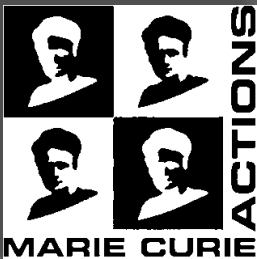


Causality with Two Metrics

Ignacy Sawicki
Université de Genève

Based on super-subluminally propagated
conversations with Alex Vikman, still forthcoming
to a journal near you



Perturbations in Cosmology

$$\phi + \bar{\phi}(\tau) + \pi$$

$$g_{\mu\nu} = \bar{g}_{\mu\nu}^{\text{FRW}} + h_{\mu\nu}$$

$$S_{\phi 2} = \int d\tau d^3x a^4 D (\dot{\pi}^2 - c_s^2 \partial_i \pi \partial^i \pi - m^2 \pi^2)$$



$$\mathcal{H} = D \left(\frac{\Pi^2}{D^2} + c_s^2 (\partial_i \pi)^2 + m^2 \pi^2 \right)$$

$D < 0$: Ghosts

$c_s^2 < 0$: Gradient
Instabilities

$m^2 < 0$: Tachyons

The Acoustic Metric $\mathcal{G}_{\mu\nu}$

$$S_\phi = \int d^4x \sqrt{-g} \mathcal{L}(\phi, \partial_\mu \phi, \nabla_\mu \nabla_\nu \phi)$$



$$S_{\phi 2} = \frac{1}{2} \int d^4x \sqrt{-\mathcal{G}} (\mathcal{G}^{\mu\nu} \partial_\mu \pi \partial_\nu \pi - m^2 \pi^2)$$



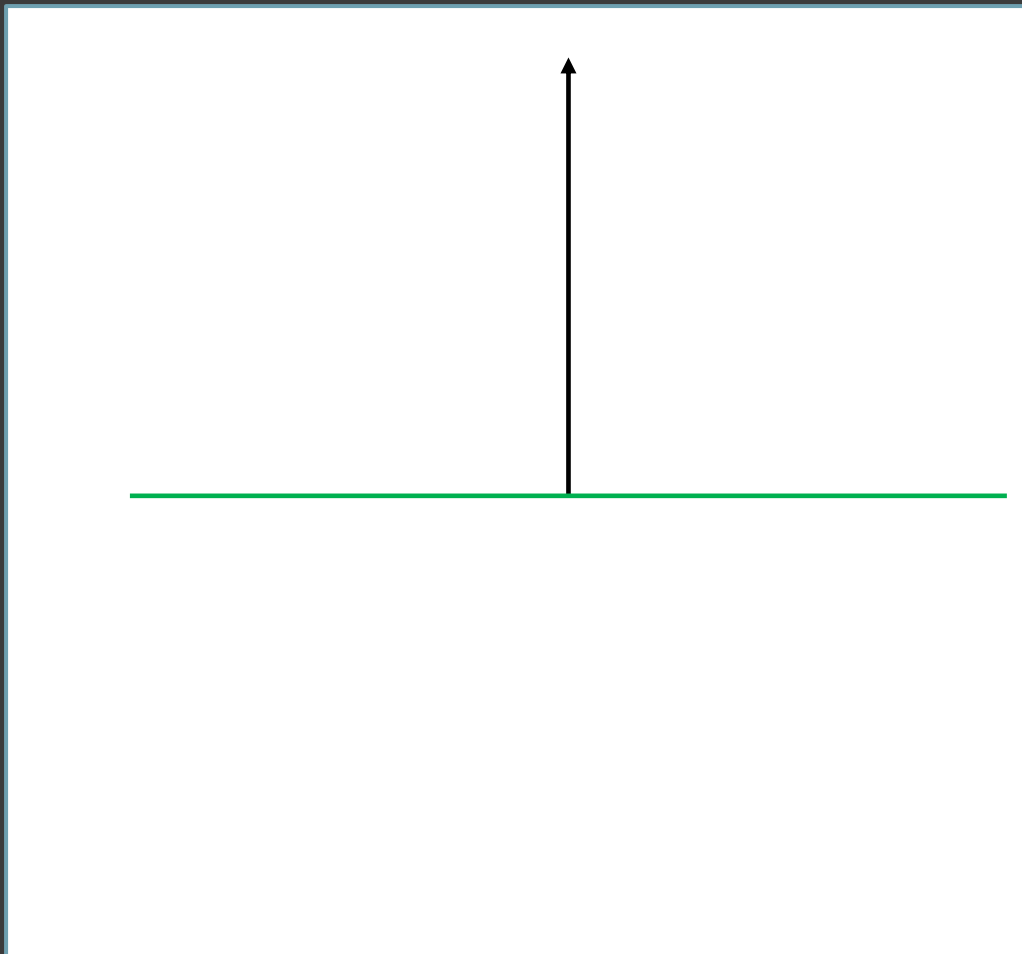
$$\mathcal{G}_{\mu\nu}^{\text{FRW}} \propto \begin{pmatrix} D & & & \\ & -Dc_s^2 & & \\ & & -Dc_s^2 & \\ & & & -Dc_s^2 \end{pmatrix}$$

