boundary conditions corresponds to a sign flip for the rightmovers,

$$X_+ \to X_+, \qquad X_- \to -X_-. \tag{6.42}$$

By worldsheet supersymmetry, we must also flip the sign of ψ_-,

$$\psi_+ \to \psi_+, \qquad \psi_- \to -\psi_-. \tag{6.43}$$

Since this only gives an overall sign for the boundary conditions at $\sigma = 0$ and $\sigma = \ell$, the modes continue to be (half-)integer moved in the (NS)/R sector.
By contrast, mixed DN or ND boundary conditions correspond to a sign flip only at one of the two boundaries. Therefore, the R-sector is now half-integer moved, and the NS-sector modes are integer moved.
In the sequel, unless stated otherwise, we refer to (DD) or (NN) boundary conditions, i.e. the NS sector is the half-integer one.

6.4 Canonical quantisation and Super-Virasoro-Algebra

We now turn to canonical quantization of the RNS string.

6.4.1 Canonical (anti-)commutation relations

- The X-sector modes continue to enjoy the familiar commutation relations $[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}$.
- The fermions ψ_A^μ satisfy the canonical anti-commutation relations

$$\left\{\psi_+^\mu(\tau,\sigma), \psi_+^\nu(\tau,\sigma')\right\} = 2\pi\,\eta^{\mu\nu}\delta(\sigma-\sigma'), \tag{6.44}$$
$$\left\{\psi_-^\mu(\tau,\sigma), \psi_-^\nu(\tau,\sigma')\right\} = 2\pi\,\eta^{\mu\nu}\delta(\sigma-\sigma'), \tag{6.45}$$
$$\left\{\psi_+^\mu(\tau,\sigma), \psi_-^\nu(\tau,\sigma')\right\} = 0. \tag{6.46}$$

This is easy to see because $\psi_\pm$ and $\frac{i}{2\pi}\psi_\pm$ are canonically conjugate variables. Since they are anti-commuting their canonical Poisson-brackets are replace by $\frac{1}{i}\times$ the anti-commutator.

The corresponding anti commutation relation for the modes is

$$\{b_m^\mu, b_n^\nu\} = \left\{\tilde{b}_m^\mu, \tilde{b}_n^\nu\right\} = \eta^{\mu\nu}\delta_{m+n,0}. \tag{6.47}$$

- The **ground state** of the Fock space is annihilated by the positive modes.

  i) **NS sector**:

$$\alpha_m^\mu\,|0\rangle_{\mathrm{NS}} = 0 \qquad \forall\, m>0 \quad (m\in\mathbb{Z}) \tag{6.48}$$
$$b_r^\mu\,|0\rangle_{\mathrm{NS}} = 0 \qquad \forall\, r>0 \quad (r\in\mathbb{Z}+\tfrac{1}{2}) \tag{6.49}$$
$$\tag{6.50}$$

  and similarly for $\tilde{\alpha}_m^\mu$, $\tilde{b}_r^\mu$ for closed strings.

  $|0\rangle_{\mathrm{NS}}$ is the **unique** ground state. It is a **spacetime** scalar.

  The negative modes $\alpha_{-|m|}^\mu$, $b_{-|r|}^\mu$ create excited states from this ground state. All states in the NS sector are thus **spacetime** bosons.

  ii) **Ramond sector**: The ground state is defined by

$$\alpha_m^\mu\,|0\rangle_{\mathrm{R}} = b_n^\mu\,|0\rangle_{\mathrm{R}} = 0, \qquad \forall\, m,n>0 \quad (m,n\in\mathbb{Z}). \tag{6.51}$$

  Unlike in the NS sector, this does not uniquely specify the ground state. The reason is that the state $b_0^\mu\,|0\rangle_{\mathrm{R}}$ is not zero, but is likewise annihilated by all positive modes. In fact we will see below that $|0\rangle_{\mathrm{R}}$ and $b_0^\mu\,|0\rangle_{\mathrm{R}}$ have the same mass, i.e. they are degenerate. Therefore $|0\rangle_{\mathrm{R}}$ is a **degenerate** ground state.

  To describe the ground state further we observe that the Ramond zero-modes satisfy the (ambient spacetime) Clifford algebra because the anti-commutation relations (6.47) imply

$$\{b_0^\mu, b_0^\nu\} = \eta^{\mu\nu}. \tag{6.52}$$

  Compared with the space-time Clifford-algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$ satisfied by d-dimensional Γ-matrices we identify

$$b_0^\mu = \frac{1}{\sqrt{2}}\Gamma^\mu. \tag{6.53}$$

This shows that the ground state $|0\rangle_{\mathrm{R}}$ furnishes a representation of the d-dimensional Clifford algebra.

$\boxed{|0\rangle_{\mathrm{R}} \text{ is a } \textbf{spinor} \text{ in } d\text{-dimensions.}}$

We therefore need to learn more about spinors in an arbitrary number of dimensions.

6.4.2 Interlude: Spinors of $SO(1, d-1)$

We focus on even-dimensional spaces and set $d = 2 + 2\,k$.

- The key idea for finding the representations of the Clifford algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2\,\eta^{\mu\nu} = 2\,\mathrm{diag}(-1, +1, \ldots, +1)$ is to build a set of $(k+1)$ raising and lowering operators as

$$\boxed{\begin{aligned} \Gamma^{0\,\pm} &= \tfrac{1}{2}(\pm\Gamma^0 + \Gamma^1), \\ \Gamma^{a\,\pm} &= \tfrac{1}{2}(\Gamma^{2a} \pm i\Gamma^{2a+1}), \qquad a = 1, \ldots, k. \end{aligned}} \tag{6.54}$$

  These satisfy the anti-commutation relations

$$\begin{aligned} \left\{\Gamma^{a\,+}, \Gamma^{b\,-}\right\} &= \delta^{ab}, \\ \left\{\Gamma^{a\,-}, \Gamma^{b\,-}\right\} &= \left\{\Gamma^{a\,+}, \Gamma^{b\,+}\right\} = 0. \end{aligned} \tag{6.55}$$

- Since $(\Gamma^{a\,\pm})^2 = 0$ we can find a state ζ such that $\Gamma^{a\,-}\zeta = 0\ \forall\, a$, if necessary by acting on a certain vector once with $\Gamma^{a\,-}$. Given such ζ we can now construct 2^{k+1} states by letting $\Gamma^{a\,+}$ act on ζ zero or one times for all $a = 0, \ldots, k$. These 2^{k+1} states assemble as the components of a state

$$|s\rangle = |s_0, s_1, \ldots, s_k\rangle = (\Gamma^{k\,+})^{s_k + \frac{1}{2}} \cdot \ldots \cdot (\Gamma^{0\,+})^{s_0 + \frac{1}{2}} \cdot \zeta \qquad \text{with} \qquad s_j = \pm\frac{1}{2}\forall j = 0, \ldots, k. \tag{6.56}$$

  The state $|s\rangle$ is called a **Dirac spinor**. It has $2^{d/2}$ complex components.

- The combination

$$\Sigma^{\mu\nu} = -\frac{i}{4}[\Gamma^\mu, \Gamma^\nu] \tag{6.57}$$

  is readily checked to form a representation of the $SO(1, d-1)$ algebra. Letting

$$S_a := i^{\delta_{a,0}}\Sigma^{2a, 2a+1} = \Gamma^{a\,+}\Gamma^{a\,-} - \frac{1}{2} \tag{6.58}$$

  the Dirac spinor $|s\rangle$, defined as a representation of the Clifford algebra, furnishes also a representation of $SO(1, d-1)$[2] given by

$$\boxed{S_a\,|s\rangle = s_a\,|s\rangle.} \tag{6.59}$$

[2]This is an abuse of notation. We really mean $Spin(1, d-1)$, the double cover of SO(1,d-1), of course.

- In an even number of dimensions, a Dirac spinor is not irreducible. Rather a Dirac spinor can be reduced further into 2 Weyl spinors of opposite chirality. To this end we define

$$\Gamma = i^{-k}\,\Gamma^0\Gamma^1\dots\Gamma^{d-1} \tag{6.60}$$

with the properties

$$\Gamma^2 = 1\,, \qquad \{\Gamma, \Gamma^\mu\} = 0\,, \qquad [\Gamma, \Sigma^{\mu\nu}] = 0, \tag{6.61}$$

$$\Gamma = 2^{k+1} S_0 S_1 \dots S_k. \tag{6.62}$$

In view of (6.59) we find that

$$\Gamma\,|s\,\rangle = \begin{cases} 1 & \text{if an even number of } s_a = \frac{1}{2}, \\ -1 & \text{if an odd number of } s_a = \frac{1}{2}. \end{cases} \tag{6.63}$$

The eigenspinors with Γ eigenvalue ± 1 are called positive and negative chirality Weyl spinors. The Dirac spinor can be decomposed as

$$[2^{d/2}]_{\mathrm{Dirac}} \to [2^{d/2-2}]_{\mathrm{Weyl}} \oplus [2^{d/2-1}]_{\mathrm{Weyl}}. \tag{6.64}$$

For example we will heavily use that in $d = 10$ dimensions, a Dirac spinor has 32 complex components and be decomposed into 16-component Weyl spinors, $[32]_{\mathrm{Dirac}} = [16] \oplus [16']$.

- For $d = 2k + 3$, we can add Γ to the set of Γ^μ of one dimension less, i.e. of $SO(1, d-2)$. Together these satisfy the Clifford algebra in $d = 2k+3$ dimensions as follows from (6.61). Thus the dimensionality of the Clifford algebra representation in $d = 3 + 2k$ is the same as in $d = 2 + 2k$. We still have $2^{k+1} \times 2^{k+1}$ square matrices and we can still construct a 2^{k+1}-component Dirac spinor. Now Γ does not commute with $\Sigma^{\mu,d-1}$ and thus the Dirac spinor is irreducible. This just means that chirality is not an available concept in odd dimensions.

- A Majorana spinor satisfies a certain reality condition. Without proof we note that this is possible only if $d = 0, 1, 2, 3, 4$ mod 8. Spinors can be both Majorana and Weyl only if $d = 2$ mod 8. for more information we strongly recommend Appendix B of [P], Volume 2.

Let us now return to the R-sector ground state $|0\rangle_{\mathrm{R}}$. Being a spinor in d dimensions it has $2^{[d/2]}$ complex components. However, the Majorana condition on $b_0^\mu = (b_0^\mu)^\dagger$ implies that these $2^{[d/2]}$ components are real. We will find further reductions momentarily.
Finally, all states in the R-sector are obtained by acting with $\alpha^\mu_{-|m|}\,|0\rangle_{\mathrm{R}}$, $\alpha^\mu_{-|m|}\,|0\rangle_{\mathrm{R}}$. They are therefore **spacetime** fermions.

6.4.3 Super-Virasoro-Algebra and physical state condition

In old canonical quantisation we must impose the super-Virasoro constraints (6.22)

$$T_{\pm\pm} \stackrel{!}{=} 0, \qquad J_\pm \stackrel{!}{=} 0. \tag{6.65}$$

We first define the modes of the super-Virasoro generators, focusing on **open strings** to avoid double-writing.

i) The energy-momentum modes take the form

$$L_m = -\frac{\ell}{2\pi^2}\int_0^\ell d\sigma\,(e^{i\frac{\pi}{\ell}m\,\sigma}\,T_{++} + e^{-i\frac{\pi}{\ell}m\,\sigma}T_{--}) = L_m^{(b)} + L_m^{(f)} \tag{6.66}$$

with

$$L_m^{(b)} = \tfrac{1}{2}\sum_n :\alpha_{-n}\cdot\alpha_{m+n}:, \qquad m\in\mathbb{Z}, \tag{6.67}$$

$$L_m^{(f)} = \tfrac{1}{2}\sum_{r\in\mathbb{Z}+\phi}(r+\frac{m}{2}) :b_{-r}b_{m+r}:, \qquad m\in\mathbb{Z}\,, \quad \phi = \begin{cases} 0 & \text{R} \\ \frac{1}{2} & \text{NS}\end{cases} \tag{6.68}$$

In particular the zero mode reads

$$L_0 = \tfrac{1}{2}\alpha_0^2 + N\,, \qquad \text{number operator} \qquad N = \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n + \sum_{\substack{r\in\mathbb{Z}+\phi \\ r>0}} r\,b_{-r}b_r. \tag{6.69}$$

ii) The modes of the supercurrent J are

$$G_r = -\frac{1}{\pi}\sqrt{\frac{\ell}{\pi}}\int_0^\ell d\sigma\,(e^{i\frac{\pi}{\ell}r\,\sigma}J_+ + e^{-i\frac{\pi}{\ell}r\,\sigma}J_-), \tag{6.70}$$

$$G_r = \sum_{m\in\mathbb{Z}}\alpha_{-m}b_{r+m} \qquad r\in\begin{cases}\mathbb{Z}+\frac{1}{2} & \text{NS,} \\ \mathbb{Z} & \text{R}\end{cases} \tag{6.71}$$

Note that while the energy momentum tensor is always integer moded, the supercurrent modes are integer in the Ramond sector and half-integer in the NS sector (for NN/DD boundary conditions; for ND/DN this is reversed).

They form the **Super-Virasoro-Algebra**:

$$\boxed{\begin{aligned} &[L_m, L_n] = (m-n)\,L_{m+n} + \frac{c}{12}m(m^2-2\,\phi)\delta_{m+n,0}, \\ &[L_m, G_r] = (\frac{m}{2}-r)\,G_{m+r}, \\ &\{G_r, G_s\} = 2\,L_{r+s} + \frac{c}{12}(4r^2-2\,\phi)\delta_{r+s,0}\end{aligned}} \tag{6.72}$$

with

$$\phi = \begin{cases} 0 & \text{R} \\ \frac{1}{2} & \text{NS}\end{cases} \tag{6.73}$$

The **superconformal anomaly** is

$$c = \frac{3}{2}d = d(\underbrace{1}_{X\text{-CFT}} + \underbrace{\frac{1}{2}}_{\psi\text{-CFT}}). \tag{6.74}$$

This reflects the familiar fact that the degrees of freedom of two fermions count as one real boson.

Let us now evaluate the **physical state conditions** that follow from the super-Virasoro constraints $T_{\pm\pm} = 0 = J_\pm$.

i) The **NS sector** super-Virasoro constraints translate into

$$(L_0 - a_{\rm NS})\,|\phi\rangle = 0, \tag{6.75}$$
$$L_m\,|\phi\rangle = 0, \qquad m > 0, \quad m \in \mathbb{Z}, \tag{6.76}$$
$$G_r\,|\phi\rangle = 0, \qquad r > 0, \quad r \in \mathbb{Z} + \tfrac{1}{2}. \tag{6.77}$$

The open string mass shell condition (focussing on NN boundary conditions for all dimensions) thus reads

$$\alpha' M^2 = N - a_{\rm NS} \tag{6.78}$$

in terms of the normal ordering constant a_{NS}, which will be determined momentarily.

ii) The **R sector** physical state condition is

$$(L_0 - a_{\rm R})\,|\phi\rangle = 0, \tag{6.79}$$
$$L_m\,|\phi\rangle = 0, \qquad m > 0, \tag{6.80}$$
$$G_n\,|\phi\rangle = 0, \qquad \mathbf{n \geq 0}. \tag{6.81}$$

In the R-sector, the super-Virasoro-algebra (6.72) implies $L_0 = G_0^2$. Thus as a consistency condition

$$\boxed{a_{\rm R} \stackrel{!}{=} 0.} \tag{6.82}$$

We will confirm momentarily that this matches the explicit computation from ζ-function regularisation.

