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(Toward) Understanding quark confinement through a gauge-invariant Higgs mechanism

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[1] K.-I. Kondo, Phys. Lett.B**762**,219–224 (2016). e-Print:arXiv:1606.06194 [hep-th]
[2] K.-I. Kondo, e-Print: arXiv:1612.05933 [hep-th]
[3] K.-I. Kondo et al, papers in preparation.

§ Introduction

⊙ **Brout-Englert-Higgs (BEH) mechanism** is one of the most well-known mechanisms by which massless gauge bosons acquire their mass.
The conventional understanding (usual explanation) of the BEH mechanism is based on the "spontaneous breaking of gauge symmetry".

- In general, if a continuous (global) symmetry G is spontaneously broken $G \to H$,

Nambu-Goldstone theorem: There appear massless particles called the **Nambu-Goldstone (NG) particles** associated to the broken part G/H of G

- Especially, for the gauge symmetry (i.e., local symmetry), e.g., $G = U(1)$

$$\phi(x) \to \begin{cases} \varphi(x): & \text{Higgs} \quad \text{(massive)} \\ \pi(x): & \text{NG} \quad \text{(massless)} \end{cases} \to \varphi(x) : \text{Higgs} \quad \text{(massive)}$$

$$\downarrow$$

$$\mathscr{A}_\mu(x) \to \qquad \mathscr{A}_\mu(x): \quad \text{(massless)} \to \mathscr{W}_\mu(x) : \text{gauge boson} \quad \text{(massive)}$$

The massless NG particles generated associated with the spontaneous breaking of the continuous gauge symmetry G are "absorbed" into the massless gauge boson.
Then the NG particles disappear and the gauge boson becomes massive.

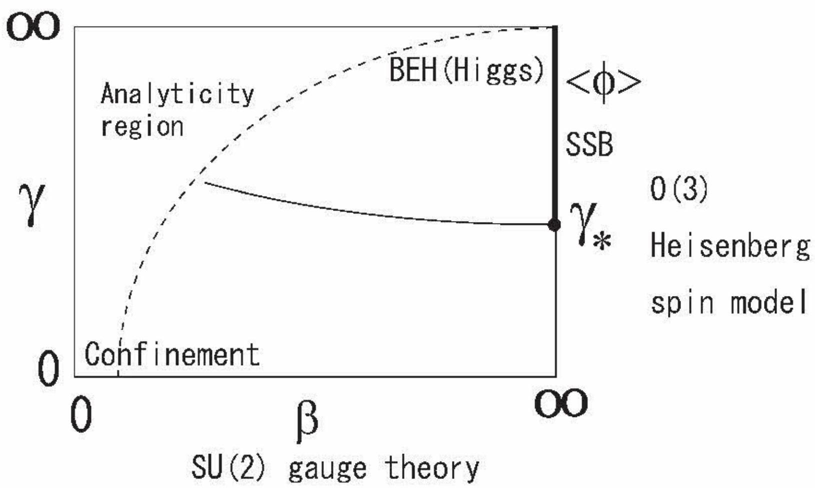
The non-vanishing **vacuum expectation value (VEV) of the scalar field** $\langle 0|\phi|0\rangle := v \neq 0$ is indispensable for spontaneous symmetry breking (SSB) to occur.