$$\square_{\mathcal{G}} \pi = m^2 \pi$$

The Gist

- ⊙ Each d.o.f. propagates in its own metric which can be different from $g_{\mu\nu}$
 - When is it consistent even to talk about a evolution?
- ⊙ Eigenvalue structure of the acoustic metric tell you about
 - Ghosts and instabilities
 - Causal evolution and Cauchy problem
 - And it's all coordinate/observer invariant
- ⊙ Clustering DE/MG typically has metrics which are not homogeneous/isotropic around non-linear sources
 - e.g Vainstein

Lightcones: characteristic surfaces for Maxwell



- Geometrical optics

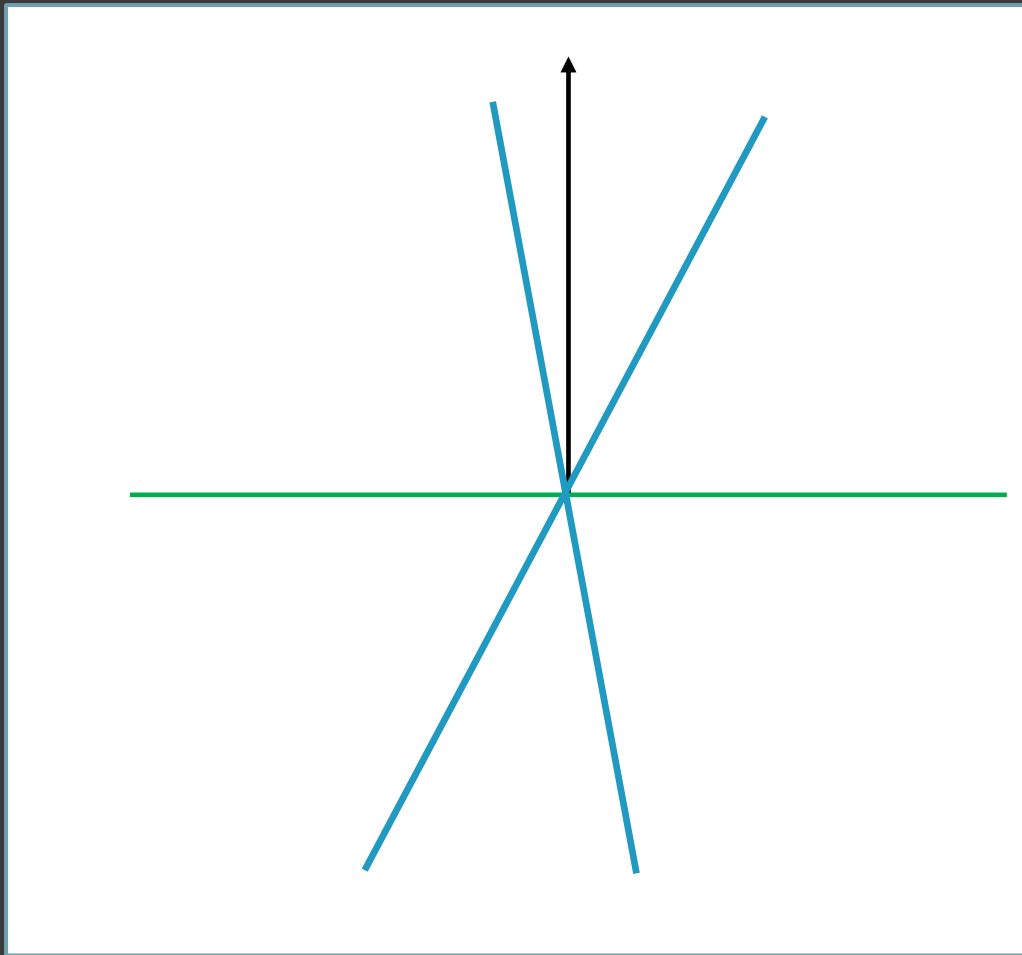
$$A_\mu = \bar{A}_\mu(x) e^{iS(x)}$$

$$p_\mu = -\partial_\mu S \quad n^\mu = g^{\mu\nu} p_\nu$$

- Maxwell: $\nabla_\mu F^{\mu\nu} = 0$

$$g^{\mu\nu} \partial_\mu S \partial_\nu S = 0$$

Sound Cones: characteristic surfaces for scalars



- Eikonal ansatz

$$\pi = \mathcal{A}(x) e^{iS(x)/\epsilon}$$

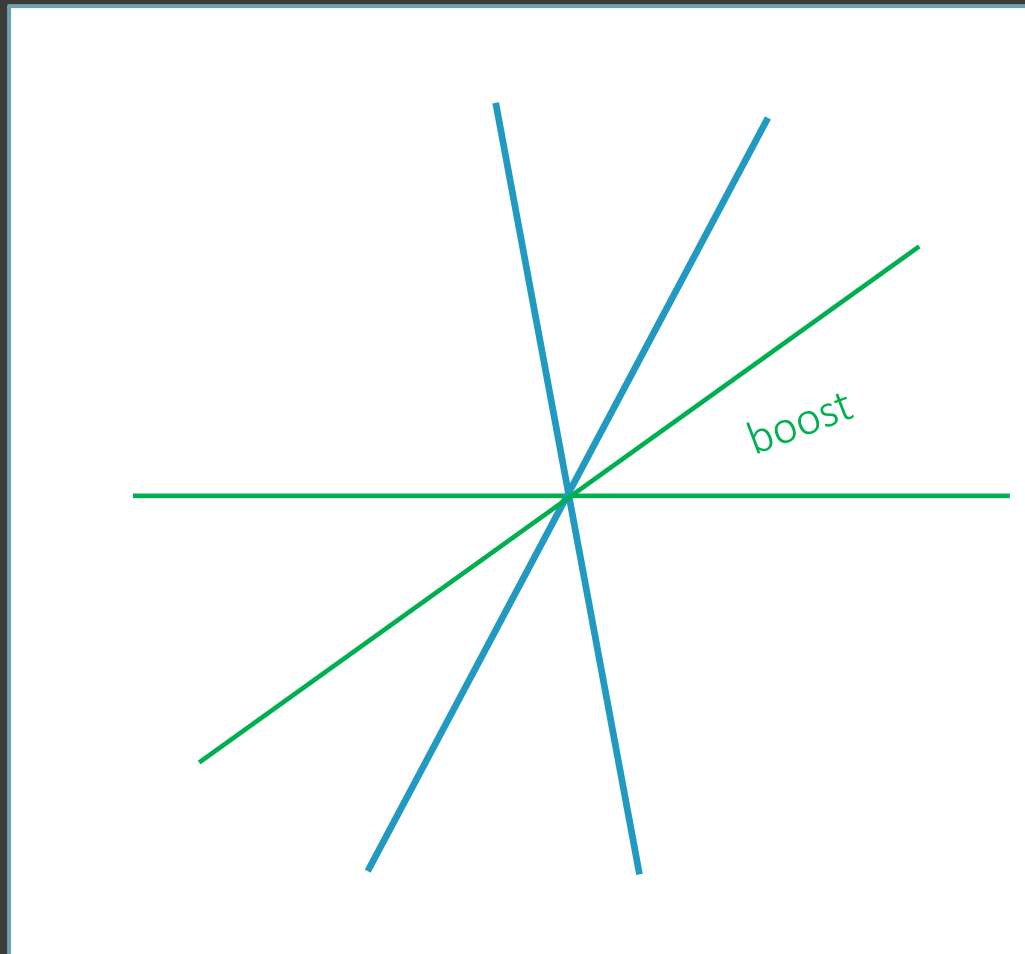
$$P_\mu \equiv -\partial_\mu S \quad N^\mu \equiv \mathcal{G}^{\mu\nu} P_\nu$$