Remark: The R-sector constraint $0 = G_0\,|\phi\rangle$, expressed in terms of the modes, takes the form

$$\left[p\cdot\Gamma + \frac{1}{\sqrt{\alpha'}}(\sum_{n=1}^{\infty}\alpha_{-n}\cdot b_n + b_{-n}\cdot\alpha_n)\right]|\phi\rangle = 0. \tag{6.83}$$

Here we identified $b_0^\mu = \frac{1}{\sqrt{2}}\Gamma^\mu$. The zero mode piece is just the Dirac equation for a massless space-time fermion. The costraint $0 = G_0\,|\phi\rangle$ thus furnishes the stringy generalisation of the Dirac equation in the same manner as the Virasoro constraint $(L_0 - a)\,|\phi\rangle = 0$ yields the stringy form of the mass shell equation $p^2 + M^2 = 0$.

6.4.4 Normal ordering constants

The contribution of each field X^μ, ψ^μ to the normal ordering constants $a_{\rm NS}$, $a_{\rm R}$ follows e.g. by ζ-function regularisation.

- From the discussion of the bosonic string we recall the following reasoning that led to the **normal ordering constant of one periodic boson**: From

$$L_0^{(b)} = \frac{1}{2}\alpha_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n + \frac{1}{2}\sum_{n=-\infty}^{-1}\alpha_{-n}\alpha_n \tag{6.84}$$

with

$$\frac{1}{2}\sum_{n=-\infty}^{-1}\alpha_{-n}\alpha_n = \frac{1}{2}\sum_{n=1}^{\infty}\alpha_n\alpha_{-n} = \frac{1}{2}\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n + \underbrace{\frac{1}{2}\sum_{n=1}^{\infty} n}_{=-a} \tag{6.85}$$

we concluded that

$$a = -\frac{1}{2}\sum_{n=1}^{\infty} n = -\frac{1}{2}\zeta(-1) = \frac{1}{24}. \tag{6.86}$$

- Likewise for **periodic fermions** we compute

$$L_0^{(f)} = \frac{1}{2}\sum_{n=1}^{\infty} r\, b_{-r} b_r + \underbrace{\frac{1}{2}\sum_{n=1}^{\infty}(-r) b_r b_{-r}}_{\frac{1}{2}\sum_{n=1}^{\infty} r\, b_{-r}b_r - \frac{1}{2}\sum_{n=1}^{\infty} r} \tag{6.87}$$

and conclude

$$a = -\frac{1}{24}. \tag{6.88}$$

- For anti-periodic bosons (upper sign) and anti-periodic fermions fermions (lower sign), the normal ordering constant is

$$a = \mp\frac{1}{2}\zeta(-1,q)|_{q=\frac{1}{2}} \equiv \mp\frac{1}{2}\sum_{n=0}^{\infty}(n+q)^{-1}|_{q=\frac{1}{2}} = \mp\frac{1}{48}. \tag{6.89}$$

This is summarised in the following table.

1 periodic boson	$a = +\frac{1}{24}$
1 anti-periodic boson	$a = -\frac{1}{48}$
1 periodic fermion	$a = -\frac{1}{24}$
1 anti-periodic fermion	$a = \frac{1}{48}$

(6.90)

6.5 Open string spectrum in light-cone quantisation (all NN)

As in the bosonic theory, the physical spectrum is most easily determined in lightcone quantisation, where the residual gauge symmetries - here the super-conformal symmetry - is exploited at a classical level to solve for the super-Virasoro constraints explicitly. We briefly outline the procedure here - focussing for simplicity to the open string with Neumann-Neumann boundary conditions - and then move on to a detailed discussion of the spectrum.

- The residual conformal symmetry allows us to set

$$X^+(\tau,\sigma) = x^+ + p^+\tau \tag{6.91}$$

in light-cone coordinates $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{d-1})$.

- In addition it is simple to see that the chiral SUSY transformations (6.15) on ψ^μ_A allow us to set

$$\psi^+_A(\tau,\sigma) = 0. \tag{6.92}$$

by a suitable choice of chiral SUSY parameter $\epsilon_\mp$.

- One can then solve for $X^-(\tau,\sigma)$ and $\psi_A^-(\tau,\sigma)$ by exploiting

$$T_{\pm\pm} = 0 \qquad \text{and} \qquad J_\pm = 0. \tag{6.93}$$

- In particular one can solve for α_m^- and b_r^- in terms of $(d-2)$ transverse modes and plug these into the Hamiltonian to find the mass formula for physical states.

i) **NS-sector**
The result of this procedure is the mass-shell condition

$$\alpha' M^2 = \underbrace{\sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i}_{=:N^{(X)}} + \underbrace{\sum_{r=\frac{1}{2}}^{\infty} r\, b_{-r}^i b_r^i}_{=:N^{(\psi)}} - a_{\mathrm{NS}}. \tag{6.94}$$

The spectrum thus organises as follows:

- The ground state $|0;k\rangle_{\mathrm{NS}}$ is a spacetime scalar state with momentum k^μ and of mass

$$\alpha' M^2 = -a_{\mathrm{NS}}.$$

- The first level excitations arise by acting not with α_{-1}^i, but with $b_{-\frac{1}{2}}^i$ on the vacuum,

$$|\psi\rangle = \zeta_i\, b_{-\frac{1}{2}}^i\, |0;k\rangle_{\mathrm{NS}} \tag{6.95}$$

with mass

$$\alpha' M^2 = \frac{1}{2} - a_{\mathrm{NS}}. \tag{6.96}$$

This is a transverse vector of $SO(d-2)$ and must thus be massless in a Lorentz invariant theory. Therefore

$$\boxed{a_{\mathrm{NS}} = \frac{1}{2}.} \tag{6.97}$$

On the other hand, in the NS sector

$$a_{\mathrm{NS}} = (d-2)\Big(\underbrace{\frac{1}{24}}_{X\text{-CFT}} + \underbrace{\frac{1}{48}}_{\psi\text{-CFT}}\Big), \tag{6.98}$$

which fixes the critical dimension of the superstring to be

$$\boxed{d = 10.} \tag{6.99}$$

Note that the ground state is still tachyonic. We will see how to construct a consistent theory without the tachyon in the context of the GSO-projection the subsequent chapters.

- The states at the second excited level,

$$|\psi\rangle = (\zeta_i \alpha_{-1}^i + \zeta_{[ij]} b_{-\frac{1}{2}}^i\, b_{-\frac{1}{2}}^j)\, |0;k\rangle_{\mathrm{NS}}, \tag{6.100}$$

comprise $8 + \binom{8}{2} = 36$ transverse components. Since this is a massive state of mass

$$\alpha' M^2 = \frac{1}{2} \tag{6.101}$$

it must organise into an irreducible representation of the little group $SO(9)$. Indeed the 36-component representation of $SO(9)$ is just the anti-symmetric as follows by counting

$$\text{numbers of d.o.f. of antisymmetric of } SO(9) = \binom{9}{2} = 36. \tag{6.102}$$

To summarise the first few level of the open NS-string tower consist of

- a tachyonic $\mathbb{1}$ of $SO(9)$,
- a massless 8_V of $SO(8)$,
- massive bosons in tensor representation of $SO(9)$.

ii) **R-sector**:
The mass-shell condition is

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{n=1}^{\infty} n b^i_{-n} b^i_n. \tag{6.103}$$

Indeed, as noted already, the normal ordering constant vanishes because $a_\mathrm{R} = (d-2)(\frac{1}{24} - \frac{1}{24}) = 0$.

- The ground state $|u^a, k\rangle_\mathrm{R}$ is a massless ($k^2 = 0$) spacetime spinor. Based on the notation $|s\rangle = |s_0, \ldots, s_k\rangle$ for a spacetime spinor as in section (6.4.2) we introduce the symbol u_s for the wavefunction or polarisation of the various spinor components

$$|u^a, k\rangle = |s, k\rangle \underbrace{u_s}_{\text{polarisation}}. \tag{6.104}$$

As a consequence of the Majorana condition on the R-sector zero modes b_0^μ, $|s\rangle$ is a priori a Majorana spinor of $SO(10)$. As discussed at the end of section (6.4.2), in 10 dimensions Majorana spinors can be decomposed further into Weyl spinors, corresponding to the splitting

$$32 = 16 \oplus 16' \qquad \text{with } \textbf{real} \text{ components.}$$

The prime denotes a negative chirality Weyl spinor.
Under the decomposition

$$SO(1,9) \quad \to \quad \underbrace{SO(1,1)}_{x^\pm} \times \underbrace{SO(8)}_{\text{trans. direct. } x^i} \tag{6.105}$$

induced by going to spacetime lightcone gauge, the Weyl spinors decompose as

$$16 \to (\tfrac{1}{2}, 8) \oplus (-\frac{1}{2}, 8'), \tag{6.106}$$

$$16' \to (\tfrac{1}{2}, 8') \oplus (-\frac{1}{2}, 8). \tag{6.107}$$

Furthermore $|s\rangle$ must satisfy the Dirac equation (6.83) due to the supercurrent zero-mode constraint,

$$G_0\,|0\rangle_\mathrm{R} = 0.$$

Since $k^2 = 0$ we can pick w.l.o.g. $k_0 = k_1$, $k^i = 0$. The Dirac equation reads $0 = k_\mu \Gamma^\mu |0\rangle = (k_0\Gamma^0 + k_1\Gamma^1)|0\rangle$, where

$$0 = k_\mu \Gamma^\mu |0\rangle = k_0\Gamma^0 + k_1\Gamma^1 = -k_1\Gamma^0(\Gamma^0\Gamma^1 - \mathbb{1}). \tag{6.108}$$

Here we used that $(\Gamma^0)^2 = -1$.
Recalling from section that (6.4.2) $S_0 = \Gamma^{0,+}\Gamma^{0,-} - \frac{1}{2}$ we rewrite (6.108) as

$$0 = -2\,k_1\Gamma^0(S_0 - \frac{1}{2}). \tag{6.109}$$

Thus the Dirac equation implies $(S_0 - \frac{1}{2})|0\rangle_{\mathrm{R}} = 0$, i.e. only the components $s_0 = \frac{1}{2}$ are kept for the on-shell vacuum. In all, we have

$$\boxed{|0\rangle_{\mathrm{R}} = (\tfrac{1}{2}, 8) \oplus (\tfrac{1}{2}, 8').} \tag{6.110}$$

- All higher excitations form massive spinors in irreducible representations of $SO(9)$.

For later purposes we define the **fermion number**

$$F = \left\{ \begin{array}{ll} \sum_{r=\frac{1}{2}}^{\infty} b^i_{-r} b^i_r & \text{(NS)} \\ \sum_{n=1}^{\infty} b^i_{-n} b^i_n & \text{(R)} \end{array} \right\} \tag{6.111}$$

with the property that

$$(-1)^F = \left\{ \begin{array}{ll} 1 \quad \text{for even number of } b\text{-excitations} & \\ \sum_{n=1}^{\infty} b^i_{-n} b^i_n \quad \text{for odd number of } b\text{-excitations} & \end{array} \right\} \tag{6.112}$$

We will also need the G**-parity** operator

$$\text{(NS)}: \qquad G = (-1)^{F+1}, \tag{6.113}$$
$$\text{(R)}: \qquad G = \Gamma(-1)^F, \qquad \Gamma = \Gamma_0\Gamma_1 \ldots \Gamma_9. \tag{6.114}$$

The lowest level open spectrum can be summarised as follows:

Sector	G-parity	SO(8)	m^2	Statistics
(NS)	+	8_v	0	boson
(NS)	-	1	$-\frac{1}{2\alpha'}$	boson
(R)	+	8_s	0	fermion
(R)	-	8_c	0	fermion

Here 8_s and 8_c represent a postive and negative chirality Wel spinor, respectively.

6.6 Closed string spectrum in LCQ

We now turn to the closed string spectrum in lightcone quantisation. Up to the level-matching condition the left- and right-moving sectors are independent. Choosing NS and R periodicity conditions and positive or negative G-parity for the left- and right-moving sectors independently would give rise to $2 \times 2 \times 2 \times 2 = 16$ independent sectors of the form $(\mathrm{R}_+, \mathrm{R}_+)$, $(\mathrm{R}_-, \mathrm{R}_+)$, $(\mathrm{NS}_+, \mathrm{R}_+)$ etc. We will now see how to construct consistent string theories out of these.
The **mass-shell condition** is

$$\frac{\alpha'}{4}M^2 = (N - a) \stackrel{!}{=} (\tilde{N} - a) \qquad \text{(with the last equality due to level matching)} \tag{6.115}$$

Here

$$N = N^{(X)} + N^{(\psi)} \qquad \text{(left-moving)}, \tag{6.116}$$
$$\tilde{N} = \tilde{N}^{(X)} + \tilde{N}^{(\psi)} \qquad \text{(right-moving)}, \tag{6.117}$$
$$\text{and } a = \begin{cases} a_{NS} = \frac{1}{2} \\ a_R = 0 \end{cases}. \tag{6.118}$$

One observes that in the (NS$_-$)-sector $(N-a) \in \frac{2\mathbb{Z}+1}{2}$, while in the (NS$_+$), (R$_+$), (R$_-$) sectors $(N-a) \in \mathbb{Z}$. Therefore the (NS$_-$) sector cannot pair with the (NS$_+$), (R$_+$), (R$_-$) sectors due to level matching. This reduces the numbers of possible sectors to only 10 = 16 - (3+3).

A consistent superstring theory is formed by combining various of these 10 sectors. The procedure how to do so goes by the name of the GSO projection and will be discussed in the next section. Before we come to this, we analyse the lowest-lying states contained in these 10 sectors.

- The lowest-lying state is in the NS$_-$ $\otimes$ NS$_-$ sector and given by the ground state

$$|0;k\rangle_{NS} \otimes |0;k\rangle_{NS}, \qquad \text{forming a} \quad 1 \quad \text{of SO(8)} \quad \text{with} \qquad m^2 = -\frac{2}{\alpha'}. \tag{6.119}$$

  Again, the ground state is tachyonic.

- At the massless level we find the following possible states (up to interchange of left- and right-movers):

sector	state	SO(8)	statistics
(NS$_+$, NS$_+$)	$b^i_{-\frac{1}{2}}\,|0\rangle_{NS} \otimes \tilde{b}^i_{-\frac{1}{2}}\,|0\rangle_{NS}$	$8_v \otimes 8_v$	boson
(R$_+$, R$_+$)	$|+\rangle_R \otimes |+\rangle_R$	$8_s \otimes 8_s$	boson
(R$_+$, R$_-$)	$|+\rangle_R \otimes |-\rangle_R$	$8_s \otimes 8_c$	boson
(R$_-$, R$_-$)	$|-\rangle_R \otimes |-\rangle_R$	$8_c \otimes 8_c$	boson
(NS$_+$, R$_+$)	$b^i_{-\frac{1}{2}}\,|0\rangle_{NS} \otimes |+\rangle_R$	$8_v \otimes 8_s$	fermion
(NS$_+$, R$_-$)	$b^i_{-\frac{1}{2}}\,|0\rangle_{NS} \otimes |-\rangle_R$	$8_v \otimes 8_c$	fermion

Decomposition into irreducible representations of SO(8)

a) The decomposition of the states in the (NS$_+$, NS$_+$) sector into irreducible representations of $SO(8)$ proceeds in the same manner as for the closed string spectrum of the bosonic string, i.e. by decomposing

$8_v \otimes 8_v =$	[0]	+ [2]	+ (2)
	dilaton Φ (scalar)	Kalb-Ramond field $B_{\mu\nu}$ (anti-symmetric 2-form)	graviton $G_{\mu\nu}$ (symmetric traceless 2-tensor)

The (NS$_+$, NS$_+$) has the same degrees of freedom the closed sector in the bosonic theory.

b) The RR sector involves **spinor bilinears.** These are bosonic states as they arise as the product of two fermions. It must therefore be possible to decompose them into irreducible tensor representations of SO(8).