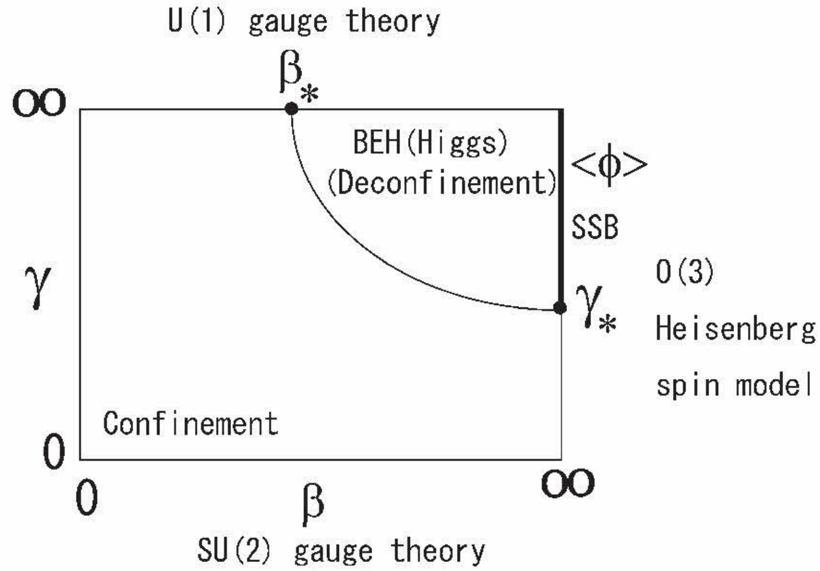
⊙ Fradkin and Shenker Continuity, Osterwalder and Seiler theorem 温故知新

gauge-scalar model in the lattice gauge theory (at zero temperature)

$$S = \beta S_{\mathrm{gauge}}[U] + \gamma S_{\mathrm{scalar}}[\phi, U] \text{ with a gauge group } G \text{ in } D \text{ spacetime dimensions}$$

1. radially fixed scalar field $||\phi(x)||^2 \equiv v^2$ [Fradkin and Shenker,1979]
   Fundamental scalar vs. Adjoint scalar
   ($G = SU(2)$, $D = 4$) ($G = SU(2)$, $D = 4$)