- EoM : $\mathcal{G}^{\mu\nu} \nabla_\nu \nabla_\mu \pi + \dots = 0$

$$\begin{aligned} \mathcal{G}^{\mu\nu} P_\mu P_\nu &= 0 \\ \mathcal{G}_{\mu\nu}^{-1} N^\mu N^\nu &= 0 \end{aligned}$$

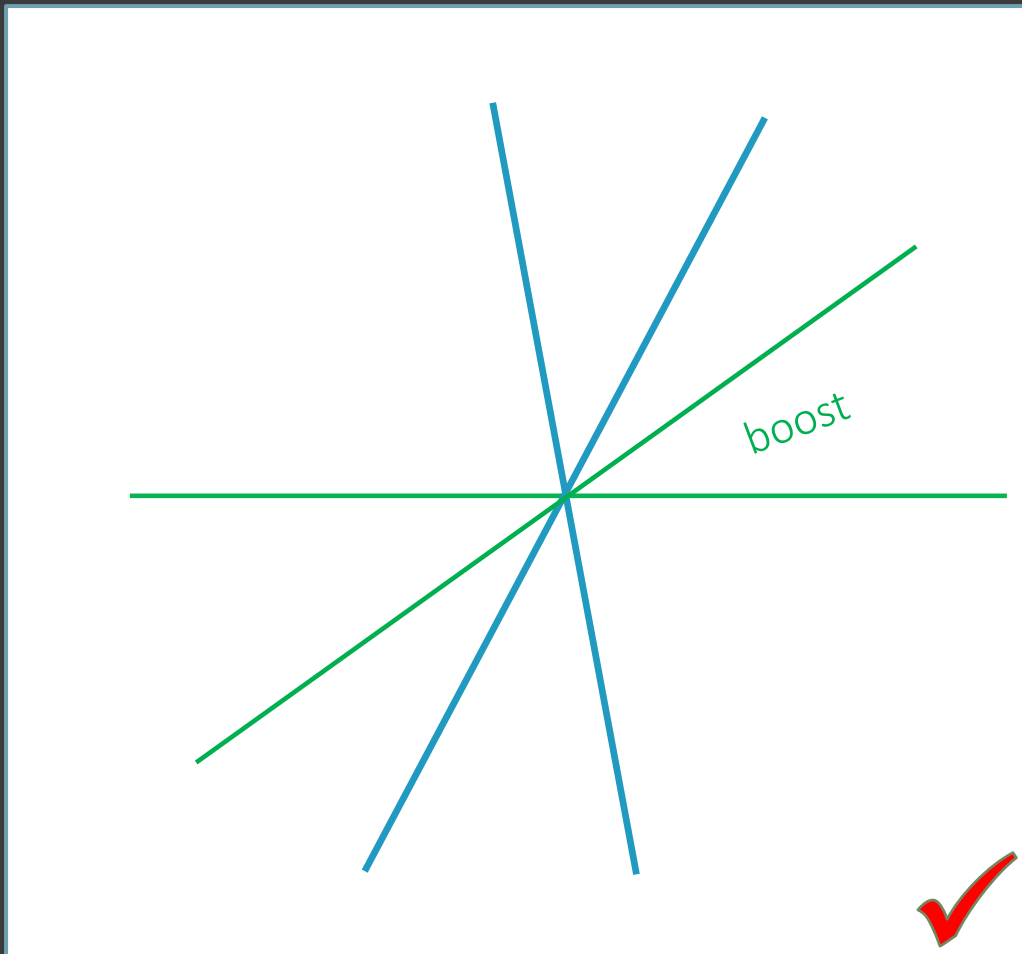
Superluminality and Causality

Subluminal

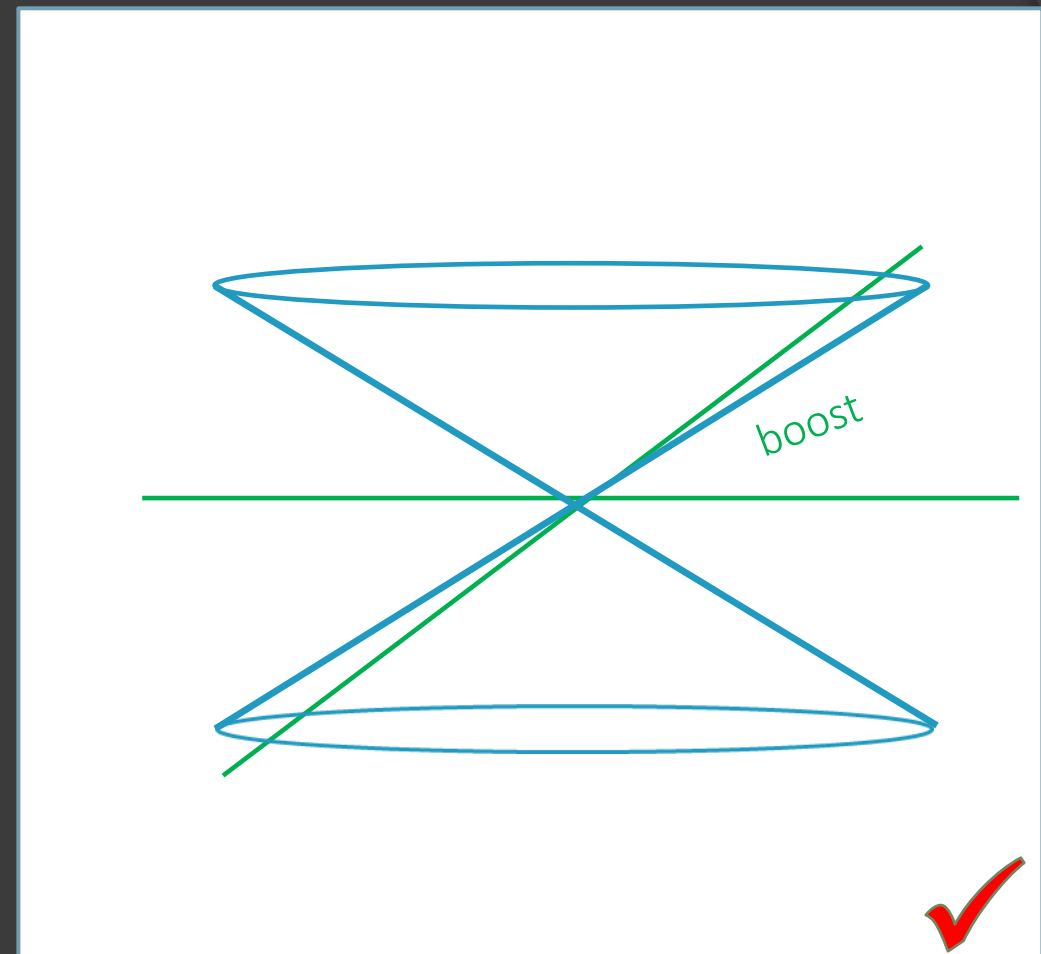