The decomposition of spinor bilinears in tensor representations is derived in [P], Appendix B.1, to which we refer for details. The general idea is to form spinor bilinears of the type $\bar{\zeta}\Gamma^{[\mu_1}\ldots\Gamma^{\mu_p]}\psi$ for suitable p, which transform as antisymmetric tensors. Suffice it here to state without proof the following result of this representation theoretic analysis: Consider a theory in $d = 2k+2$ spacetime dimensions with chiral Weyl spinors (2^k) and anti-chiral Weyl spinors $(2^k)'$. Then,

$$(2^k) \otimes (2^k) = \begin{cases} [1] + [3] + \ldots + [k+1]_+ \,, & k \text{ even.} \\ [0] + [2] + \ldots + [k+1]_+ \,, & k \text{ odd,} \end{cases} \tag{6.120}$$

$$(2^k) \otimes (2^k)' = \begin{cases} [0] + [2] + \ldots + [k] \,, & k \text{ even,} \\ [1] + [3] + \ldots + [k] \,, & k \text{ odd,} \end{cases} \tag{6.121}$$

where

- $[n]$ denotes a fully antisymmetric n-tensor, i.e. $[n] \equiv C^{(n)}$ with components $C^{(n)}_{[\mu_1 \ldots \mu_n]}$,
- $[k+1]_+$ denotes the self-dual part of the tensor with respect to Hodge $*$ duality. $C^{(+)}_{\mu_1 \ldots \mu_n} = *C^{(+)}_{\mu_1 \ldots \mu_n}$.

As a reminder, the Hodge $*$ operator maps

$$\begin{aligned} *: \quad & p - \text{form} \to (d-p) - \text{form} \\ & C^{(p)} \to *C^{(p)} = \frac{\sqrt{|g|}}{p!(d-p)!} C_{\mu_1 \ldots \mu_p} \epsilon^{\mu_1 \ldots \mu_p}{}_{\nu_{p+1} \ldots \nu_d} dx^{\nu_{p+1}} \wedge \ldots \wedge dx^{\nu_d}. \end{aligned} \tag{6.122}$$

It satisfies:

$$** = \begin{cases} (-1)^{p(d-p)+1}, & \text{Lorentzian signature} \\ (-1)^{p(d-p)}, & \text{Euclidean signature} \end{cases} \tag{6.123}$$

This leads to the following bosonic states the different sectors:

$$\begin{aligned} (\mathrm{R}_+, \mathrm{R}_+) \;:\; 8 \times 8 \quad & = \quad [0] + [2] + [4]_+ \\ \text{number of d.o.f.} \quad & \quad 1 + 28 + 35 = 1 + 28 + \frac{1}{2}\binom{8}{4} \\ (\mathrm{R}_+, \mathrm{R}_-) \;:\; 8 \times 8' \quad & = \quad [1] + [3] \\ \text{number of d.o.f.} \quad & \quad 8 + 56 \\ (\mathrm{R}_-, \mathrm{R}_-) \;:\; 8' \times 8' \quad & = \quad [0] + [2] + [4]_- \\ \text{number of d.o.f.} \quad & \quad 1 + 28 + 35 = 1 + 28 + \frac{1}{2}\binom{8}{4} \end{aligned}$$

c) The mixed R-NS sector contains **spinor** $\otimes$ **vector-bilinears**. Such objects are **spacetime fermions**.
A detailed representation theoretic analysis, which we do not carry out here, gives the following decomposition:

$(\mathrm{NS}_+, \mathrm{R}_+)$:	$8_v \otimes 8$	=	8'	+ 56
			spin 1/2 **dilatino** λ^a	spin 3/2 **gravitino** ψ^i_a
$(\mathrm{NS}_+, \mathrm{R}_-)$:	$8_v \otimes 8'$	=	8	+ 56'
			$\tilde{\lambda}^a$	$\tilde{\psi}^i_a$

The appearance and chirality of the dilatino can be seen as follows: Let $|i, s\rangle$ be the state $8_v \otimes 8$ with i a vector index of $SO(8)$. Then one can form the combination $\Gamma_i |i, s\rangle$. This is a spinor with $\Gamma \Gamma_i |i, s\rangle = -\Gamma_i \Gamma |i, s\rangle = -\Gamma_i |i, s\rangle$, corresponding to the $8'$ dilatino. The gravitino ψ^i_a - which should be confused with the worldsheet RNS fields ψ^μ_A - assembles then the remaining 56 degrees of freedom. It caries both a vector index i and a spinor index a. We will say more about the physical interpretation of the gravitino at the end of the next section.

6.7 The GSO projection: Type IIA and Type IIB

Our task is now to define consistent closed superstring theories by combining the 10 different sectors discussed in the previous section in a manner that leads to a well-defined CFT on the worldsheet. A priori, each of the 10 sectors may or may not be included in the final theory. This would lead to 2^{10} different theories. However, it turns out that only a few of these theories lead to consistent interactions on the worldsheet.
To understand the possible restrictions we first need to fill in a gap in our discussion of the RNS theory so far and formulate it in the language as a CFT on the sphere. We will be rather brief and only summarise the main points we will need in the sequel.

- The fermionic action S_F, eq. (6.10), is conformally invariant if the RNS fields ψ^μ_+ and ψ^μ_- are taken to be primary fields of conformal weight $h = \frac{1}{2}$.
- To define the theory on the sphere, we perform a Wick rotation and introduce the coordinate $w = \tau - i\sigma$ as in the bosonic theory. Along the way we switch to the notation $\psi^\mu(w)$ and $\tilde{\psi}^\mu(\bar{w})$ instead of $\psi^\mu_\pm$.
- On the sphere with coordinate $z = e^{\frac{2\pi}{\ell}w}$ the mode expansion of the primary fields can compactly be written as
$$\psi^\mu(z) = \sum_{r\in\mathbb{Z}+\phi} \frac{b^\mu_r}{z^{r+\frac{1}{2}}}, \qquad \tilde{\psi}^\mu(\bar{z}) = \sum_{r\in\mathbb{Z}+\phi} \frac{\tilde{b}^\mu_r}{\bar{z}^{r+\frac{1}{2}}} \tag{6.124}$$
with $\phi = 0\,(\frac{1}{2})$ in the Ramond (NS) sector. Importantly, an extra $\frac{1}{2}$ appears in the Laurent expansion because $\psi^\mu(z)$ is a primary of weight $h = \frac{1}{2}$.
- Therefore, on the sphere **the fields in the Ramond sector have a branch cut**, while the fields in the NS sector do not have such a branch cut. This feature carries over to the vertex operators for physical states. In particular, one can show that the vertex operator associated with the vacuum state in the Ramond sector introduces a branch cut on the sphere.

While we are at it let us at least briefly mention the following famous speciality of 2-dimensional CFT on the sphere that goes by the name of **bosonisation** - the fact that the fermionic RNS theory can be expressed entirely in terms of bosons.
Consider two RNS fields, say,$\psi^1(z)$ and $\psi^2(z)$ and combine them into the the complexified field
$$\psi(z) = \psi^1(z) + i\psi^2(z), \qquad \bar{\psi}(z) = \psi^1(z) - i\psi^2(z). \tag{6.125}$$
From our general discussion of CFT, the OPE of the RNS theory is
$$\psi^\mu(z_1)\psi^\nu(z_2) \simeq \frac{\eta^{\mu\nu}}{z_1 - z_2} \tag{6.126}$$
and similarly for the anti-holomorphic fields. This translates into
$$\psi(z)\bar{\psi}(0) \simeq \frac{1}{z}, \qquad \psi(z)\psi(0) \simeq \mathcal{O}(z) = \bar{\psi}(z)\bar{\psi}(0) \tag{6.127}$$
for the complexified field.
The idea is now that this OPE can be reproduced in terms of a bosonic field $H(z)$ with OPE
$$H(z)H(0) \simeq -\ln z. \tag{6.128}$$

Then the operator $: e^{iH(z)} :$ has the OPE (we omit the normal ordering symbol to reduce clumsiness)

$$e^{iH(z)}\, e^{-iH(0)} \simeq \frac{1}{z}, \qquad e^{iH(z)}\, e^{iH(0)} = \mathcal{O}(z), \qquad e^{-iH(z)}\, e^{-iH(0)} = \mathcal{O}(z). \tag{6.129}$$

Obviously the OPEs match. This suggests the identification

$$\psi(z) \equiv e^{iH(z)}, \qquad \bar{\psi}(z) \equiv e^{-iH(z)} \tag{6.130}$$

and similarly for the anti-holomorphic fields. Furthermore, it turns out that in all OPEs we identify

$$: \psi\bar{\psi} : (z) \equiv i\partial H(z). \tag{6.131}$$

This formulation is useful e.g. in the construction of vertex operators. The vertex operators associated with the Ramond vacuum $|s_0, s_1, \ldots, s_4\rangle$ is the spin field

$$e^{i\sum_a s_a H^a}. \tag{6.132}$$

To introduce more general vertex operators we need the superpartners of the bc-ghost system. These commuting $\beta\gamma$ ghosts and their bosonisation is described e.g. in [P], 10.4.

Let us now come back to our task of combining the different sectors into a consistent superstring theory. In fact, there are the following sources for **potential inconsistencies:**

a) As motivated above, a vertex operator in the R-sector has a square-root branch cut. This results in a monodromy as 2 such operators encircle each other. These **monodromies** must be absent in a consistent theory so that OPEs are single-valued. This imposes severe constraints on which sectors can be combined with each other into a consistent theory.

b) If one computes superstring one-loop amplitudes by generalising the technology we got to know in the bosonic theory for CFT computations on the torus, one notes that the result is only **modular invariant** if there is at least one left- and one right-moving Ramond sector in the theory.

One can show that these consistency conditions - together with closure of the OPE - imply that only the following combinations of sectors lead to consistent theories (see [P], 10.6. for a proof):
In **Type IIB theory the following four sectors are in the spectrum:**

$$\begin{aligned}
(\mathrm{NS}_+;\ \mathrm{NS}_+) &\ :\ \Phi,\ B_{[\mu\nu]},\ G_{(\mu\nu)} \\
(\mathrm{R}_+;\ \mathrm{R}_+) &\ :\ C^{(0)},\ C^{(2)}_{[\mu_1\mu_2]},\ C^{(4)+}_{[\mu_1\ldots\mu_4]} \\
(\mathrm{NS}_+;\ \mathrm{R}_+) &\ :\ \lambda_a,\ \psi^\mu_a \\
(\mathrm{R}_+;\ \mathrm{NS}_+) &\ :\ \lambda_a,\ \psi^\mu_a
\end{aligned}$$

Thus the theory is chiral because left- and right-movers have the **same chirality**.

In **Type IIA theory these four sectors are in the spectrum:**

$$\begin{aligned}
(\mathrm{NS}_+;\ \mathrm{NS}_+) &\ :\ \Phi,\ B_{[\mu\nu]},\ G_{(\mu\nu)} \\
(\mathrm{R}_+;\ \mathrm{R}_-) &\ :\ C^{(1)}_{\mu_1},\ C^{(3)}_{[\mu_1\mu_2\mu_3]} \\
(\mathrm{NS}_+;\ \mathrm{R}_-) &\ :\ \tilde{\lambda}_a,\ \tilde{\psi}^\mu_a \\
(\mathrm{R}_+;\ \mathrm{NS}_+) &\ :\ \lambda_a,\ \psi^\mu_a
\end{aligned}$$

Here left- and right-movers have **opposite chirality**.
In addition the consistency conditions allow for the following two types of theories:

- Type IIB' is as IIB, but with $\mathrm{R}_+ \to \mathrm{R}_-$.
  Type IIA' is as IIA, but with $\mathrm{R}_\pm \to \mathrm{R}_\mp$.
  These are equivalent to Type IIB/IIA.

- **Type 0A** consists of $(\mathrm{NS}_+;\ \mathrm{NS}_+)$, $(\mathrm{NS}_-;\ \mathrm{NS}_-)$, $(\mathrm{R}_+;\ \mathrm{R}_-)$, $(\mathrm{R}_-;\ \mathrm{R}_+)$.
  Type 0B consists of $(\mathrm{NS}_+;\ \mathrm{NS}_+)$, $(\mathrm{NS}_-;\ \mathrm{NS}_-)$, $(\mathrm{R}_+;\ \mathrm{R}_+)$, $(\mathrm{R}_-;\ \mathrm{R}_-)$.

The projection leading to the above consistent theories is called **GSO (Gliozzi-Scherk-Olive) projection**.
We note an important difference between the Type II and the Type 0 theories:

- In Type IIA/B, the $(\mathrm{NS}_-;\ \mathrm{NS}_-)$ sector is projected out. Since this is the sector that contains the tachyonic ground state, these theories are tachyon-free.

- On the other hand, the Type 0A/B theories are fully consistent from a worldsheet CFT perspective, but still contain the a tachyon in the $(\mathrm{NS}_-;\ \mathrm{NS}_-)$ sector. These theories are therefore dynamically unstable: Even if we allowed for these theories, a universe described by it immediately decays and thus plays no role. We can therefore discard these theories.

Conclusion:

CFT consistency + stability of vacuum (= absence of tachyon) $\Downarrow$ Type IIA or Type IIB as closed oriented superstring theories

Important remarks:

- Type IIA and Type IIB both contain an equal number of bosonic and fermionic degrees of freedom, e.g. 128 + 128 at the massless level. This is a necessary condition for **spacetime supersymmetry**, which exchanges bosonic and fermionic fields. Indeed, the Green-Schwarz formalism makes this spacetime SUSY manifest.

- Both IIA/B contain two massless spin 3/2 fields ψ^μ_a $(\tilde{\psi}^\mu_a)$ called **gravitinos**.
  In Type IIA, ψ^μ_a, $\tilde{\psi}^\mu_a$ have opposite chirality.
  In Type IIB, ψ^μ_a, ψ^μ_a have the same chirality.
  The **gravitino** is the **superpartner of the graviton.** Just like the presence of a massless vector boson in QFT implies a gauge symmetry since otherwise no consistent quantisation is possible, the presence of the gravitino ψ^μ_a implies that spacetime SUSY is local. The low-energy limit of the Type II theories is therefore a **supergravity**.

- The presence of 2 independent gravitinos implies the existence of **2 independent SUSY algebras.** The notation for this is $\mathcal{N} = 2$ **SUSY** in d=10, thereby explaining the name Type II. The superalgebras realise what is known on general grounds to be the maximal amount of supersymmetry in ten dimensions.

- As a consequence we have established the far reaching fact that **in string theory worldsheet consistency and stability of vacuum imply local SUSY in 10 dimensions.**

- To make contact with observation one considers string theory on a spacetime where the extra 6 spatial dimensions are small and compact, corresponding to the compactification ansatz $\mathbb{R}^{1,9} \to \mathbb{R}^{1,3} \times M^6$. In this process SUSY may or may not be broken at low energies in the non-compact four dimensions. There is no prediction from string theory at which scale SUSY is broken.