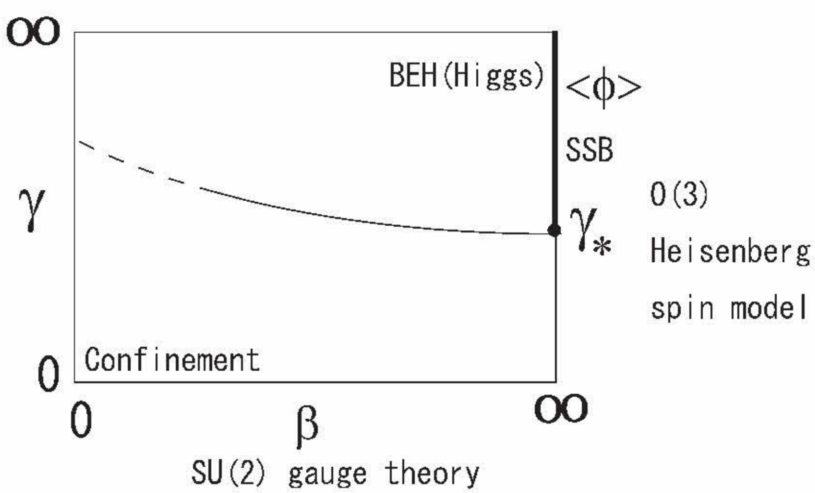


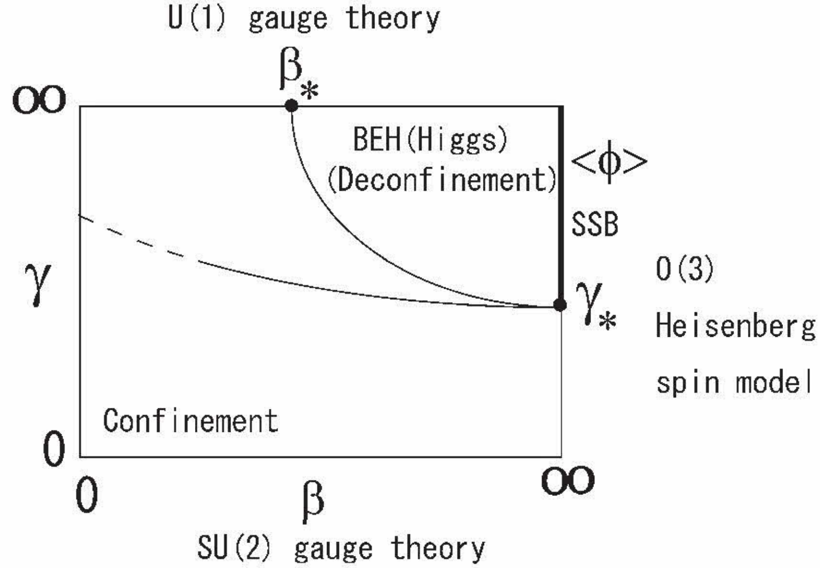


In the gauge-scalar model with the fundamental scalar field, Higgs phase and Confinement phase are not separated by the phase transition and are analytically continued in the phase diagram. (“**complementarity**” between Higgs and Confinement) This holds for any compact group (continuous and discrete) $G = SU(N), U(1), Z(N)$.

2. Fradkin-Shenker continuity still hold for almost radially fixed scalar field $\lambda \gg 1$ with the potential term $V = \lambda(||\phi(x)||^2 - v^2)^2$ [Osterwalder and Seiler,1978]

3. This is not the case for the radially variable scalar field $0 < \lambda \ll 1$ with the potential term $V = \lambda(||\phi(x)||^2 - v^2)^2$ [Munehisa and Munehisa,1986] for $Z(2), U(1), SU(2)$ [Kondo,1988] for $Z(2)$ [Jersak,1989] for a review
Higgs phase and Confinement phase are separated by the phase transition and are not analytically continued in the phase diagram.





⊙ We want to realize the Fradkin-Shenker continuity in quantum field theory on the continuum spacetime. If this is possible, we can understand confinement and gluon mass generation from a different viewpoint of the BEH mechanism.
[' t Hooft,197?][Fröhlich, Morchio and Strocchi,1980,1981]

⊙ However, we immediately encounter the obstructions.

Confinement phase respects the gauge symmetry (with no SSB) and confinement is believed to be understood in the gauge-invariant (or gauge-independent) way.

The usual description of BEH mechanism is based on spontaneous breaking of the gauge symmetry.

How the BEH phase with spontaneously broken gauge symmetry can be continued to the confinement phase with the unbroken gauge symmetry?

⊙ We must reconsider the BEH mechanism.

Indeed, **spontaneous breaking of gauge symmetry** usually assumed for the BEH mechanism to occur is a rather misleading terminology.

The lattice gauge theory a la Wilson is a formulation of quantum gauge theory keeping the gauge symmetry manifest.
[**Elitzur theorem**] local continuous symmetry cannot break spontaneously, if no gauge fixing is introduced.

The VEV $\langle 0|\phi|0\rangle$ of the scalar field which is not gauge invariant is zero $\langle 0|\phi|0\rangle = 0$ unless we fix the gauge. $\langle \phi(x)\rangle \neq 0$ is possible only in the non-gauge theory $g = 0$ (or $\beta = \infty$).

Whether the gauge symmetry is spontaneously broken or not in this sense is a concept depending clearly on the gauge choice. This prevents us from establishing the BEH mechanism for gauge boson mass generation in a gauge-independent way as a true physical phenomenon.

[The global symmetry H' which remains unbroken even after fixing the local gauge symmetry, the **remnant global gauge symmetry**, can break spontaneously. But the remnant global symmetry is not unique and depends on the choice of the gauge fixing condition. We must discuss the BEH mechanism gauge by gauge.]

$\odot$ To avoid these difficulties, we give a **manifestly gauge-invariant description of the BEH mechanism** in the operator level from the beginning.

Consequently, the massless gauge boson can acquire the mass without relying on the spontaneous breaking of the gauge symmetry.

• We do not need to introduce the the VEV $\langle 0|\phi|0\rangle$ of the scalar field which depends on the gauge fixing condition.

• The gauge symmetry is always kept without breaking and we do not need to introduce the intermediate steps for generating the Nambu-Goldstone bosons and subsequently absorbing them.

• We can discuss confinement from the viewpoint of complementarity between Higgs and Confinement. Because confinement is realized in the phase with a mass gap in which gauge symmetry is kept unbroken.

