Bound states in the ϕ^4 model

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Bound states in the Ising model: State of the art

d=2: Theory

- many exact results close to criticality from conformal theory and S-matrix: A.B. Zamolodchikov, Int. J. Mod. Phys. A 3 743 (1988)
- At $T = T_c$, with $B \neq 0$ (and small) seven "bound states"
- only two below the threshold $2m_0$ of the multi-particle continuum
- $m_1/m_0 = (1 + \sqrt{5})/2$ (golden ratio).
- No bound state for $T < T_c$ and $B = 0$.

d=2: Experiment

Quasi-1d quantum Ising ferromagnet: $CoNb_2O_6$, first bound state seen by neutron scattering R. Coldea et al. Science 327 177 (2010).

Open question: what about $T < T_c$ and $B \neq 0$?

Bound states in the Ising model: State of the art

d=3: Theory

- one bound state for $T < T_c$ ($B = 0$)
- simple argument from the quantum (2+1) system at $T = 0$,
- $m_1/m_0 \sim 1.8$ for $T \to T_c^-$
- many theoretical and numerical approaches: Bethe-Salpeter, exact diagonalization, Monte-Carlo.

Bethe-Salpeter at leading order is OK but very large (and unphysical) correction at next order.

$\Rightarrow$ need for nonperturbative methods.

NPRG and the BMW approximation

Naive answer from perturbation theory: the ratio between the two first excited levels is an integer: $m_0, 2m_0, \cdots$

$\Rightarrow$ Need to go beyond naive perturbation theory to describe bound states (e.g. resummation of infinitely many diagrams).

But "impossible" within the derivative expansion of the NPRG.

$\Rightarrow$ Need to go beyond the derivative expansion and keep the full momentum dependence of the two-point function.

$\Rightarrow$ Need BMW (Blaizot-Mendez-Wschebor) approximation.

Signature of a bound state in the spectral function

Instead of the lattice Ising model, we consider the ϕ^4 theory:

$$S[\varphi] = \int d^d x \left\{ \frac{1}{2} \left(\nabla \varphi(x)\right)^2 + \frac{r_0}{2} \varphi^2(x) + \frac{u_0}{4!} \varphi^4(x) \right\}. \qquad (1)$$

Monte Carlo simulations: bound states detected by studying $\langle \varphi(x) \varphi(0) \rangle_c$ in the broken phase.

Usually:

$$\langle \varphi(x) \varphi(0) \rangle_c \underset{x \to \infty}{\sim} A e^{-mx}, \quad \text{with} \quad m = \xi^{-1} \qquad (2)$$

Non trivial spectrum: sub-leading exponential(s) as well:

$$\langle \varphi(x) \varphi(0) \rangle_c \underset{x \to \infty}{\sim} A_0 e^{-mx} + A_1 e^{-Mx} + \dots \qquad (3)$$

Non trivial spectrum:

$$\langle \varphi(x)\varphi(0)\rangle_c \underset{x\to\infty}{\sim} A_0 e^{-mx} + A_1 e^{-Mx} + \ldots \tag{4}$$

In Fourier space:

$$\begin{aligned} G(p) &= \int d^d x \, \langle \varphi(x)\varphi(0)\rangle_c \, e^{-ipx} \\ &\underset{p\to 0}{\sim} \frac{A_0'}{p^2+m^2} + \frac{A_1'}{p^2+M^2} + \cdots \end{aligned} \tag{5}$$

$\Rightarrow$ analytic continuation $G(\omega = ip)$ has poles at the values of the masses of the system.

Work Plan:

- Compute the momentum dependence of the two-point function $\Gamma^{(2)}(p)$ and invert it to get $G(p)$;
- Analytically continue it: $p \to ip$;
- Find the poles.

BMW does point 1 for us.

Padé approximants followed by an evaluation on the complex axis ($G(ip - \epsilon)$) do point 2.

BMW approximation

$$\partial_k \Gamma_k^{(2)}(p,\phi) = \int_q \partial_k R_k(q^2) G_k^2(q) \Big[\Gamma_k^{(3)}(p,-p-q,q) \times \\ G_k(p+q) \Gamma_k^{(3)}(-p,p+q,-q) - \tfrac{1}{2} \Gamma_k^{(4)}(p,-p,q,-q) \Big]. \tag{6}$$

with the full propagator

$$G_k(p,\phi) = \left(\Gamma_k^{(2)}(p,\phi) + R_k(p) \right)^{-1} \tag{7}$$

Problem: The hierarchy of flow equations is not closed
$\Rightarrow$ need for a closure that preserves the full momentum dependence of $\Gamma_k^{(2)}(p,\phi)$
$\Rightarrow$ approximations on $\Gamma_k^{(3)}$, $\Gamma_k^{(4)}$.