The low-energy effective action

The 10-dimensional low-energy effective action keeping only the massless modes for Type IIA and Type IIB theory can be computed order by order in spacetime and worldsheet perturbation theory, by generalising the methods we got to know in the bosonic theory. For completeness we collect here the bosonic sector to lowest order in α' and at string tree-level. As noted already, these Type IIA/B supergravity actions have the maximal possible amount of supersymmetry in ten dimensions. Their form is in fact completely fixed by supersymmetry.
The action takes the form

$$S_{IIA/B} = S_{NS} + S_R + S_{CS}, \tag{6.133}$$

where S_{NS} is the same for Type IIA/B and in fact coincides in form with the bosonic low-energy effective action

$$S_{NS} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} \Big(R + 4\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \Big) \tag{6.134}$$

with $H = dB$ the field strength of the Kalb-Ramond B-field. S_R contains the kinetic terms of the field strengths of the respective RR gauge potentials and S_{CS} certain topological terms. In **Type IIA** these are

$$\begin{aligned} S_R &= -\frac{1}{4\kappa_{10}^2} \int F_2 \wedge *F_2 + \tilde{F}_4 \wedge *\tilde{F}_4, && (6.135) \\ S_{CS} &= -\frac{1}{4\kappa_{10}^2} \int B_2 \wedge F_4 \wedge F_4 && (6.136) \end{aligned}$$

with

$$F_2 = dC_1, \qquad F_4 = dC_3, \qquad \tilde{F}_4 = F_4 - C_1 \wedge F_3. \tag{6.137}$$

In **Type IIB** we have

$$\begin{aligned} S_R &= -\frac{1}{4\kappa_{10}^2} \int F_1 \wedge *F_1 + \tilde{F}_3 \wedge *\tilde{F}_3 + \tilde{F}_5 \wedge *\tilde{F}_5, && (6.138) \\ S_{CS} &= -\frac{1}{4\kappa_{10}^2} \int C_4 \wedge H_3 \wedge F_3 && (6.139) \end{aligned}$$

with

$$F_1 = dC_0, \qquad F_3 = dC_2, \qquad F_5 = dC_4, \qquad \tilde{F}_3 = F_3 - C_0 \wedge H_3, \tag{6.140}$$

$$\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3. \tag{6.141}$$

Self-duality is imposed at the level of equations of motion as

$$\tilde{F}_5 = *\tilde{F}_5. \tag{6.142}$$

6.8 Digression: Differential forms

For your convenience we here collect a number of useful mathematical facts about differential forms and some aspects of their appearance in physics.
A p-form $C^{(p)}$ is a totally antisymmetric tensor of rank p:

$$C^{(p)} = \frac{1}{p!} C_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}. \tag{6.143}$$

- The wedge product between two forms is defined by

$$A^{(p)} \wedge B^{(q)} = C^{(p+q)} \tag{6.144}$$

$$C^{(p+q)}_{\mu_1\ldots\mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1\ldots\mu_p} B_{\mu_{p+1}\mu_{p+q}]}. \tag{6.145}$$

It satisfies

$$A^{(p)} \wedge B^{(q)} = (-1)^{pq} B^{(q)} \wedge A^{(p)}. \tag{6.146}$$

- In n dimensions, the integral of an n-form is defined as follows:

$$\int C^{(n)} = \int d^n x\, C^{(n)}_{01\ldots n}. \tag{6.147}$$

More generally, given an n-dimensional manifold $\mathcal{M}$, one can integrate a p-form over a p-dimensional submanifold Γ_p of $\mathcal{M}$ as $\int_{\Gamma_p} C^{(p)}$.

- The **exterior derivative** is defined as an operator

$$d : p - \text{form} \to (p+1) - \text{form}, \tag{6.148}$$

$$dC^{(p)} = \frac{1}{p!} \partial_{\mu_1} C_{\mu_2\ldots\mu_{p+1}} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{p+1}} \tag{6.149}$$

so that

$$(dC^{(p)})_{\mu_1\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1} C_{\mu_2\ldots\mu_{p+1}]}. \tag{6.150}$$

Here [] denotes normalised antisymmetrisation:

$$[\mu_1 \ldots \mu_p] = \frac{1}{p!} \left((\text{even perm. of } (12\ldots p)) - (\text{odd perm. of } (12\ldots p)\right). \tag{6.151}$$

The exterior derivative is nilpotent:

$$d^2 C^{(p)} = 0. \tag{6.152}$$

- One can therefore define the **cohomology** of the exterior derivative in analogy to the cohomology of the nilpotent BRST operator. First define

$$C^{(p)} \text{ closed} \leftrightarrow dC^{(p)} = 0, \quad C^{(p)} \text{ exact} \leftrightarrow C^{(p)} = dQ^{(p-1)}. \tag{6.153}$$

Then the p-th cohomology group of the n-dimensional manifold $\mathcal{M}$ is defined as the quotient

$$H^p(\mathcal{M}) = \frac{\text{closed } p - \text{forms}}{\text{exact } p - \text{forms}}. \tag{6.154}$$

Exact forms are said to be cohomologically trivial.
The dimension of $H^p(\mathcal{M})$,

$$b^p = \dim H^p(\mathcal{M}), \tag{6.155}$$

is called the p-th **Betti number** and is an important topological invariant of $\mathcal{M}$. For $\mathcal{M}$ compact, b^p is finite.

- **Stoke's theorem** expressed in form language takes the suggestive form

$$\int_{\Gamma_{p+1}} dC^{(p)} = \int_{\partial\Gamma_{p+1}} C^{(p)}, \tag{6.156}$$

where $\partial\Gamma_{p+1}$ denotes the p-dimensional boundary of the $(p+1)$-dimensional submanifold $\Gamma_{(p+1)}$.

- The operation of taking the boundary of a p-dimensional submanifold of $\mathcal{M}$ is therefore dual, in the above sense, to taking the exterior derivative of a p-form. A p-fold with $\partial\Gamma_p = 0$ is called a **p-cycle**, and a p-fold which is the boundary of another $p+1$-dimensional submanifold, $\Gamma_p = \partial\Omega_{p+1}$, is called a **$p$-boundary**. The object Ω_{p+1} is a **$p+1$-chain**. One then defines the p-th **homology** of $\mathcal{M}$ as the set
$$H_p(\mathcal{M}) = \frac{p - \text{cycles}}{p - \text{boundaries}}, \tag{6.157}$$
with dimension $b_p = \text{dim}H_p(\mathcal{M})$.
- The **de Rham dual** of a p-fold Γ_p is defined as the $(n-p)$-form $\delta^{(n-p)}(\Gamma_p)$ such that for each p-form $\omega^{(p)}$ one has
$$\int_{\Gamma_p} \omega^{(p)} = \int_{\mathcal{M}} \omega^{(p)} \wedge \delta^{(n-p)}(\Gamma_p). \tag{6.158}$$
This gives a **1-to-1 pairing between homology and cohomology**. In particular,
$$b_p = b^p. \tag{6.159}$$
- All quantities defined so far are topological in that they exist without reference to any metric on $\mathcal{M}$. By contrast, the **Hodge $*$-operator** makes use of a metric $g_{\mu\nu}$, in the following sense: Define first the antisymmetric tensor
$$\epsilon_{\mu_1\ldots\mu_n} = \begin{cases} \pm 1 \text{ for } (\mu_1 \ldots \mu_n) \text{ even/odd perm. of } (1, 2, \ldots n) \\ 0 \text{ else} \end{cases} \tag{6.160}$$
Then
$$\begin{aligned} *: \quad & p - \text{form} \to (n-p) - \text{form} \\ & C^{(p)} \to *C^{(p)} = \frac{\sqrt{|g|}}{p!(n-p)!} C_{\mu_1\ldots\mu_p} \epsilon^{\mu_1\ldots\mu_p}{}_{\nu_{p+1}\ldots\nu_n} dx^{\nu_{p+1}} \wedge \ldots \wedge dx^{\nu_n}. \end{aligned} \tag{6.161}$$
It satisfies:
$$** = \begin{cases} (-1)^{p(n-p)+1}, \text{ Lorentzian signature} \\ (-1)^{p(n-p)}, \text{ Euclidean signature} \end{cases} \tag{6.162}$$
One can show:
$$\omega^{(p)} \wedge *\eta^{(p)} = \frac{1}{p!} \omega_{\mu_1\ldots\mu_p} \eta^{\mu_1\ldots\mu_p} \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n, \tag{6.163}$$
or, more importantly,
$$\int_{\mathcal{M}} \omega^{(p)} \wedge *\eta^{(p)} = \frac{1}{p!} \int_{\mathcal{M}} \omega_{\mu_1\ldots\mu_p} \eta^{\mu_1\ldots\mu_p} \sqrt{|g|} dx^1 \wedge \ldots \wedge dx^n. \tag{6.164}$$

6.8.1 p-form potentials in physics

- In string theory we find various exmples of p-form fields. In the bosonic string the only such example is the Kalb-Ramond 2-form potential $B = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$. In superstring theory, we have found in addition the Ramond-Ramond form potentials $C^{(p)}$ with $p = 1, 3$ in Type IIA and $p = 0, 2, 4$ in Type IIB.
- These higher rank form fields can be interpreted as generalisations of the electromagnetic 1-form gauge potential $A = A_\mu dx^\mu$.
This is because the field strength
$$F^{(p+1)} = dC^{(p)} \tag{6.165}$$
is invariant under the abelian gauge transformation
$$C^{(p)} \to C^{(p)} + d\chi^{(p-1)} \tag{6.166}$$

by virtue of nilpotency of the exterior derivative.
The so-defined field strength is closed because

$$dF^{(p+1)} = d\,(dC^{(p)}) = 0. \tag{6.167}$$

This is the **Bianchi identity**.

- The canonical kinetic term of the field strength can compactly be written

$$S_{\text{kin}} = -\frac{1}{2(p+1)!}\int d^n x\sqrt{|g|}F_{\mu_1\cdots\mu_{p+1}}F^{\mu_1\cdots\mu_{p+1}} = -\frac{1}{2}\int F\wedge *F. \tag{6.168}$$

- Hodge $*$-duality shows that in n dimensions, a $(p+1)$-field strength $F^{(p+1)} = dC^{(p)}$ describes the same number of degrees of freedom as an $(n-p-1)$-field strength $\tilde{F}^{(n-p-1)} = d\tilde{C}^{(n-p-2)}$. This can be expressed as duality in the sense

$$F^{(p+1)} = *\tilde{F}^{(n-p-1)}. \tag{6.169}$$

Thus **a p-form potential is dual to an** $(n-p-2)$ **form potential**. This is the generalisation of electric-magnetic duality in four dimensions: If $n=4$, a vector potential $A^{(1)}$ is dual to a vector potential $\tilde{A}^{(1)}$ (see exercise 1).

- A p-form couples naturally to a p-fold via

$$S_{coup} = \mu_p\int_{\Gamma_p} C^{(p)}, \tag{6.170}$$

where μ_p is the charge of Γ_p. This is the natural generalisation of the coupling of the 1-form potential $A^{(1)}$ to a point particle.

6.9 Type I theory

As we have seen, Type IIA and Type IIB are two consistent, closed, oriented superstring theories in ten dimensions. In this section we will get to know Type I theory as the only consistent ten-dimensional string theories with open strings.
Our starting point is the observation that the chiral Type IIB theory is invariant under the exchange of right- and left movers. From the worldsheet perspective this symmetry is worldsheet parity,

$$\Omega:\ \sigma\to\ell-\sigma. \tag{6.171}$$

On the string fields this symmetry is realised by the unitary operator Ω such that

$$\begin{aligned}\Omega^\dagger\, X^\mu(\tau,\sigma)\,\Omega &= X^\mu(\tau,\ell-\sigma), &(6.172)\\ \Omega^\dagger\,\psi^\mu(\tau,\sigma)\,\Omega &= \psi^\mu(\tau,\ell-\sigma). &(6.173)\end{aligned}$$

The induced action on the closed string oscillators has already been discussed on Assignment 6 for the bosonic theory. Generalising the logic gives

$$\begin{aligned}\Omega^\dagger\,\alpha^\mu_n\,\Omega &= \tilde{\alpha}^\mu_n. &(6.174)\\ \Omega^\dagger\, b^\mu_n\,\Omega &= e^{2\pi i\phi}\,\tilde{b}^\mu_n, \quad \phi = \begin{cases} 0, & \text{R},\\ \frac{1}{2}, & \text{NS}.\end{cases} &(6.175)\end{aligned}$$

One can therefore (preliminarily) define a new unoriented theory by taking the quotient

$$(\text{Type I})_{\text{closed}} := \frac{\text{Type IIB}}{\Omega} \tag{6.176}$$

as follows:

1) **Bosonic sector:**
The action of Ω on the groundstate is

$$(\mathrm{NS},\mathrm{NS}) \quad : \quad \Omega\,|0\rangle_L \otimes |0\rangle_R = |0\rangle_L \otimes |0\rangle_R, \tag{6.177}$$
$$(\mathrm{R},\mathrm{R}) \quad : \quad \Omega\,|a\rangle_L \otimes |b\rangle_R = -\,|b\rangle_L \otimes |a\rangle_R. \tag{6.178}$$

The minus sign in the R-R sector is due the fermionic nature.

i) The (NS, NS) sector contains, at the massless level the following fields: $B_{\mu\nu} = -B_{\nu\mu}$ forming an antisymmetric 2-tensor of of SO(8). Therefore $B_{\mu\nu}$ is Ω-odd and projected out upon taking the quotient by Ω. The remaing fields $(\Phi, G_{\mu\nu})$ transform as the symmetric representation of SO(8) with 36 degrees of freedom. They are Ω-even and and thus projected in.

ii) In the (R, R)-sector the massless fields behave as follows: The 2-form $C^{(2)}_{[\mu\nu]}$ is in the antisymmetric representation of SO(8), but due to the extra (-1) from the action of Ω on the RR sector ground state, this field is effectively Ω-even. The remaining $C^{(0)}$, $C^{(4)+}$ contain 36 degrees of freedom and thus combine into the symmetric representation of SO(8). They are in turn Ω-odd.

$\Rightarrow$ The bosonic sector of Type $\mathrm{I}_{\text{closed}}$ contains, at the massless level, the fields $G_{\mu\nu},\ \Phi,\ C^{(2)}_{[\mu\nu]}$.

2) **Fermionic sector:**
The action of Ω on the R-NS and NS-R groundstate is simply

$$\Omega\,|0\rangle_{L,\mathrm{NS}} \otimes |a\rangle_{R,\mathrm{Ramond}} = |a\rangle_{L,\mathrm{Ramond}} \otimes |0\rangle_{L,\mathrm{NS}}, \tag{6.179}$$

i.e. it exchanges the NS-R and the R-NS sector. Thus the diagonal combination of the 2 copies of $\lambda^a,\ \psi^\mu_a$ transforming as $(8_c, 56)$ of $SO(8)$ remains in the spectrum.
Consequenctly, the fermionic sector contains, at the massless level, one copy of $(\lambda^a,\ \psi^\mu_a)_{(\text{NS-R + R-NS})}$ in the $(8_c, 56)$ of $SO(8)$.

As a result, our preliminary Type $\mathrm{I}_{\text{closed}}$ theory preserves only 1/2 of the SUSY of Type IIB and thus realises $\mathcal{N} = 1$ SUSY in 10D.