$\odot$ Due to the gauge-invariant BEH mechanism, we can extract the massive modes $\mathscr{W}_\mu$ from the original gauge field $\mathscr{A}_\mu$ and the scalar field ϕ in the gauge-independent way.
• The massive mode $\mathscr{W}_\mu$ will rapidly fall off in the distance and hence it is identified with the short-distance (or high-energy) mode.
• The residual mode must exist, $\mathscr{V}_\mu = \mathscr{A}_\mu - \mathscr{W}_\mu$. The residual mode is identified with the long-distance (or low-energy) mode. In Confinement phase, the residual mode will mediate the long-range force which is responsible for quark confinement, e.g., area law of the Wilson loop average or linear quark potential.

To obtain the Yang-Mills theory from the "complementary" gauge-scalar model, we must discriminate the fundamental scalar and the adjoint scalar. The adjoint scalar case was already discussed in [1]. Now we treat the fundamental scalar case.

Contents

For details of the reformulation of the Yang-Mills theory, see

Physics Report Vol.579, pp.1–226 (2015), e-Print: arXiv:1409.1599 [hep-th]

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§ Radially fixed $SU(2)$ gauge-adjoint scalar model

⊙ We consider $G = SU(N)$ gauge-scalar model with the gauge-invariant Lagrangian:

$$\mathscr{L}_{\rm YMH} = -\frac{1}{4}\mathscr{F}^{\mu\nu}(x) \cdot \mathscr{F}_{\mu\nu}(x) + \frac{1}{2}(\mathscr{D}^{\mu}[\mathscr{A}]\boldsymbol{\phi}(x)) \cdot (\mathscr{D}_{\mu}[\mathscr{A}]\boldsymbol{\phi}(x)). \tag{1}$$

The scalar field $\boldsymbol{\phi}(x)$ and the Yang-Mills field $\mathscr{A}_\mu(x) = \mathscr{A}_\mu^A(x)T_A$ obey the gauge transformation:

$$\begin{aligned} \boldsymbol{\phi}(x) &\to U(x)\boldsymbol{\phi}(x)U^{-1}(x), \quad U(x) \in G = SU(N), \\ \mathscr{A}_\mu(x) &\to U(x)\mathscr{A}_\mu(x)U^{-1}(x) + ig^{-1}U(x)\partial_\mu U^{-1}(x). \end{aligned} \tag{2}$$

The covariant derivative $\mathscr{D}_\mu[\mathscr{A}] := \partial_\mu - ig[\mathscr{A}_\mu, \cdot]$ transforms $\mathscr{D}_\mu[\mathscr{A}] \to U(x)\mathscr{D}_\mu[\mathscr{A}]U^{-1}(x)$.

In what follows, we assume that the adjoint scalar field $\boldsymbol{\phi}(x) = \phi^A(x)T_A$ $(A = 1, \ldots, N^2 - 1)$ has the fixed radial length,

$$\boldsymbol{\phi}(x) \cdot \boldsymbol{\phi}(x) \equiv \boldsymbol{\phi}^A(x)\boldsymbol{\phi}^A(x) = v^2. \tag{3}$$

Notice that $\boldsymbol{\phi}(x) \cdot \boldsymbol{\phi}(x)$ is a gauge-invariant combination.
Notation: For the Lie-algebra valued quantities $\mathscr{A} = \mathscr{A}^A T_A$ and $\mathscr{B} = \mathscr{B}^A T_A$

$$\mathscr{A} \cdot \mathscr{B} := 2{\rm tr}(\mathscr{A}\mathscr{B}) = \mathscr{A}^A\mathscr{B}^B 2{\rm tr}(T_A T_B) = \mathscr{A}^A\mathscr{B}^A \quad (A = 1, \ldots, N^2 - 1). \tag{4}$$