BMW approximation

Based on two remarks:

1. $q < k$ because of $\partial_k R_k(q^2)$

$\Rightarrow$ replace $q \to 0$ in the vertex functions $\Gamma_k^{(3)}$, $\Gamma_k^{(4)}$

$\Rightarrow$ replace

$$\Gamma_k^{(3)}(p, q-p, -q; \phi) \to \Gamma_k^{(3)}(p, -p, 0; \phi)$$

$$\Gamma_k^{(4)}(p, -p, q, -q; \phi) \to \Gamma_k^{(4)}(p, -p, 0, 0; \phi)$$

2. $\Gamma_k^{(n)}(p_1, \cdots, p_{n-1}, 0; \phi) = \dfrac{\partial}{\partial \phi} \Gamma_k^{(n-1)}(p_1, \cdots, p_{n-1}; \phi)$

$$\begin{aligned} \partial_k \Gamma_k^{(2)}(p, \phi) \simeq \int_q \partial_k R_k(q^2) G_k^2(q) \Big[\Gamma_k^{(3)}(p, -p, 0; \phi) \times \\ G_k(p+q) \Gamma_k^{(3)}(-p, p, 0; \phi) - \tfrac{1}{2} \Gamma_k^{(4)}(p, -p, 0, 0; \phi) \Big]. \end{aligned} \tag{8}$$

BMW approximation

$$\partial_k \Gamma_k^{(2)}(p,\phi) \simeq \int_q \partial_k R_k(q^2) G_k^2(q) \Big[\Gamma_k^{(3)}(p,-p,0;\phi) \times G_k(p+q)\Gamma_k^{(3)}(-p,p,0;\phi) - \tfrac{1}{2}\Gamma_k^{(4)}(p,-p,0,0;\phi) \Big]. \tag{9}$$

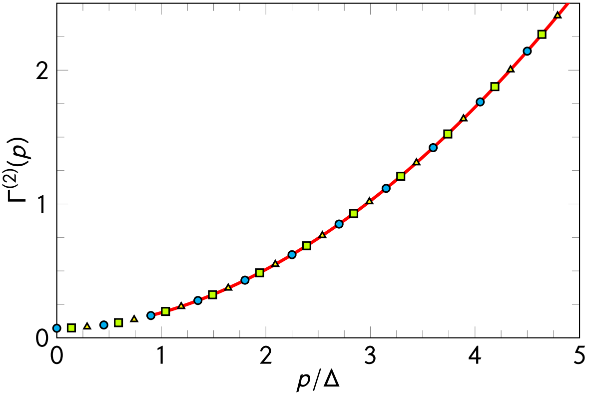
"finally"

$$\partial_k \Gamma_k^{(2)}(p,\phi) \simeq J_3(p,\phi)\left(\partial_\phi \Gamma_k^{(2)}(p,\phi)\right)^2 - \frac{1}{2} J_2(p,\phi)\, \partial_\phi^2 \Gamma_k^{(2)}(p,\phi)$$

and

$$J_n(p,\phi) = \int_q \partial_k R_k(q^2) G_k^{n-1}(q,\phi) G_k(p+q,\phi)$$

$\Gamma^{(2)}_{k=0}(p;\phi=0)$ for $T < T_c$



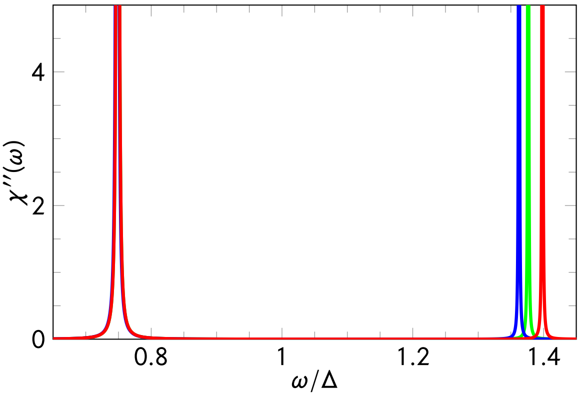
Δ is the mass of the fundamental particle (the inverse correlation length) at the LPA'.

Padé approximants

Necessary to perform an analytic continuation.
Procedure:

- We compute $G(p)$ for $N=$30 to 50 values p_i of p equally spaced in an interval $\omega_{\min} \sim \Delta$ and $\omega_{\max} \sim 10\Delta$,
- We construct a [(N-2)/N] Padé approximant $F(p)$ of $G(p)$, even in p, that satisfies $F(p_i) = G(p_i)$ for all i,
- We compute $\mathrm{Im}[F(\omega = ip - \epsilon)]$ which is an approximation of $\mathrm{Im}\, G(ip)$,
- The peaks of F correspond to the poles of $G(ip)$.

Results in $d = 3$



Very good resolution of the main peak, small dispersion of the second peak.

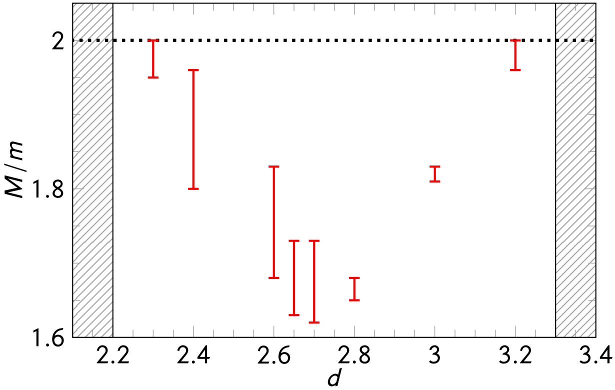
In $d = 3$ and for $T \to T_c$, we find $m_1/m_0 = 1.82(2)$.

Monte Carlo: 1.83(3),

Continuous unitary transformations: 1.84(3)

Exact diagonalization: 1.84(1).

Results in other dimensions



Results in agreement with exact results in $d = 2$.

Conclusions and perspectives

BMW + analytic continuation works remarkably well, at least for Ising.

Possible to study "non integrable perturbations" in $d = 2$: $T < T_c$ together with a magnetic field.

More difficult: 3-state Potts model in $d = 2$ and $d = 3$ where a bound state is expected.