Due to the Ω-projection it must be checked anew if the theory is really fully consistent at the level of interactions. We cannot go through these computations in detail for reasons of time, but we can understand the logic and the consequences of this sanity check as follows.

i) Computation of the 1-loop amplitude requires summing over a torus amplitude together with a Klein bottle amplitude, which is the unoriented Riemann surface at Euler characteristic $\chi = 0$ with no boundaries. As mentioned in section 5.5.3, the Klein bottle partition function exhibits an infrared (IR) divergence. Thus, **the preliminary Type $\mathbf{I}_{\text{closed}}$ theory is inconsistent at 1-loop level due to appearance of a tadpole.**

ii) The closed string divergence can be cancelled in a Lorentz invaraint manner in ten dimensions if we include open string degrees of freedom with NN boundary conditions in all 10 dimensions from the sectors (NS_+) and (R_+). In particular at massless level we must include

$$A^\mu \ \text{ vector boson } \overset{\text{SUSY}}{\leftrightarrow} \ \psi_a \ \text{ gaugino.} \tag{6.180}$$

Inclusion of open strings with NN boundary conditions in 10 dimensions means inclusion of D9-branes.

iii) As in the bosonic theory the open unoriented theory contributes an annulus and Möbius amplitude to the full partition function, each of which contain IR-divergences. Similar to the bosonic discussion in section 5.5.3, for a stack of N coincident D9-branes the overall tadpole turns out to be proportional to

$$(2^{d/2} \pm N)^2 \int_0^\infty ds = (2^5 \pm N)^2 \int_0^\infty ds. \tag{6.181}$$

The two signs correspond to the two possible actions of Ω on the Chan-Paton labels for the massless gauge bosons

$$\Omega\,|ij,p\rangle = \pm\,|ij,p\rangle \qquad \text{with basis } \lambda^a_{ij} = \pm(\lambda^a_{ij})^{\mathrm{T}}. \tag{6.182}$$

where $+/-$ generates symplectic (SP)/orthogonal (SO) groups.

This comes about as follows (for details see [P] I, 6.5, pg. 189-192):

A general open string state with CP labels ij transforms as

$$\Omega|k;ij\rangle = (-1)^{1+\alpha' M^2}\gamma_{jj'}|k;j'i'\rangle\gamma^{-1}_{i'i}. \tag{6.183}$$

The sign factor results from the action of worldsheet parity on the open string oscillators. If we restrict ourselves for simplicity to the bosonic string theory the action is

$$\Omega\,\alpha^\mu_n\,\Omega^{-1} = (-1)^n\alpha^\mu_n. \tag{6.184}$$

One can argue similarly in the superstring, where however some complications arise that we do not discuss here. The γ-matrices, which are in general allowed, square to 1 because $\Omega^2 = 1$. This requires

$$\gamma^T = \pm\gamma. \tag{6.185}$$

For the upper sign, a $U(N)$ gauge transformation on the CP factors can be used to arrive at $\gamma = 1$. In this case, the massless gauge bosons must have antisymmetric Chan-Paton factors with basis

$$(\lambda^a_{ij})^T = -\lambda^a_{ij}. \tag{6.186}$$

For the lower sign, N must be even and a $U(N)$ gauge transformation can be used to bring γ in the form of a symplectic matrix. Eventually,

$$(\lambda^a_{ij})^T = +\lambda^a_{ij}. \tag{6.187}$$

This defines Type I theory as the unoriented 10-dimensional theory by adding to Type $\mathrm{I}_{\mathrm{closed}}$ the degrees of freedom from the R_+ and NS_+ open sector of 32 D9-branes subject to the orientifold projection (6.182) with upper sign. Its gauge group is SO(32) (more precisely $Spin(32)/\mathbb{Z}_2$.)

Comment: In 10 dimensions all other brane configurations except for the one of Type I theory are inconsistent. Upon compactification absence of tadpoles can be used to construct lower-dimensional consistent brane configurations. These are no more unique.

Final remarks on consistent theories in 10 dimensions:

- We have established 3 consistent superstring theories in 10 dimensions: Type IIA/B theory contains closed oriented strings only, while Type I theory is a theory of closed plus open unoriented strings with gauge group SO(32).

- In addition, one can construct two variants of the so-called **heterotic string theory** in ten dimensions, which owes its name to the fact that the left- and the right-moving sectors consist of different chiral CFTs. Concretely one combines ten left-moving copies of the RNS CFT with fields X_L^i and ψ_L^i, $i = 0, 1, \ldots, 9$ with a right-moving bosonic theory X_R^μ, $\mu = 0, \ldots, 25$. Extra constraints arise by requiring modular-invariance of the 1-loop amplitudes. Heterotic theory is a theory of closed strings only, but nonetheless gives rise to gauge symmetry in 10 dimensions. The gauge symmetry is a consequence of compactification of the bosonic sector along the additional $26 - 10 = 16$ spatial dimensions, giving rise to a rank 16 gauge group.[3] There are two possible gauge symmetries respecting all consistency conditions: The SO(32) heterotic string and the $E_8 \times E_8$ heterotic string.

- Until 1995 it seemed that all these 5 consistent theories in 10 dimensions are independent. As a consequence of the duality revolution it was realised, however, that they are related by dualities. Thus they should be interpreted as different manifestations of one underlying theory.

[3] We will see in the next section that for toroidal compactifications, each compactified spacetime dimension gives rise to one abelian gauge boson, with non-abelian enhancements at special radii.

Chapter 7

Compactification, T-duality, D-branes

In the previous chapters we have learned how to formulate a fully consistent theory of quantum gravity and Yang-Mills theory in 10 dimensions, unique up to dualities. This theory has all the prerequisites one expects a theory of everything to have - except for one little detail: The spacetime we live in does not exhibit 10 large dimensions.
If we are interested in making contact with experiment we have two options: Either we discard superstring theory completely and start searching for a fundamental theory anew. So far this search has not lead to anything comparable to string theory, which of course is not a proof that this must remain so. On the other hand, if we restrict ourselves to the class of theories which are already known to exist, we can make contact with observations by requiring that the extra 6 spatial dimensions be compact and small - so small that they have not been discovered in any experiment so far. This strategy goes by the name of compactification and follows, in fact, an old idea by Kaluza and Klein.

7.1 Kaluza-Klein compactification in field theory

Before discussing compactification in string theory we review the idea of Kaluza-Klein compactification in field theory, which goes back to the work of Kaluza in 1914 and was further extended by Klein.
Consider therefore a field theory in D = 1+d spacetime dimensions. Let dimension x^d be "rolled up on a circle", i.e. identify

$$x^d \equiv x^d + 2\pi R. \tag{7.1}$$

This corresponds to the compactification ansatz

$$\mathbb{R}^{1,d-1} \to \mathbb{R}^{1,d-2} \times S^1. \tag{7.2}$$

The compactified space - here S^1 - is in general called the **internal space** or the **compactification manifold**.
For a general diffeomorphism invariant field theory this has 3 consequences:

a) There appears a Kaluza-Klein tower of massive states in $(D-1)$ dimensions.

b) There appears an extra $U(1)$ symmetry in (D-1) dimensions.

c) There appear massless scalar fields called modulus fields in $(D-1)$ dimensions.

Ad a):
Let $M, N = 0, 1, ..., d-1, d$ and $\mu, \nu = 0, 1, ..., d-1$. For simplicity we consinder a massless scalar field in D dimensions

$$\Phi(x^M) \ : \quad \partial_M \partial^M \Phi(x^M) = 0, \tag{7.3}$$

but the reasoning can be applied to all kind of field theories. To respect the spacetime periodicity (7.1), Φ must be periodic in x^d.
The most general ansatz for $\Phi(x)$ is to take

$$\Phi(x^M) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu) e^{i \frac{n}{R} x^d}, \tag{7.4}$$

which corresponds to expansion in a complete set of periodic functions in x^d. Plugging this into (7.3) yields

$$\boxed{\partial_\mu \partial^\mu \phi_n(x^\mu) = \frac{n^2}{R^2} \phi_n(x^\mu) \qquad \forall n.} \tag{7.5}$$

Thus, the n-th Fourier mode $\phi_n(x^\mu)$ appears as a scalar field of mass $m_n^2 = \frac{n^2}{R^2}$ from the perspective of the $(D-1)$-dimensional theory. The collection of these massive scalars are called the **Kaluza-Klein (KK) tower of states**. Note that the zero-mode ϕ_0 is massless and independent of x^d.

As $R \to 0$, the mass of the lowest-lying state $m_1^2 \to \infty$ and the KK tower disappears from the low-energy spectrum. At energies $E << \frac{1}{R}$ the theory looks $(D-1)$-dimensional - we are in the realm of the **low-energy effective field theory**.

Ad b):
The extra $U(1)$ **gauge potential** arises from the components $G^{(D)}_{\mu d}$ of the D-dimensional metric. The most general metric ansatz is

$$ds^2 = G^{(D)}_{MN} dx^M dx^N = G_{\mu\nu} dx^\mu dx^\nu + G_{dd} \left(dx^d + A_\mu dx^\mu \right)^2, \tag{7.6}$$

i.e. we parametrise

$$G^{(D)}_{\mu d} = 2\, G_{dd}\, A_\mu. \tag{7.7}$$

Consider for simplicity the zero modes of $G_{\mu\nu}$, G_{dd}, A_μ, i.e. let all components depend only on x^μ. The subgroup of diffeomorphisms in D spacetime dimensions compatible with the ansatz (7.2) transforms as follows:

- Diffeomorphism invariance in $(d-1)$-dimensions implies $x^\mu \to x'^\mu(x^\nu)$.
- Diffeomorphism invariance along the circle S^1 formed by x^d implies $x^d \to x'^d = x^d + \lambda(x^\mu)$. This entails

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \lambda. \tag{7.8}$$

Therefore A_μ is a gauge potential in $D-1$ dimensions. The associated KK $U(1)$ symmetry descends from diffeomorphism invariance in D dimensions. Under the compactification ansatz (7.2) the group of diffeomorphisms decomposes as

$$Gl(D,\mathbb{R}) \rightarrow Gl(D-1,\mathbb{R}) \times U(1). \tag{7.9}$$

Ad c):
The metric component G_{dd} is a scalar field from the perspective of the $(D-1)$-dimensional theory. It sets the volume of the internal space

$$\mathrm{Vol}(S^1) = \int_0^{2\pi R} dx^d \sqrt{G_{dd}} = \sqrt{G_{dd}} \cdot 2\pi R, \tag{7.10}$$

i.e. the vacuum expectation value (VEV) of the scalar field G_{dd} determines a geometric property of the internal space - here the volume of S^1. The appearance of the scalar field G_{dd} in the D_1 dimensional effective action follows by dimensional reduction, i.e. expansion of the D-dimensional Einstein action. It turns out that $\sqrt{G_{dd}}$ is not constrained by a potential. One says that the scalar field is flat. In particular, it is massless. Such **flat scalar fields whose VEV determine geometric properties of the compactification space are called moduli fields**.

7.2 KK compactification of closed bosonic strings

Let us now apply this simple idea of KK compactification to string theory. We will see that important new features appear due to the stringy nature of the theory.
Recall the closed bosonic mode expansion (setting from now on $2\pi/\ell \equiv 1$)

$$X^\mu(\tau,\sigma) = \frac{x^\mu}{2} + \frac{\tilde{x}^\mu}{2} + \sqrt{\frac{\alpha'}{2}}\left(\alpha_0^\mu + \tilde{\alpha}_0^\mu\right)\tau + \sqrt{\frac{\alpha'}{2}}\left(\alpha_0^\mu - \tilde{\alpha}_0^\mu\right)\sigma + N + \tilde{N} \tag{7.11}$$

where

$$\sqrt{\frac{\alpha'}{2}}\tilde{p}^\mu = \alpha_0^\mu, \quad \sqrt{\frac{\alpha'}{2}}\tilde{p}^\mu = \tilde{\alpha}_0^\mu, \qquad \mu = 0,1,...,D \tag{7.12}$$

Under $\sigma \rightarrow \sigma + 2\pi$

$$X^\mu(\tau,\sigma) \rightarrow X^\mu(\tau,\sigma) + 2\pi\sqrt{\frac{\alpha'}{2}}\left(\alpha_0^\mu - \tilde{\alpha}_0^\mu\right). \tag{7.13}$$

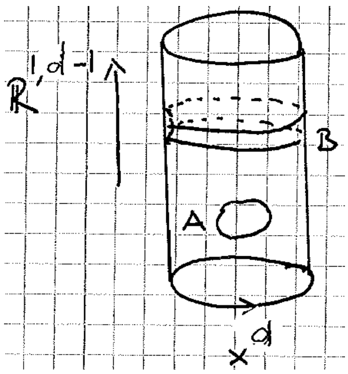
Imposing periodic boundary conditions thus lead us to $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\tilde{p}^\mu$.
Now consider KK compactification of x^d along an S^1 by identifying $x^d \cong x^d + 2\pi R$.
In string theory this has 2 consequences:

i) The momentum in direction x^d is quantised as before: $p^d = \frac{n}{R}$. Therefore

$$(\alpha_0^d + \tilde{\alpha}_0^d) = 2\sqrt{\frac{\alpha'}{2}}\frac{n}{R}. \tag{7.14}$$

This is a field theory effect.

ii) In addition something new appears: We can have **winding strings** looping w times around the compact S^1.



String A is non-winding:

$$x^d(\tau, \sigma + 2\pi) = x^d(\tau, \sigma).$$

String B is winding ω-times around S^1:

$$x^d(\tau, \sigma + 2\pi) = x^d(\tau, \sigma) + \omega\, 2\pi\, R.$$

Here we think of S^1 as $\mathbb{R}/2\pi\, R$, i.e. work in covering space.

Comparison with (7.13) yields

$$\alpha_0^d - \tilde{\alpha}_0^d = \sqrt{\frac{2}{\alpha'}}\omega\, R. \tag{7.15}$$

Thus for winding strings the **left- and right-moving "momenta" are independent**:

$$\alpha_0^d = \left(\frac{m}{R} + \frac{\omega\, R}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}} = p_L^d\sqrt{\frac{\alpha'}{2}}, \tag{7.16}$$

$$\tilde{\alpha}_0^d = \left(\frac{m}{R} - \frac{\omega\, R}{\alpha'}\right)\sqrt{\frac{\alpha'}{2}} = p_R^d\sqrt{\frac{\alpha'}{2}}. \tag{7.17}$$

Note that we still have that the centre-of-mass momentum is given by

$$p_L^d + p_R^d = \frac{n}{R}. \tag{7.18}$$

The mass-shell condition (for simplicity again for the bosonic string) follows as always from the Virasoro constraints

$$(L_0 - 1)\,|\phi\rangle = 0, \qquad (\tilde{L}_0 - 1)\,|\phi\rangle = 0. \tag{7.19}$$

With $L_0 = \frac{1}{2}\alpha_0^2 + N$ and $\tilde{L}_0 = \frac{1}{2}\tilde{\alpha}_0^2 + \tilde{N}$ this implies

$$\Rightarrow \qquad \underbrace{-p^\mu p_\mu}_{\mu=0,1,\ldots,d-1} = (p_L^d)^2 + \frac{4}{\alpha'}(N-1) = (p_R^d)^2 + \frac{4}{\alpha'}(\tilde{N}-1). \tag{7.20}$$

This leads to an effective mass in $(D-1)$ dimensions of the form

$$\boxed{m^2 = -p_\mu p^\mu = \frac{n^2}{R^2} + \frac{\omega^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2).} \tag{7.21}$$

The level-matching condition $0 = (L_0 - \tilde{L}_0)\,|\phi\rangle$ now relates the left- and right-moving oscillation numbers as

$$\boxed{N - \tilde{N} = n\,\omega.} \tag{7.22}$$

We observe the following structure:

- The sector $n = \omega = 0$ gives rise to the familiar states present also for $R \to \infty$.
- The sector $\omega = 0$, $n \neq 0 \Rightarrow$ contains the KK tower of massive KK excitations present also in point particle theory on an S^1.