Superluminality and Causality

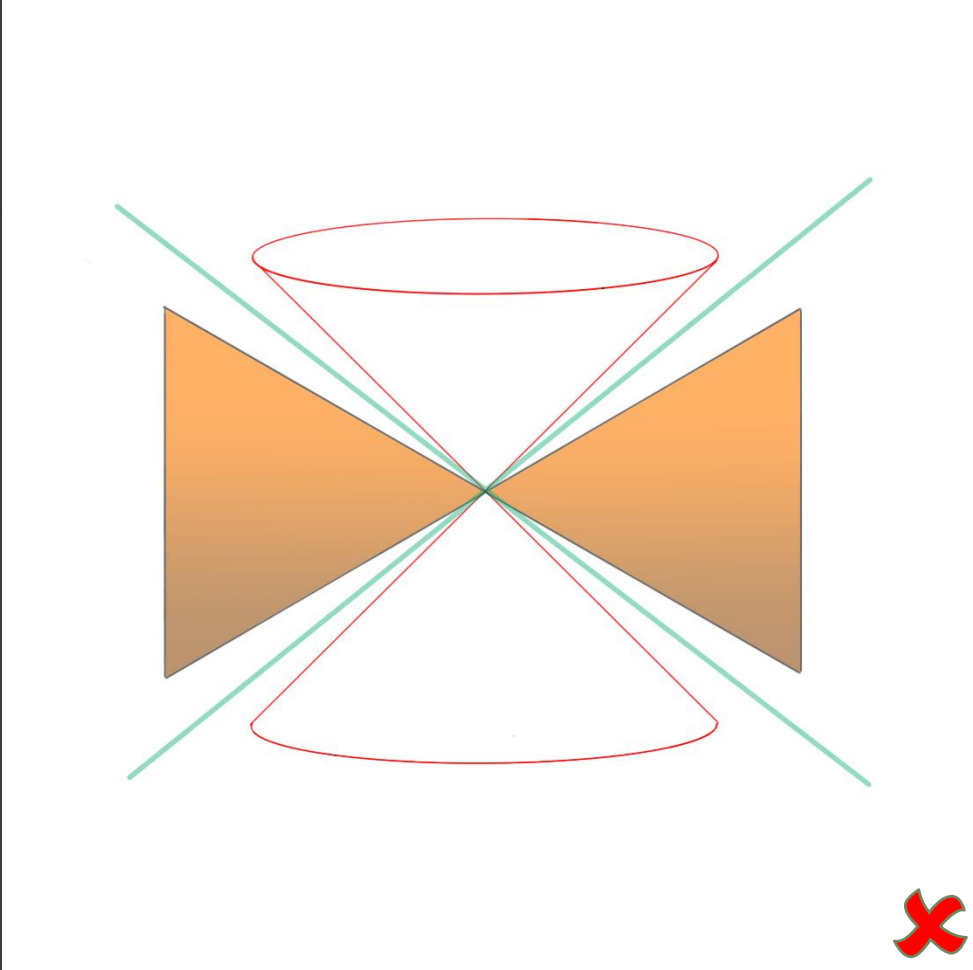
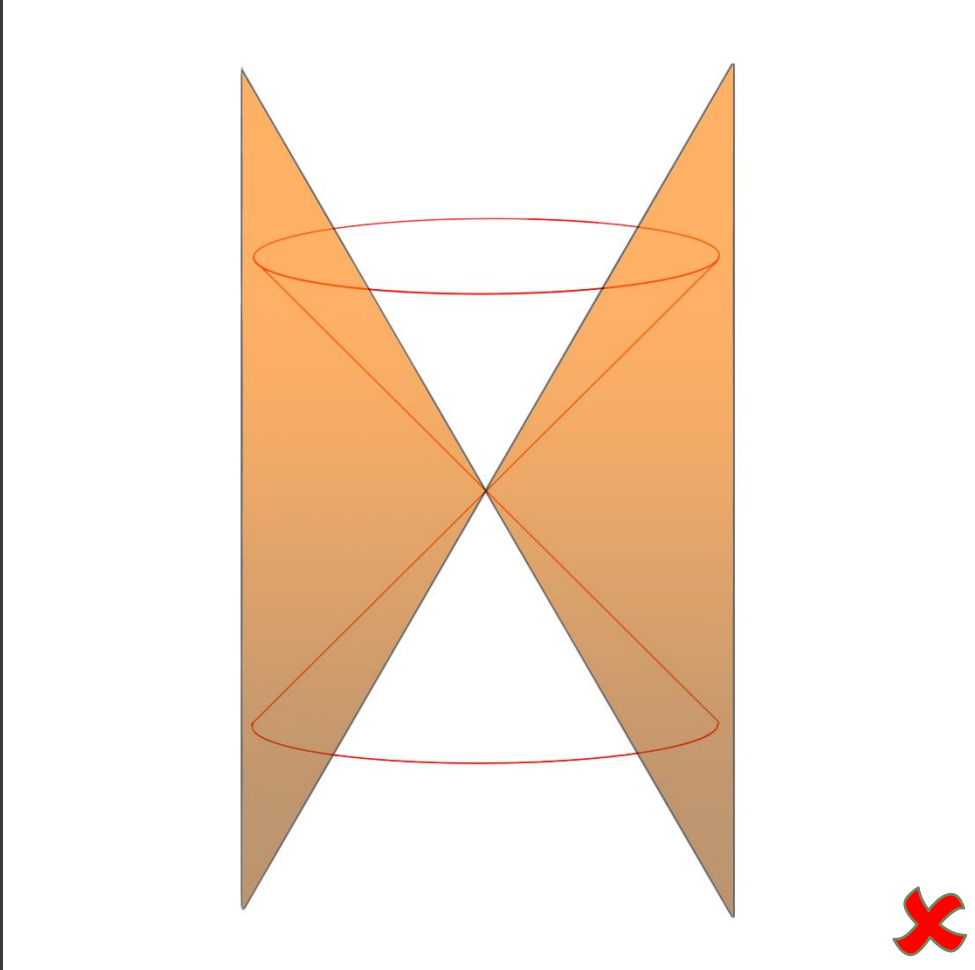
Subluminal