⊙ First, we recall the conventional description for the BEH mechanism.
If $\boldsymbol{\phi}(x)$ acquires a non-vanishing VEV $\langle \boldsymbol{\phi}(x) \rangle = \langle \boldsymbol{\phi} \rangle = \langle \boldsymbol{\phi}^A \rangle T_A$, then

$$\mathscr{D}_\mu[\mathscr{A}]\phi(x) := \partial_\mu \boldsymbol{\phi}(x) - ig[\mathscr{A}_\mu(x), \boldsymbol{\phi}(x)] \to -ig[\mathscr{A}_\mu(x), \langle \boldsymbol{\phi} \rangle] + ..., \tag{5}$$

and

$$\begin{aligned}\frac{1}{2}(\mathscr{D}^\mu[\mathscr{A}]\boldsymbol{\phi}(x)) \cdot (\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}(x)) \to & - g^2 \mathrm{tr}_G\{[\mathscr{A}^\mu(x), \langle \boldsymbol{\phi} \rangle][\mathscr{A}_\mu(x), \langle \boldsymbol{\phi} \rangle]\} + \ldots \\ = & - g^2 \mathrm{tr}_G\{[T_A, \langle \boldsymbol{\phi} \rangle][T_B, \langle \boldsymbol{\phi} \rangle]\} \mathscr{A}^{\mu A}(x) \mathscr{A}_\mu^B(x).\end{aligned} \tag{6}$$

To break spontaneously the local continuous gauge symmetry G by realizing the non-vanishing VEV $\langle \boldsymbol{\phi} \rangle$ of the scalar field $\boldsymbol{\phi}$, we choose the unitary gauge in which $\boldsymbol{\phi}(x)$ is pointed to a specific direction $\boldsymbol{\phi}(x) \to \boldsymbol{\phi}_\infty$ uniformly over the spacetime.

By this procedure the original gauge symmetry G is not completely broken. Indeed, there may exist a subgroup H (of G) which does not change $\boldsymbol{\phi}_\infty$.
This is the partial SSB $G \to H$: the mass is provided for the coset G/H (broken parts), while the mass is not supplied for the H-commutative part of $\mathscr{A}_\mu$:

$$\mathscr{L}_{\mathrm{YMH}} \to -\frac{1}{2}\mathrm{tr}_G\{\mathscr{F}^{\mu\nu}(x)\mathscr{F}_{\mu\nu}(x)\} - (gv)^2 \mathrm{tr}_{G/H}\{\mathscr{A}^\mu(x)\mathscr{A}_\mu(x)\}. \tag{7}$$

Thus the theory reduces to a gauge theory with the **residual gauge group** H.

For $G = SU(2)$, by taking the usual **unitary gauge**

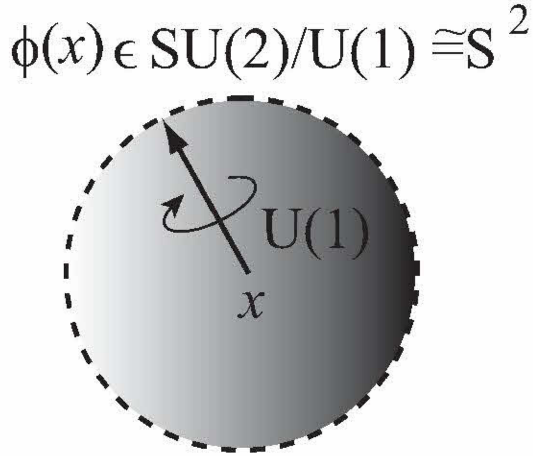
$$\langle \boldsymbol{\phi}_\infty \rangle = vT_3, \quad \Longleftrightarrow \quad \langle \phi_\infty^A \rangle = v\delta^{A3}, \tag{8}$$

the kinetic term generates the mass term,

$$-g^2 v^2 \mathrm{tr}_G\{[T_A, T_3][T_B, T_3]\}\mathscr{A}^{\mu A}\mathscr{A}_\mu^B = \frac{1}{2}(gv)^2(\mathscr{A}^{\mu 1}\mathscr{A}_\mu^1 + \mathscr{A}^{\mu 2}\mathscr{A}_\mu^2). \tag{9}$$

- The off-diagonal gluons $\mathscr{A}_\mu^1, \mathscr{A}_\mu^2$ acquire the same mass $M_W := gv$,
- The diagonal gluon $\mathscr{A}_\mu^3$ remains massless.