- The sector $\omega \neq 0$ contains **winding states** of mass $\frac{\omega^2 R^2}{\alpha'}$.
  Note that the mass formula reflects the fact that winding costs energy due to the string tension. The winding sector represents a **truly string effect** not present for point particles.

Consider now the limit $R \to 0$.

i) The KK tower disappears from the low-energy spectrum. For a point particle theory this would be the end of the story and we would conclude that the theory becomes effectively $(1+(d-1))$ dimensional.

ii) However, the winding states become light because their mass scales as $m^2 \sim \frac{1}{\alpha'}\omega^2 R^2$.

We conclude that unlike a point particle theory, string theory on S^1 remains effectively $(1+(d-1))$-dimensional. We will further discuss this stunning observation below, but first let us analyze in more detail the massless spectrum.

For **generic values of** R the massless spectrum corresponds to the sectors $n \stackrel{!}{=} \omega \stackrel{!}{=} 0$, $N \stackrel{!}{=} \tilde{N} \stackrel{!}{=} 1$. We find the following states:

- $\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}\,|0,k\rangle$ gives rise to $G_{(\mu\nu)}$, $B_{[\mu\nu]}$ and Φ in the non-compact $1+(d-1)$ dimensions.
- $(\alpha^\mu_{-1}\tilde{\alpha}^d_{-1} + \alpha^d_{-1}\tilde{\alpha}^\mu_{-1})\,|0,k\rangle$ corresponds to a vector from the perspective of the $1+(d-1)$ non-compact dimensions. This is just the $U(1)$ potential from $G^{(D)}_{\mu\, d}$ encountered also for a point particle theory.
- $(\alpha^\mu_{-1}\tilde{\alpha}^d_{-1} - \alpha^d_{-1}\tilde{\alpha}^\mu_{-1})\,|0,k\rangle$ gives another vector, cooresponding to the component $B_{\mu\, d}$.
- $\alpha^d_{-1}\tilde{\alpha}^d_{-1}\,|0,k\rangle$ gives a scalar corresponding to the component $G^{(D)}_{dd}$.

Thus we find a $U(1)\times U(1)$ gauge symmetry and a modulus in the non-compact dimensions. ✓

At **special values of** R extra states appear. In particular for $R=\sqrt{\alpha'}$

$$p^d_{L,R} = \frac{1}{\sqrt{\alpha'}}(n \pm \omega). \tag{7.23}$$

Massless states, $m^2=0$, now require

$$(n+\omega)^2 + 4\,N = (n-\omega)^2 + 4\tilde{N} = 4 \quad \text{together with level-matching} \quad N - \tilde{N} = n\,\omega. \tag{7.24}$$

This includes the following new states:

$$\text{The sector} \quad n=\omega=\pm 1, \quad N=0, \quad \tilde{N}=1 \qquad \text{gives 2 more vectors.} \tag{7.25}$$

$$\text{The sector} \quad n=-\omega=\pm 1, \quad N=1, \quad \tilde{N}=0 \qquad \text{gives another 2 more vectors.} \tag{7.26}$$

These are in addition to the two gauge bosons present for generic values of R. This suggest that **at** $R=\sqrt{\alpha'}$ **the symmetry** $U(1)\times U(1)$ **is enhanced to** $SU(2)\times SU(2)$, where each $SU(2)$ factor accounts for three gauge bosons. Indeed this assertion can be checked at the level of interactions.

Remarks:

- This appearance of non-abelian enhancements at $R_c = \sqrt{\alpha'}$ is a stringy effect not present in field theory.

- The compactification on S^1 can be generalised to compactification of several dimensions on a torus $T^d = S^1 \times \ldots \times S^1$.
  The resulting **toroidal compactification** is a special case of compactification of several dimensions on more general internal spaces.

- Recall from the closing remarks of the previous chapter how the **heterotic string theory** is defined:

  a) Combine a left-moving superstring theory $(X^\mu, \psi^\mu)_L$, $\quad \mu = 0, \ldots, 9$ with a right-moving bosonic string theory X_R^m, $m = 0, \ldots, 25$.
  b) Compactify X_R^n, $n = 10, \ldots, 25$ on T^{16} to form a 10-dimensional theory.

  The compactified right-moving sector gives rise to gauge bosons in the ten-dimensional theory - arising now from the right-moving closed sector. At generic radius the gauge group would be $U(1)^{16}$, corresponding to 16 independent vector bosons (note that each dimension yields only $U(1)$ as opposed to $U(1) \times U(1)$ because only the right-moving sector contributes.) However, modular invariance of the 1-loop partition function poses strong constraints on the theory and in particular enforces that the radii of the T^{16} be at their critical value, for which one finds non-abelian enhancement of $U(1)^{16}$. The two consistent ways to compactly T^{16} turn out to lead to gauge groups

  $$G = SO(32) \qquad \text{or} \qquad G = E_8 \times E_8, \tag{7.27}$$

  corresponding to the two possible heterotic string theories in 10 dimensions.

7.3 T-duality

Let us first consider closed superstrings of Type IIA or Type IIB type.

a) **Bosonic sector**
From the mass formula

$$m^2 = \frac{n^2}{R^2} \frac{1}{\alpha'^2} R^2 \omega^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2) \tag{7.28}$$

for closed bosonic strings compactified on a circle of radius R we observe that the spectrum is invariant under the operation

$$n \leftrightarrow \omega \qquad\qquad R \leftrightarrow R' = \frac{\alpha'}{R}, \tag{7.29}$$

which exchanges the KK momentum and the winding momentum.

This is a truly **stringy** feature that relies on the extended, non-local nature of a string. The transformation (7.29) is called **T-duality**; it extends to an exact symmetry of the closed CFT, including interactions. To see this we note that exchanging $n \leftrightarrow \omega$ corresponds to the transformation

$$p_L^d \rightarrow p_L \qquad\qquad p_R^d \rightarrow -p_R^d. \tag{7.30}$$

More generally, T-duality is defined by extending this to a fully fledged parity transformation not only of the right-moving momenta, but of the full string field including the oscillations,

$$X_L^d(z) \rightarrow X_L^d(z) \qquad X_R^d(z) \rightarrow -X_R^d(z), \tag{7.31}$$

i.e. we map

$$X^d(z,\bar{z}) = X^d_L(z) + X^d_R(\bar{z}) \to \qquad X'^d(z,\bar{z}) = X^d_L(z) - X^d_R(\bar{z}). \tag{7.32}$$

Now, one can convince oneself that replacing $X^d \to X'^d$ is indeed a symmetry of the X-CFT.

Physical conclusions:

Since the spectrum and all interactions are left invariant by the transformation (7.29) we have established that in string theory

$$\text{Physics at} \quad R < \sqrt{\alpha'} \quad \cong \quad \text{Physics at} \quad R > \sqrt{\alpha'}. \tag{7.33}$$

Consequently there is a minimal distance $R = \sqrt{\alpha'}$, corresponding to the self-dual radius. It does not make sense to define distances smaller than this minimal radius, to the extent that we can always map all processes at such distances back to radii bigger than $\sqrt{\alpha'}$.

Note that precisely at the self-dual radius $R = \sqrt{\alpha'}$ we have gauge enhancement $U(1)\times U(1) \to SU(2) \times SU(2)$.

b) **T-duality for Type IIA/B superstrings**

T-duality is readily generalized to the superstring as follows: Consider T-duality along x^d, $d = 9$.

- The bosonic fields transform as derived above, i.e. $\tilde{X}^9_R(\bar{z}) \to -\tilde{X}^9_R(\bar{z})$.
- By worldsheet superconformal invariance also $\tilde{\psi}^9_R(\bar{z}) \to -\tilde{\psi}^9_R(\bar{z})$.

In the R-sector, this implies in particular for the right-moving zero modes:

$$\tilde{b}^8_0 \pm i\,\tilde{b}^9_0 \quad \to \quad \tilde{b}^8_0 \mp i\,\tilde{b}^9_0. \tag{7.34}$$

From our previous identification $\tilde{b}^\mu_0 = \frac{1}{\sqrt{2}}\,\tilde{\Gamma}^\mu$ with $\tilde{\Gamma}$ acting on right-movers this means that $\tilde{\Gamma}^{4\pm} \to \tilde{\Gamma}^{4\mp}$.

We conclude that T-duality flips the chirality for right-moving spinors and therefore transforms the various superstring sectors as

$$(\mathrm{R}^+, \mathrm{R}^\pm) \quad \to \quad (\mathrm{R}^+, \mathrm{R}^\mp), \tag{7.35}$$

$$(\mathrm{NS}^+, \mathrm{R}^\pm) \quad \to \quad (\mathrm{NS}^+, \mathrm{R}^\mp). \tag{7.36}$$

This exchanges Type IIB and Type IIA theory.

The precise statement of this T-dulaity between both theories is:

$$\text{Type IIB on } S^1 \text{ with radius } R \quad \cong \quad \text{Type IIA on } \tilde{S}^1 \text{ with radius } \tilde{R} = \frac{\alpha'}{R}$$

Indeed, an analysis of the vertex operators of the RR states confirms that T-duality removes or adds a form index,

$$\begin{array}{ccc} \text{IIA} & & \text{IIB} \\ C_{\mu_1\mu_2\mu_3} & \to & C_{\mu_1\mu_2\mu_3 9}, \\ C_{\mu_1\mu_2 9} & \to & C_{\mu_1\mu_2}, \\ C_{\mu} & \to & C_{\mu 9}, \\ C_9 & \to & C. \end{array}$$

Remark on T-duality for open strings

As established above, T-duality is nothing by parity on the right-movers. In a theory with open string, this operation exchanges Neumann and Dirchlet boundary conditions. Suppose we start with a theory in 10 dimensions with NN boundary conditions, i.e. with D9-branes. There is only one consistent theory of this type, Type I theory, which is built from Type IIB theory in the way described before. T-duality along x^9 transforms Type IIB into Type IIA and leads to DD boundary conditions in x^9. Thus we arrive at a variant of Type IIA theory with D8-branes.[1] This process can be repeated, suggesting that that Type IIB theory contains Dp-branes with p odd and Type IIA theory contains Dp-branes with p even:

Type IIB	Type IIA
Dp-branes with p odd	p even
D9, D7, D5, D3, D1, D(-1)	D8, D6, D4, D2, D0

We will now understand this feature better by taking a closer look at the dynamical nature of D-branes.

7.4 Dp-branes as dynamical objects

D-branes are more than just hyperplanes on which open strings end. They are by themselves **dynamical objects** that

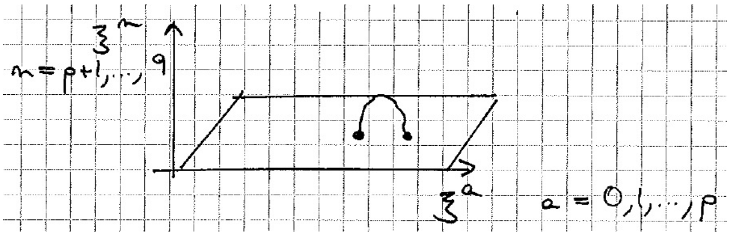
- **gravitate** by coupling to closed strings in the NS-NS sector; i.e. they have a **mass**.
- are **charged** under RR p-form potentials.

By supersymmetry, they interact, of course, with the respective fermionic superpartners. In the sequel we restrict ourselves to describing the bosonic interactions.

There is the following evidence for the claimed dynamical nature of D-branes:

i) In the full quantum theory the worldvolume of a Dp-brane is not static, but it undergoes quantum fluctuations. These brane fluctuations in the normal directions are, in fact, described by the light open-string excitations.

[1] We say 'variant' because we started from Type I, which includes quotienting by worldsheet parity Ω. Thus also the resulting Type IIA theory is an orientifold.

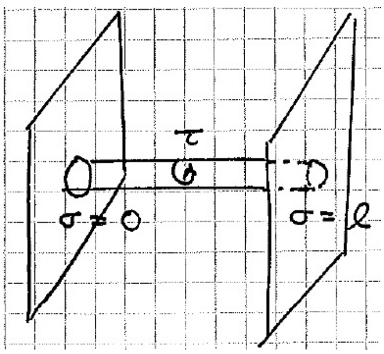


The excitations $\psi^n_{-1/2}\,|0,k\rangle$ normal to the Dp-brane describe a massless scalar field propagating along the Dp-brane. It is interpreted as a modulus field whose VEV determines the position of the brane. Its quantum fluctuations describe brane fluctuations. This is implied by the fact that there **is** non-zero momentum exchange between the DD string and the D-brane, as can be checked explicitly for the DD-solution.

The necessity to include such quantum fluctuations can be understood by the following analogy with the situation in the closed string sector:

- We start from a theory of closed strings in flat spacetime. This theory gives rise to gravitons in its massless spectrum, which are nothing but the quantum fluctuations of the dynamical metric.
- Similarly we start with an open string sector along an initially rigid hypersurface, which gives rise to scalar fields. Their fluctuations represent the fluctuations of the dynamical Dp-brane.

ii) A very direct argument was achieved in a seminal paper by Polchinski (1996), which computed the tree-level exchange of RR and NS-NS closed strings between two Dp-branes by considering instead a 1-loop open diagram.



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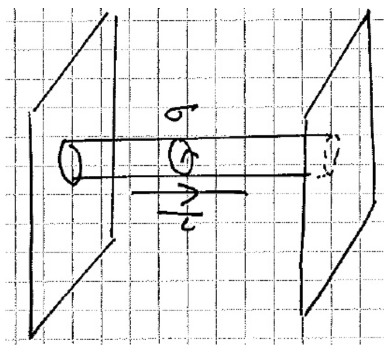


open 1-loop

tree-level closed

As in the computation of the string-loop amplitude with NN boundary conditions in all dimensions, we can transform the 1-loop open string channel into a tree-level closed string channel amplitude. This allows one to compare the amplitude for exchange of NS-NS and RR states with the one ontained in an effective action of extended objects with mass and RR-charge.

More precisely, this dynamics is captured by a **low-energy effective action for the world-volume of the Dp-brane** of the form

$$S_{\text{eff}} = \underbrace{S_{\text{DBI}}}_{\text{coupling to NS-NS}} + \underbrace{S_{\text{CS}}}_{\text{coupling to R-R}} . \tag{7.37}$$

a) The **Dirac-Born-Infeld action** for the Dp-brane reads

$$S_{\text{DBI}} = -T_p \int d^{p+1}\xi e^{-\Phi} \left[-\det\left(G_{ab} + 2\pi\alpha' F_{ab} + B_{ab}\right)\right]^{\frac{1}{2}} . \tag{7.38}$$

- Here,
  $$G_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} G_{\mu\nu}\left(X(\xi)\right) \tag{7.39}$$
  is the pullback of the ambient space metric onto the brane worldvolume. Note that ξ^a $a = 0, 1, ..., p$ represent brane coordinates, while $X^\mu(\xi)$ describes the embedding of brane world-volume in 10D. Thus, $\int d^{p+1}\xi\sqrt{-\det G_{ab}}$ is the higher-dimensional generalisation of the Nambu-Goto action and appears naturally.
- The factor of $e^{-\Phi}$ shows that closed strings couple at tree-level to the disk in the open-closed CFT.
- The field strength of the U(1) gauge field propagating along a single Dp-brane, $2\pi\alpha' F_{ab}$, appears only in combination with the pullback of the Kalb-Ramond field,
  $$B_{ab} = \frac{\partial X^\mu}{\partial \xi^a}\frac{\partial X^\nu}{\partial \xi^b} B_{\mu\nu}\left(X(\xi)\right). \tag{7.40}$$
  As we recall from Assignment 12, only the combination $2\pi\alpha'\mathcal{F}_{\mu\nu} = 2\pi\alpha' F_{\mu\nu} + B_{\mu\nu}$ is invariant under the closed string U(1) symmetry
  $$\delta B_{\mu\nu} = \partial_\mu\xi_\nu - \partial_\nu\xi_\mu, \qquad \delta A_\mu = -\frac{1}{2\pi\alpha'}\xi_\mu \tag{7.41}$$
  due to the worldsheet coupling
  $$\frac{i}{4\pi\alpha'}\int_\Sigma d^2\xi\sqrt{h}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu} + i\int_{\partial\Sigma} dX^\mu A_\mu. \tag{7.42}$$
- The coupling strength is governed by the brane tension
  $$T_p = \frac{2\pi}{\ell_s^{p+1}}, \qquad \ell_s = 2\pi\sqrt{\alpha'}. \tag{7.43}$$

Note that expanding the square root in the DBI action leads to the kinetic term of Yang-Mills theory plus higher order curvature corrections. These match with an explicit computation of scattering results.

b) **Chern-Simons action**
So far the massless RR-sector of Type II superstrings contains the following p-forms: Type IIA: $C^{(1)},\ C^{(3)}$ Type IIB: $C^{(0)},\ C^{(2)},\ C^{(4)+}$.