Superluminal



Inescapable Violations of Causality



Cone Existence + Causality \Rightarrow Stability

$$g_{\mu\nu}p^\mu p^\nu = 0$$

Diagonalise*

$$g_{IJ}p^I p^J = \lambda_0(p^0)^2 + \sum_1^3 \lambda_i(p^i)^2$$

Cone exists

No $\lambda_I = 0$
Exactly **one** $\lambda_I > 0$

Common TL
directions

$$\lambda_0 > 0$$



$$c_s^2 = -\frac{\lambda_i}{\lambda_0}$$

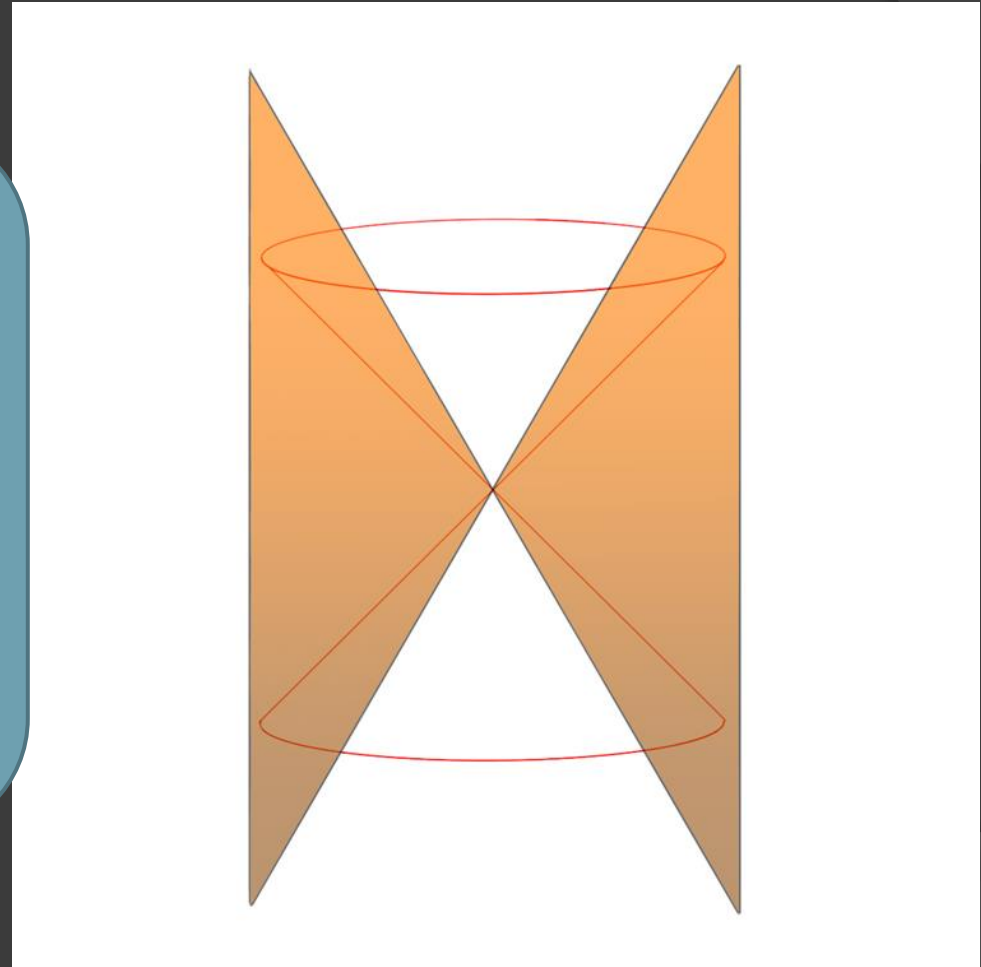
Why $\lambda_0 > 0$?