Even after taking the unitary gauge (8),
$U(1)$ gauge symmetry described by $\mathscr{A}_\mu^3$ still remains
as the residual local gauge symmetry $H = U(1)$,
which leaves $\boldsymbol{\phi}_\infty$ invariant
(the local rotation around the axis of the scalar field $\boldsymbol{\phi}_\infty$).



Thus, the SSB is sufficient for the BEH mechanism to take place.
But, it is not clear whether the SSB is necessary or not for the BEH mechanism to work. This description depends on the specific gauge.

⊙ Gauge-invariant BEH mechanism

Next, we give a gauge-invariant description for the BEH mechanism.

- We construct a composite vector field $\mathscr{W}_\mu(x)$ from $\mathscr{A}_\mu(x)$ and $\boldsymbol{\phi}(x)$ by

$$\mathscr{W}_\mu(x) := -ig^{-1}[\hat{\boldsymbol{\phi}}(x), \mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}(x)], \quad \hat{\boldsymbol{\phi}}(x) := \boldsymbol{\phi}(x)/v. \tag{10}$$

- We find that the kinetic term of the gauge-scalar model is identical to the "mass term" of the vector field $\mathscr{W}_\mu(x)$:

$$\frac{1}{2}\mathscr{D}^\mu[\mathscr{A}]\boldsymbol{\phi}(x) \cdot \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}(x) = \frac{1}{2}M_W^2 \mathscr{W}^\mu(x) \cdot \mathscr{W}_\mu(x), \quad M_W := gv, \tag{11}$$

as far as the constraint ($\boldsymbol{\phi} \cdot \boldsymbol{\phi} = 1$) is satisfied. This fact is shown explicitly $G = SU(2)$,

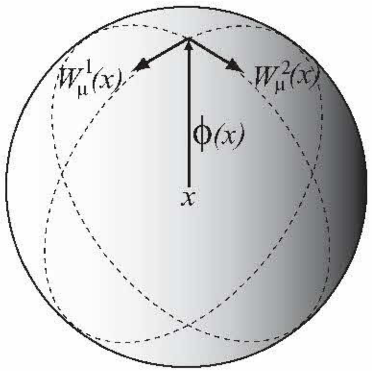
$$\begin{aligned} g^2 v^2 \mathscr{W}^\mu \cdot \mathscr{W}_\mu =& v^{-2} 2\mathrm{tr}([\boldsymbol{\phi}, \mathscr{D}^\mu[\mathscr{A}]\boldsymbol{\phi}][\boldsymbol{\phi}, \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}]) \\ =& v^{-2}\{(\boldsymbol{\phi}\cdot\boldsymbol{\phi})(\mathscr{D}^\mu[\mathscr{A}]\boldsymbol{\phi} \cdot \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}) - (\boldsymbol{\phi} \cdot \mathscr{D}^\mu[\mathscr{A}]\boldsymbol{\phi})(\boldsymbol{\phi} \cdot \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi})\} \\ =& (\mathscr{D}^\mu[\mathscr{A}]\boldsymbol{\phi}) \cdot (\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}), \end{aligned} \tag{12}$$

where we have used the constraint ($\boldsymbol{\phi} \cdot \boldsymbol{\phi} \equiv v^2$) and $\boldsymbol{\phi} \cdot \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi} = \boldsymbol{\phi} \cdot \partial_\mu \boldsymbol{\phi} + \boldsymbol{\phi} \cdot (g\mathscr{A}_\mu \times \boldsymbol{\phi}) = g\mathscr{A}_\mu \cdot (\boldsymbol{\phi} \times \boldsymbol{\phi}) = 0$, with $\boldsymbol{\phi} \cdot \partial_\mu \boldsymbol{\phi} = 0$ following from differentiating the constraint.