By Hodge duality in 10 dimensions we can dualise the associated field strengths as

$$* F^{(q+1)} = *dC^{(q)} = \tilde{F}^{(9-q)} = d\tilde{C}^{(8-1)}. \tag{7.44}$$

Note that in 10 dimensions the field strengths, not the potentials are dualised. Alternatively, we can dualise the potentials in the 8 transverse dimensions of light-cone quantisation. Recall that it was in this framework that we had found a self-dual 4-form.

In any case, the above argument shows that $C^{(q)}$ and $\tilde{C}^{(8-q)}$ describe the same degrees of freedom. Thus, we can switch to a so-called "democratic formulation" of Type II supergravity and consider the following field content in the massless RR sector,

$$\begin{array}{ll} \text{Type IIA}: & C^{(1)},\ C^{(3)},\ C^{(5)},\ C^{(7)}, \\ \text{Type IIB}: & C^{(0)},\ C^{(2)},\ C^{(4)},\ C^{(6)},\ C^{(8)}. \end{array} \tag{7.45}$$

Now, a $(p+1)$-form couples naturally to the worldvolume of a Dp-brane via

$$\int_{Dp} C^{(p+1)} = \int d\xi^0 \dots d\xi^p C_{01\dots(p+1)}. \tag{7.46}$$

Indeed, to lowest order the Chern-Simons coupling is just

$$S_{CS} = -\mu_p \int_{Dp} C^{(p+1)}. \tag{7.47}$$

Further curvature terms can be inferred, e.g., by T-duality. The charge of a Dp-brane under $C^{(p+1)}$ is therefore

$$\mu_p = \frac{2\pi}{\ell_s^{p+1}}. \tag{7.48}$$

This explains the spectrum of D-branes observed at the end of the previous section:

$$\text{IIB}: \quad D(2p+1) \leftrightarrow C^{(2p+2)} \quad p = -1, 0, ..., 4 \tag{7.49}$$

$$\text{IIA}: \quad D(2p) \leftrightarrow C^{(2p+1)} \quad p = 0, ..., 4. \tag{7.50}$$

Only those Dp-branes exist as stable objects which have the matching RR-forms available. E.g. a D7-brane in IIB cannot decay because it carries $C^{(8)}$ charge; in IIA a D7-brane would decay (at least in $\mathbb{R}^{1,9}$). In fact the dynamics between Dp-branes is a rich and exciting topic by itself.

Remarks

- For a Dp-brane of the above type, the tension (mass) and charge coincide:

$$T_p = \mu_p. \tag{7.51}$$

  Such objects are called BPS because they are extremely with respect to the **Bogomolny'i-Prasad-Sommerfeld (BPS) bound**

$$M \geq Z \tag{7.52}$$

  with Z the charge.

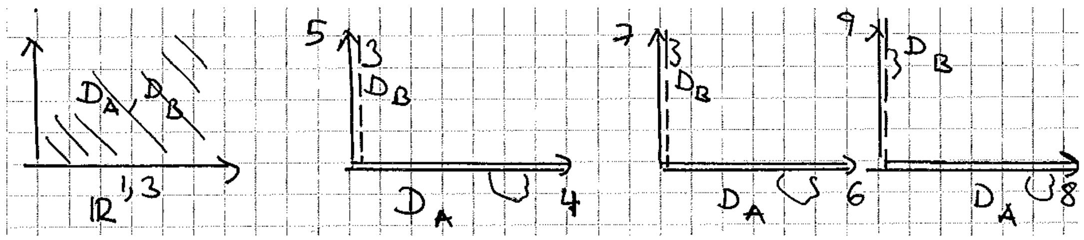
- The description of D-branes with the help of open string+closed string CFT is adequate if g_s is small so that a perturbative expansion makes sense. For large g_s the Dp-branes back-react substantially on the geometry of the ambient spacetime due to their mass. They form so-called **black brane solutions in supergravity**, which are higher-dimensional generalisations of black hole solutions of 4-dimensional Einstein or Einstein-Maxwell theory. In fact, these solutions had been known entirely form a SUGRA persepctive before it was realised in 1996 by Polchinski that they describe the same objects as the hyperplanes associated with DD boundary conditions.

7.5 Intersecting Brane Worlds

We now describe an important application of Dp-branes: Intersecting Brane Worlds.[2]

The general idea can already be understood even without compactifying the extra six dimensions. Various Dp-branes can extend along different dimensions and intersect along some subspace that contains $\mathbb{R}^{1,3}$. This way, interesting gauge theories and matter content arise along the dimensions common to all branes. In fact, the structure we find is naturally that of the Standard Model of Particle Physics! Thus, Intersecting Brane Worlds are important ingredients of string phenomenology, the subfield of string theory that tries to make contact between string theory in 10 dimensions and our 4-dimensional world.
We will first be working in $\mathbb{R}^{1,9}$ and consider configurations of branes intersecting along $\mathbb{R}^{1,3}$, temporarily ignoring complications due to compactification. Among the various possibilities we choose as the probably simplest example a configuration of intersecting D6-branes in Type IIA theory. Let D_A and D_B be two such D6-branes which fill the following dimensions:

Dimension	0	1	2	3	4	5	6	7	8	9
D_A	✓	✓	✓	✓	✓	-	✓	-	✓	-
D_B	✓	✓	✓	✓	-	✓	-	✓	-	✓



Remarks:

- The two branes intersect along $\mathbb{R}^{1,3} \times x^i = 0, i = 4, \ldots, 9$. We will be interested in the physical theory along these common dimensions.
- Even though the two branes intersect along $\mathbb{R}^{1,3}$, this setup is not yet a satisfactory effectively 4-dimensional theory. This is because all states propagating along D_A and D_B propagate not only in 4 dimensions, but also in the remaining dimensions of the brane. The effect of this is negligible to first order only if the extra dimensions are small. Indeed, generalisations to models with 6 compact dimensions are simple.
- The above setup corresponds to an angle of $\frac{\pi}{2}$ in the three planes spanned by $x^4 - x^5$, $x^6 - x^7$, $x^8 - x^9$. This can be generalised to angles φ_i, $i = 1, 2, 3$ in the three planes.
- In the presence of a Dp-brane, Poincaré invariance of $\mathbb{R}^{1,9}$ is obviously broken in the normal directions. This corresponds to a spontaneous breakdown in the vacuum described by the branes. Since Poincaré symmetry and supersymmetry are related via $\{Q, \bar{Q}\} \simeq \gamma P$, also

[2] For further reading we suggest e.g. [Z], Chapter 21, or the pedagogical review "Toward Realistic Intersecting D-Brane Models", http://arXiv.org/abs/hep-th/0502005. A more advanced and very comprehensive text is also "Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes", http://arXiv.org/abs/hep-th/0610327.

some amount of SUSY must be broken. It turns out that a single Dp-brane in Type II theories (of the appropriate type) preserves only $\frac{1}{2}$ of the original amount of supercharges.

- If 2 branes intersect as above, they will in general preserve different supercharges. The total amount of supersymmetry is then generated by the supercharges preserved by both of them. The amount of SUSY depends on the sum of angles φ_i between the branes. You can find a detailed discussion e.g. in [P], Chapter 13.4. In the case considered here, supersymmetry is broken completely. This is just one example of how the theory in 4 dimensions - here arising as the common locus of the intersecting branes - can enjoy much less supersymmetry than the original theory in 10 dimensions!
- We have not taken into account any of the various subtle consistency conditions that arise at the quantum level and that are comparable to the tadpole cancellation conditions of Type I theory. For compact models, these conditions severely constrain the allowed brane setups. Unlike in pure field theory model building it is not possible to simply assemble all ingredients one would like for phenomenological reasons into a model. Rather one must show that the string equations of motion are satisfied. In other words, each model corresponds to a new effectively 4-dimensional vacuum of the unique 10-dimensional theory.

Consider now stacks of N_A and N_B coincident branes of type D_A and D_B respectively. We have the following 2 different sectors in the open string spectrum:

1) Strings starting and ending on the same brane ($A-A$ sector and $B-B$ sector)

These contain the massless gauge bosons of gauge group $U(N_A)$ and $U(N_B)$, respectively (plus their superpartners, depending on the amount of SUSY). The important feature is that along the dimensions common to D_A and D_B, both types of gauge bosons propagate! Therefore along the common $\mathbb{R}^{1,3}\times x^i=0, i=4,\dots,9$ the gauge group is $U(N_A)\times U(N_B)$.

2) Strings stretched between different brane stacks

This sector is due to strings starting on A and ending on B (i.e. in the $A\to B$ sector) as well as strings starting on B and ending on A (i.e. in the $B\to A$ sector).

- Due to their tension, these are localised at the intersection of the branes, i.e. they propagate only along $\mathbb{R}^{1,3}$.
- As can be seen from their Chan-Paton factors, they transforms as bi-fundamentals of $U(N_A)\times U(N_B)$. The convention is that strings in $A\to B$ sector transform as $(\bar{N}_A, N_B)$. The difference between the fundamental and and the anti-fundamental is that they are complex conjugates. We take $\bar{N}_A$ to have charge -1_A under the diagonal $U(1)_A$ in $U(N_A)=SU(N_A)\times U(1)_A$ (and N_A to have charge $+1_A$.) Then, the states in the sector $B\to A$ are in representation $(N_A,\bar{N}_B)$.
- To determine the details of the string spectrum we need to quantised an open string with mixed boundary conditions. For example, for the setup at hand, these are in the $A\to B$ sector:

$$\begin{array}{lll} \sigma=0 & \partial_\sigma X^n(\tau,\sigma=0)=0, & n=0,\dots 3,4,6,8, \\ & \partial_\tau X^m(\tau,\sigma=0)=0, & m=5,7,9, \\ \sigma=\ell & \partial_\sigma X^n(\tau,\sigma=\ell)=0, & n=0,\dots 3,5,7,9, \\ & \partial_\tau X^m(\tau,\sigma=\ell)=0, & m=4,6,8. \end{array} \tag{7.53}$$

This corresponds to DN boundary conditions in dimensions $4, \dots, 9$ and be generalised to arbitrary angles φ_i between the branes.

- The mixed boundary conditions modify the oscillator modings. For DN strings this has been discussed. For general angles one arrives at fractional modings by shifting the NN moding to

$$n \to n + \frac{\varphi}{\pi}. \tag{7.54}$$

Consider now the $A \to B$ sector:

- The massless Ramond sector contains one fermion corresponding to a Dirac spinor, i.e. one chiral and one anti-chiral Weyl spinor. This is just the fermionic ground state along the extended 4 dimensions. The GSO projection will keep only one of the two, say the chiral one $(R, +)$. Thus we have one ψ^{α}_{AB}, where $\alpha = 1, 2$ denotes a 4-dimensional Weyl spinor index.
- In the NS sector the sign of M^2 of the lowest state is determined by the normal ordering constant. This in turn depends on the angle between the branes as these shift the modings of the fields. For the above DN boundary conditions we find one boson of positive M^2. This reflects the fact the brane intersection breaks supersymmetry completely so that the massless fermion has no massless superparter.

The $B \to A$ sector follows by letting $\sigma \to \ell - \sigma$. This is just worldsheet parity. As discussed in the context of T-duality this flips chirality.
We therefore obtain the following massless spectrum (after GSO projection)

$$\psi^{\alpha}_{AB} : (\bar{N}_A, N_B), \qquad \psi^{\dot{\alpha}}_{BA} : (N_A, \bar{N}_B) \tag{7.55}$$

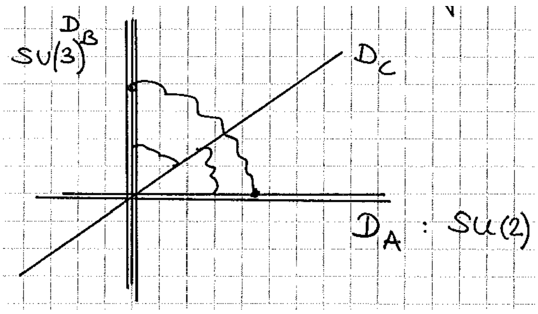
The two fermions correspond to particle and anti-particle and thus describe the same degrees of freedom.
Let us summarise our findings:

A stack of two branes D_A and D_B intersecting along $\mathbb{R}^{1,3} \times pt.$ gives rise to a $U(N_A) \times U(N_B)$ Yang-Mills theory plus one chiral fermion transforming in the bi-fundamental $(\bar{N}_A, N_B)$.

But wait a minute - this is just the structure of the Standard Model of Particle Physics (SM)! Namely, the SM gauge group is the product $SU(3) \times SU(2) \times U(1)_Y$ and the particle content is given by 3 generations of chiral fermions in various bifundamentals:

Particle	$SU(3)$	$SU(2)$	$U(1)_Y$
Q_L	3	2	$\frac{1}{6}$
u^c_R	$\bar{3}$	1	$-\frac{2}{3}$
d^c_R	$\bar{3}$	1	$\frac{1}{3}$
L	1	2	$-\frac{1}{2}$
e^c_R	1	1	$+1$
ν^c_R	1	1	0

Our notation is that u^c_R etc. are the charge conjugate of the right-handed fields and thus left-handed.
Very crudely, this can be realised by in terms of 3 intersecting brane stacks with $N_A = 3$, $N_B = 2$, $N_C = 1$.



Remarks:

- One notices that in the SM only $SU(N)$ groups appear, not $U(N)$ (apart from $U(1)_Y$). In Intersecting Brane Worlds, the diagonal $U(1) \subset U(N)$ turns out to decouple - its gauge boson is massive. In suitable configurations precisely one linear combination of $U(1)$s is massless. This must be identified with $U(1)_Y$.
- The remaining U(1)s remain as perturbative global symmetries and account for the presence of accidental symmetries such as baryon and lepton number in the SM. This is particularly attractive because in the SM no explanation for the existence of these symmetries can be given.
- To account for the correct charges of all particles, more complicated configurations than just the above 3-stack model are required. Indeed suitable brane setups can be classified.