$$g_{\mu\nu} = dt^2 - dx^2 - dy^2 - dz^2$$

$$f_{\mu\nu} = -dt^2 + dx^2 - dy^2 - dz^2$$

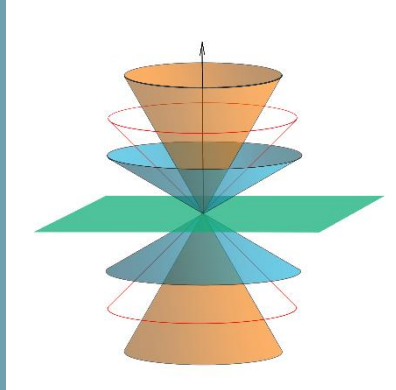
$$\lambda_0 > 0, \lambda_1 < 0$$

$$\lambda_0 < 0, \lambda_1 > 0$$

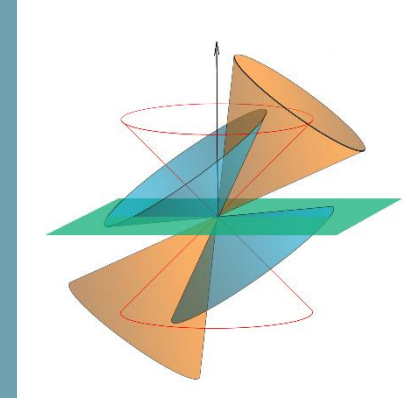


*Classification of Cone Geometries

Type I: 4 real λ , no null



Type II: 2 λ complex,
no null



$$|\mathcal{G}_{\mu\nu} - \lambda g_{\mu\nu}| = 0$$

Type III: 2 repeated λ
2 null e-vectors

Type IV: 3 repeated λ
3 null e-vectors

Example: $\mathcal{L} = K(X)$

$$2X \equiv (\partial_\mu \phi)^2$$

$$g_{\mu\nu} = K_{,X} g_{\mu\nu} + K_{,XX} \partial_\mu \phi \partial_\nu \phi$$

⊙ $\partial_\mu \phi$ timelike: type I

$$\lambda_0 = K_X + 2XK_{XX} \quad \lambda_i = -K_X$$

⊙ $\partial_\mu \phi$ spacelike: type I

$$\lambda_0 = K_X \quad \lambda_{2,3} = -K_X \quad \lambda_1 = -(K_X + 2XK_{XX})$$

⊙ $\partial_\mu \phi$ null ($X = 0$): type III

Example: $\mathcal{L} = K(X) + G(X)\square\phi$

$$u_\mu \equiv \partial_\mu \phi / \sqrt{2X}$$

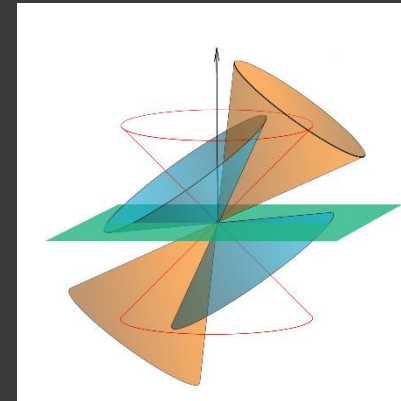
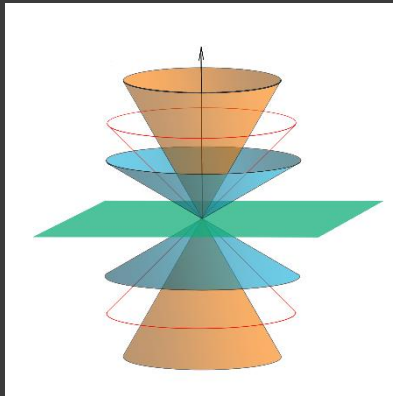
$$G_{\mu\nu} = Du_\mu u_\nu + \Omega \perp_{\mu\nu} - G_X(\alpha\sigma_{\mu\nu} + \beta u_{(\mu} a_{\nu)})$$

⊙ $a_\mu = 0$: Type I anisotropic

⊙ $\sigma_{\mu\nu} = 0$, Shear-free: type I, II, III

$$Dc_s^2 = \Omega - \alpha G_X s_i$$

$$4\beta G_X a^2 \leftrightarrow \Omega$$



The Take Away

- ⊙ Eigenvalues of acoustic metrics tell you everything about the free theory
 - Cauchy problem, stability
 - Non-linear/non-FRW backgrounds: (an)isotropisation?
- ⊙ Causality constrains the relationship between the two metrics
 - Are there dynamical mechanisms which ensure it be preserved?