- Remarkably, the above "mass term" of $\mathscr{W}_\mu$ is gauge invariant, since $\mathscr{W}_\mu$ obeys the adjoint gauge transformation:

$$\mathscr{W}_\mu(x) \to U(x)\mathscr{W}_\mu(x)U^{-1}(x). \tag{13}$$

Therefore, $\mathscr{W}_\mu$ becomes massive without breaking the original gauge symmetry.
The above $\mathscr{W}_\mu$ gives a gauge-independent definition of the massive gluon mode in the operator level.
The SSB of gauge symmetry is not necessary for generating the mass of $\mathscr{W}_\mu$.
(We do not need to choose a specific vacuum from all possible degenerate ground states distinguished by the direction of $\boldsymbol{\phi}$.)



How is this description related to the conventional one?

- Despite its appearance of $\mathscr{W}_\mu$,

the independent internal degrees of freedom of $\mathscr{W}_\mu = (\mathscr{W}_\mu^A)$ $(A = 1, 2, 3)$ is equal to $\dim(G/H) = 2$, since

$$\mathscr{W}_\mu(x) \cdot \hat{\boldsymbol{\phi}}(x) = 0. \tag{14}$$

Notice that this is a gauge-invariant statement. Thus, $\mathscr{W}_\mu(x)$ represent the massive modes corresponding to the coset space G/H components as expected. [We understand the **residual gauge symmetry** left in the partial SSB: $G = SU(2) \to H = U(1)$.]

In fact, by taking the unitary gauge $\boldsymbol{\phi}(x) \to \boldsymbol{\phi}_\infty = v\hat{\boldsymbol{\phi}}_\infty$, $\mathscr{W}_\mu$ reduces to

$$\mathscr{W}_\mu(x) \to -ig^{-1}[\hat{\boldsymbol{\phi}}_\infty, \mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}_\infty] = [\hat{\boldsymbol{\phi}}_\infty, [\hat{\boldsymbol{\phi}}_\infty, \mathscr{A}_\mu(x)]] = \mathscr{A}_\mu(x) - (\mathscr{A}_\mu(x) \cdot \hat{\boldsymbol{\phi}}_\infty)\hat{\boldsymbol{\phi}}_\infty. \tag{15}$$

Then $\mathscr{W}_\mu$ agrees with the off-diagonal components for the specific choice $\hat{\boldsymbol{\phi}}_\infty^A = \delta^{A3}$:

$$\mathscr{W}_\mu^A(x) \to \begin{cases} \mathscr{A}_\mu^a(x) & (A = a = 1, 2) \\ 0 & (A = 3) \end{cases}. \tag{16}$$

This suggests that the original gauge field $\mathscr{A}_\mu$ is separated into two pieces:

$$\mathscr{A}_\mu(x) = \mathscr{W}_\mu(x) + \mathscr{V}_\mu(x). \tag{17}$$

By definition, $\mathscr{V}_\mu(x)$ transforms under the gauge transformation just like $\mathscr{A}_\mu(x)$:

$$\mathscr{V}_\mu(x) \to U(x)\mathscr{V}_\mu(x)U^{-1}(x) + ig^{-1}U(x)\partial_\mu U^{-1}(x). \tag{18}$$

According to the expression of $\mathscr{W}_\mu(x)$, $\mathscr{W}_\mu(x) = 0$ is equivalent to

$$\mathscr{D}_\mu[\mathscr{V}]\hat{\boldsymbol{\phi}}(x) = 0. \tag{19}$$

We find that $\mathscr{V}_\mu$ is constructed from $\mathscr{A}_\mu$ and $\boldsymbol{\phi}$ as

$$\mathscr{V}_\mu(x) = c_\mu(x)\hat{\boldsymbol{\phi}}(x) + ig^{-1}[\hat{\boldsymbol{\phi}}(x), \partial_\mu\hat{\boldsymbol{\phi}}(x)], \quad c_\mu(x) := \mathscr{A}_\mu(x)\cdot\hat{\boldsymbol{\phi}}(x). \tag{20}$$

In the unitary gauge $\boldsymbol{\phi}(x) \to \boldsymbol{\phi}_\infty = v\hat{\boldsymbol{\phi}}_\infty$, $\hat{\boldsymbol{\phi}}_\infty^A = \delta^{A3}$, $\mathscr{V}_\mu$ agrees with the diagonal component

$$\mathscr{V}_\mu(x) \to (\mathscr{A}_\mu(x)\cdot\hat{\boldsymbol{\phi}}_\infty)\hat{\boldsymbol{\phi}}_\infty \to \begin{cases} 0 & (A = a = 1,2) \\ \mathscr{A}_\mu^3(x) & (A = 3) \end{cases}. \tag{21}$$

Thus, the above arguments go well in the topologically trivial sector.
The topologically non-trivial sector is discussed later.