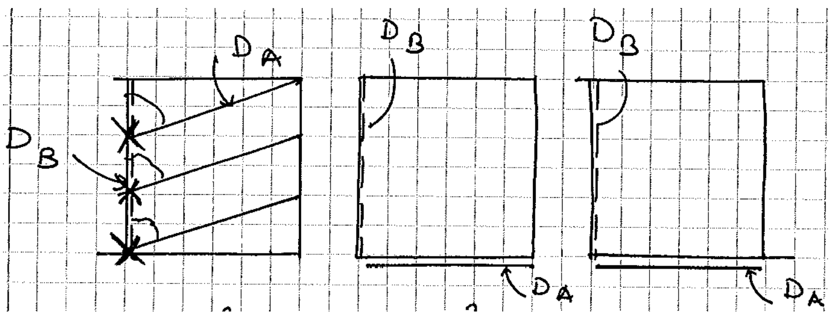
Toroidal Intersecting Brane Worlds

So far we have been working in $\mathbb{R}^{1,9}$. To obtain a truly 4 dimensional effective field theory, the extra six dimensions must be compact. The logic behind this compactification will be discussed in a more general context in the next chapter. Here let us focus on the simplest possibility and make a toroidal compactification ansatz

$$\mathcal{M}^{1,9} = \mathbb{R}^{1,3} \times T^6 \tag{7.56}$$

with T^6 a six-torus.

- It is convenient to represent T^6 as a factorisable 6-torus of the form $T^6 = T^2 \times T^2 \times T^2$. If we embed the brane configuration presented at the beginning of this section into such a compact model, the two D6-branes fill $\mathbb{R}^{1,3}$ and wrap a 1-cycle in each of the 2-tori given by one of the axes.
- More generally we can also consider a configuration as below:



- The 2 branes D_A and D_B now intersect in 3 points on T^6. At each intersection point one chiral bifundamental fermion "lives". Thus, in the effective theory in $\mathbb{R}^{1,3}$ we now find 3 chiral bifundamental fermions. This gives a beautiful way to think about family replication - the fact that we have 3 chiral generations of bifundamental matter in the SM.

As alluded to above, not every brane configuration leads to a fully consistent CFT. As in Type I theory one must check that the tadpoles of all 1-loop amplitudes cancel. This implies that we must actually consider unoriented open strings (such that the Möbvious amplitudes can cancel the tadpoles of the annulus amplitude). Such models are called orientifolds. Finding consistent solutions which are compatible with the physics of the SM is an active field of modern day string theory.

7.6 Elements of Calabi-Yau compactification

In this section we discuss more general compactifications of string theory to four dimensions.[3] Our starting point is the general warped compactification ansatz

$$\mathcal{M}^{1,9} = \mathcal{M}^{1,3} \times_{\rm w} \mathcal{M}^6. \tag{7.57}$$

We take $\mathcal{M}^{1,3}$ to be a maximally symmetric four-dimensional space, i.e. Minkowski space, deSitter (dS) or Anti-deSitter (AdS), while $\mathcal{M}^6$ is the six-dimensional internal space. The metric corresponding to the above ansatz takes the form

$$G_{MN} = \begin{pmatrix} A(y) g_{\mu\nu} & 0 \\ 0 & g_{mn}(y) \end{pmatrix}, \tag{7.58}$$

where $\mu, \nu = 0, \ldots 3$, $m, n, = 4, \ldots, 9$ and $y \equiv y^m$ are internal coordinates. Note that $A(y)$ represents a so-called warp-factor. If $A(y) \equiv 1$ we have a direct product, while more generally one speaks of a **warped compactification**.
We now specialise to one of the five incarnations of 10-dimensional string theory and focus on the low-energy effective field theory. For definiteness let us consider Type IIA or Type IIB SUGRA. In order for the ansatz (7.57) to furnish a consistent compactification, the equations of motion of all fields in the effective action must be satisfied. As discussed these arise as the string consistency conditions which ensure that the beta-functions of all fields in the non-linear sigma model vanish, i.e. that conformal invariance on the worldsheet is preserved.
In particular, we should view the Einstein's equation for the metric as the equation of motion for the graviton. This justifies the interpretation that **a consistent compactification manifold $\mathcal{M}^6$ gives rise to a 4-dimensional vacuum of the the 10-dimensional theory**. From a conceptual point of view this is an important insight:

While the string consistency conditions single out a **unique theory (up to dualities) in 10 dimensions**, every **4-dimensional effective theory** obtained from this by compactification corresponds to a choice of vacuum, i.e. to a **dynamical solution of the 10-dimensional theory**.

The uniqueness of the theory in ten dimensions is not in conflict with the existence of many consistent $\mathcal{M}^6$. Of course there is nothing surprising about the fact that a given theory - here

[3] We strongly recommend [GSW], Vol II, Chapter 15 for further reading on Calabi-Yau compactifications. A detailed review on flux compactifications is e.g. "Flux compactifications in string theory: A Comprehensive review", http://arXiv.org/abs/hep-th/0509003.

superstring theory in 10 dimensions - can have many different solutions. Consider e.g. Einstein gravity in 4 dimensions: It is one theory with many different solutions! Just as gravity theorists cannot predict which of these solutions is relevant for our solar system - we must observe the distance from the Earth to the sun, we cannot compute it from Einstein's equations - in string theory we must find the correct 4-dimensional solution corresponding to our world and then use this solution to gain further insight e.g. about particle physics.

The set of 4-dimensional solutions of string theory is called the landscape of string vacua.

The most general solution of the equations of motion will break supersymmetry completely, i.e. at the compactification scale. While this is fine in principle, for reasons of stability and of computational control we try instead to preserve supersymmetry at the compactification scale; then SUSY must be broken dynamically at a lower scale.

The condition for unbroken supersysmmetry is that the supersymmetry variation of all fields in the vacuum must vanish. For bosonic fields this is automatic, because a boson b transforms into a fermion f, $\delta_{\rm SUSY} b = f$ and thus $\langle \delta_{\rm SUSY} b\rangle = \langle f \rangle \equiv 0$ as fermions have no VEV. The non-trivial constraint is therefore that also

$$\langle \delta_{\rm SUSY} f\rangle = 0. \tag{7.59}$$

The supersymmetry charges of Type IIA/B supergravity form two independent supersymmetry algebras. The SUSY variation associated with each of these algebras is expressed in terms of one spacetime-dependent Majorana-Weyl spinor ϵ in 10 dimensions. From supergravity texts we quote the following ten-dimensional SUSY variations of the fermonic fields:

- The gravitino variations give

$$\begin{aligned}
\delta\psi^{(i)}_\mu &= \tilde{\nabla}_\mu \epsilon^{(i)} = \Big(\partial_\mu + \frac{1}{4}\tilde{\omega}_{\mu\nu\rho}\Gamma^{\nu\rho}\Big)\epsilon^{(i)}, && (7.60)\\
\delta\psi^{(i)}_m &= \nabla^{(H,F)}_\mu \epsilon^{(i)} = \Big(\partial_m + \frac{1}{4}(\omega_{mnp} - H_{mnp})\Gamma^{np} + ...\Big)\epsilon^{(i)} && (7.61)
\end{aligned}$$

  in terms of the spin connection $\omega_{\mu\nu\rho}$ and the 3-form field strength H. The ... represent further terms dependent on the Ramond-Ramond field strengths. The tilde reminds us that we must take the covariant derivative with respect to the warped metric

$$\tilde{g}_{\mu\nu} = A(y) g_{\mu\nu}. \tag{7.62}$$

- The dilatino variation is

$$\delta\chi^{(i)} = \Big(\Gamma^m\partial_m\Phi - \frac{1}{12}\Gamma^{mnp}H_{mnp}\Big)\epsilon^{(i)}. \tag{7.63}$$

For the most general SUSY vacuum the NSNS and RR field strengths will have a non-zero VEV, $\langle H\rangle \neq 0 \neq \langle F^p\rangle$. Vacua with such non-vanishing field strengths are called **flux vacua** and have gained a lot of attention in recent years. For simplicity, however, let us consider the special case $\langle H\rangle = 0 = \langle F^p\rangle$. The dilation variation implies $\Phi = const.$ and the SUSY conditions are

$$\tilde{\nabla}_\mu \epsilon^{(i)} = 0 = \nabla_m \epsilon^{(i)}. \tag{7.64}$$

In general the existence of a covariantly constant spinor poses strong conditions because it implies

$$\nabla_A \epsilon = 0 \Longrightarrow 0 = [\nabla_A, \nabla_B]\epsilon = R_{ABMN}\Gamma^{MN}\epsilon = 0. \tag{7.65}$$

In particular one can show that $\tilde{\nabla}_\mu \epsilon^{(i)} = 0$ with respect to the metric given by $\tilde{g}_{\mu\nu} = A(y) g_{\mu\nu}$ and $g_{\mu\nu}$ the metric of AdS, dS or Minkowski space requires

$$A(y) = 0, \qquad g_{\mu\nu} = \eta_{\mu\nu}. \tag{7.66}$$

This is because it implies the conditions $[\nabla_\mu, \nabla_\nu]\epsilon^{(i)} = -\frac{1}{2}(\nabla_m A)\,(\nabla^m A)\,\Gamma_{\mu\nu}\epsilon^i$, where ∇_μ is with respect to the maximally symmetric metric $g_{\mu\nu}$. For these spaces we have $[\nabla_\mu, \nabla_\nu]\epsilon^{(i)} = \frac{\lambda}{2}\Gamma_{\mu\nu}\epsilon^{(i)}$ with $\lambda < 0$ for AdS, $\lambda = 0$ for Minkowski and $\lambda > 0$ for dS. Thus $k + \nabla_m A \nabla^m A = 0$, but the only constant value for $(\nabla_m A)(\nabla^m A)$ on the compact internal space is zero.

Thus, supersymmetric compactifications to AdS are possible only in the presence of fluxes (de-Sitter space breaks supersymmetry anyways and more elaborate techniques are required to construct vacua with positive cosmological constant). What remains is to analyze the condition that $0 = \nabla_m \epsilon^{(i)}$.

In Type IIA, $\epsilon^{(1)}$ transforms as a 16-component chiral Majorana Weyl spinor $\mathbf{16}$ of $SO(1,9)$ and $\epsilon^{(2)}$ as the anti-chiral $\mathbf{16'}$, while in Type IIB both spinors are chiral. In view of the decomposition

$$SO(1,9) \to SO(1,3) \times SO(6): \qquad \mathbf{16} \to (\mathbf{2}, \mathbf{4}) + (\mathbf{2'}, \mathbf{4'}) \tag{7.67}$$

we make the ansatz (in Type IIA)

$$\begin{aligned} \epsilon^{(1)} &= \epsilon_+^{(1)} \otimes \eta_+ + \epsilon_-^{(1)} \otimes \eta_-, & (7.68)\\ \epsilon^{(2)} &= \epsilon_+^{(2)} \otimes \eta_- + \epsilon_+^{(2)} \otimes \eta_-, & (7.69) \end{aligned}$$

and correspondingly in Type IIB. Here $\epsilon_+^{(1,2)}$ denote two independent 4-dimensional Weyl spinors and η_+ is a spinor of SO(6) that must satisfy

$$\nabla_m \eta_+ = 0. \tag{7.70}$$

The spinor η_- is just the conjugate of η_+. Thus, for each covariantly constant spinor η_+ we obtain $4 + 4$ supercharges in 4 dimensions (because to each component of the 4-dimensional SUSY parameters $\epsilon_+^{(i)}$ and $\epsilon_-^{(i)}$ we associate one Noether charge). This corresponds to $\mathcal{N} = 2$ SUSY in 4 dimensions.

In fact, the existence of a covariantly constant spinor on $\mathcal{M}^6$ is equivalent to the statement that $\mathcal{M}^6$ has **SU(3) holonomy**. Let us recall what this means: Consider a general Riemannian 6-fold $\mathcal{M}^6$. Such a space has $SO(6)$ holonomy: If we transport a vector field v_m around a closed loop γ, the field transforms as

$$v_m \to v'_m = (U_\gamma \cdot v)_m, \qquad\qquad U_\gamma \in SO(6). \tag{7.71}$$

The existence of a covariantly constant v_m implies that $v_m = (U \cdot v)_m$. This reduces the holonomy group to a subgroup of $SO(6)$.

Now, it is useful to consider the isomorphism $SO(6) \simeq SU(4)$ (as always we really mean: $Spin(6)$ because we care about spinors). A chiral Weyl spinor $\mathbf{4}$ of SO(6) corresponds to the vector representation $\mathbf{4}$ of SU(4) (with $\mathbf{4'}$ corresponding to the conjugate $\bar{\mathbf{4}}$). This is similar to the familiar $SO(1,3) \simeq SU(2)_L \times SU(2)_R$ and the associated representation of the chiral and anti-chiral Weyl spinors in terms of vectors under $SU(2)_{L/R}$.

The existence of a covariantly constant spinor therefore requires that the holonomy be reduced from $SU(4)$ to $SU(3)$. Then the spinor in question is

$$\eta = \begin{pmatrix} \eta_0 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{7.72}$$

Manifolds of SU(3) holonomy are famous:

Definition 7.1. *A $2n$-dimensional manifold of holonomy $SU(n)$ is called a Calabi-Yau n-fold.*

One can also characterise a Calabi-Yau manifold as follows: First make the

Definition 7.2. *A Kähler manifold is a complex manifold whose metric is hermitian (i.e. the only non-vanishing metric components are $g_{i\bar{j}}$) and can be derived from Kähler potential K as $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$.*

One can show

Theorem 7.1. *A manifold is Calabi-Yau if and only if it is Ricci flat and Kähler.*

In particular, a Calabi-Yau manifold automatically satisfies the vacuum Einstein equations. Thus, our supersymmetric compactification solves the equations of motion.

Compactification of Type II string theory on a Calabi-Yau 3-fold corresponds to a consistent vacuum. The 4-dimensional effective theory reduces to $\mathcal{N} = 2$ supergravity.

The explicit construction of compact Ricci flat Calabi-Yau metrics is very hard. But according to a famous theorem conjectured by Calabi and proven by Yau, one can check a much simpler algebraic criterion to determine if a given space allows for such a Ricci flat metric:

Theorem 7.2. *A Kähler manifold is Calabi-Yau if and only if the first Chern class of the tangent space vanishes, i.e. $c_1(\mathcal{M}_6) = 0$.*

The study of Calabi-Yau manifolds is a typical subject of algebraic geometry, which provides many tools of significant importance to model builders.

Note that the simplest example of a Calabi-Yau 3-fold space is indeed a torus T^6. This justifies in retrospect our toroidal ansatz of the previous section. However, the holonomy group of a T^6 is trivial, i.e. it is contained in $SU(3)$. Therefore compactification of Type II string theory on T^6 gives a theory with all 32 supercharges, corresponding to $\mathcal{N} = 8$ supergravity.

So far we have only been dealing with the closed string sector. To arrive at interesting theories including gauge dynamics one can combine the idea of Intersecting Brane Worlds with compactification on a genuine Calabi-Yau 3-fold. To stick with our example of Type II A theory with D6-branes, the latter fill $\mathbb{R}^{1,3}$ and wrap an internal 3-cycle Σ, i.e. a submanifold with $\partial\Sigma = 0$ and $\Sigma \neq \partial\Gamma$. In the presence of a brane the amount of supersymmetry is at best 1/2 of the original SUSY. This corresponds to 4 conserved supercharges in 4 dimensions, i.e. $\mathcal{N} = 1$ supersymmetry. For this $\mathcal{N} = 1$ SUSY to be actually preserved, the 3-cycle must itself satisfy certain geometric conditions.

This way realistic gauge theories can be obtained in a systematic manner from string compactifications. A detailed study of the landscape of string vacua is the subject of **string phenomenology**.