• First, we introduce $\mathscr{V}_\mu(x)$ and $\mathscr{W}_\mu(x)$ as composite field operators of $\mathscr{A}_\mu(x)$ and $\hat{\boldsymbol{\phi}}(x)$.
• Then we regard a set of field variables $\{c_\mu(x), \mathscr{W}_\mu(x), \hat{\boldsymbol{\phi}}(x)\}$ as obtained from $\{\mathscr{A}_\mu(x), \hat{\boldsymbol{\phi}}(x)\}$ based on **change of variables**:

$$\{c_\mu(x), \mathscr{W}_\mu(x), \hat{\boldsymbol{\phi}}(x)\} \leftarrow \{\mathscr{A}_\mu(x), \hat{\boldsymbol{\phi}}(x)\}. \tag{22}$$

• Finally, we identify $c_\mu(x)$, $\mathscr{W}_\mu(x)$ and $\hat{\boldsymbol{\phi}}(x)$ with the fundamental field variables for describing the massive Yang-Mills theory anew.
(Here fundamental means that the quantization should be performed with respect to those variables $\{c_\mu(x), \mathscr{W}_\mu(x), \hat{\boldsymbol{\phi}}(x)\}$ which appear in the path-integral measure.)

⊙ In the gauge-scalar model, $\mathscr{A}_\mu(x)$ and $\boldsymbol{\phi}(x)$ are independent field variables. However, the Yang-Mills theory should be described by $\mathscr{A}_\mu(x)$ alone and hence $\boldsymbol{\phi}$ must be supplied by the gauge field $\mathscr{A}_\mu(x)$ due to the strong interactions. [In other words, the scalar field $\boldsymbol{\phi}$ should be given as a (complicated) functional of the gauge field.]

This is achieved by imposing the constraint which we call the **reduction condition**. We choose e.g.,

$$\boldsymbol{\chi}(x) := [\hat{\boldsymbol{\phi}}(x), \mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}(x)] = \mathbf{0} \Longleftrightarrow \mathscr{D}^\mu[\mathscr{V}]\mathscr{W}_\mu(x) = 0. \tag{23}$$

This condition is gauge covariant, $\boldsymbol{\chi}(x) \to U(x)\boldsymbol{\chi}(x)U^{-1}(x)$.
The reduction condition plays the role of eliminating the extra degrees of freedom introduced by the radially fixed scalar field into the Yang-Mills theory, since

$$\boldsymbol{\chi}(x) \cdot \hat{\boldsymbol{\phi}}(x) = 0. \tag{24}$$

Following the way similar to the Faddeev-Popov procedure, we insert the unity to the functional integral:

$$1 = \int \mathcal{D}\boldsymbol{\chi}^{\theta}\delta(\boldsymbol{\chi}^{\theta}) = \int \mathcal{D}\boldsymbol{\theta}\delta(\boldsymbol{\chi}^{\theta})\Delta^{\mathrm{red}}, \tag{25}$$

where $\boldsymbol{\chi}^{\theta} := \boldsymbol{\chi}[\mathscr{A}, \Phi^{\theta}]$ is the reduction condition written in terms of $\mathscr{A}$ and Φ^{θ} which is the local rotation of Φ by $\boldsymbol{\theta}$ and $\Delta^{\mathrm{red}} := \det\left(\frac{\delta\boldsymbol{\chi}^{\theta}}{\delta\boldsymbol{\theta}}\right)$ denotes the Faddeev-Popov determinant associated with the reduction condition $\chi = 0$. See Kondo et al., Phys. Report 579, 1–226 (2015),

$$Z_{\mathrm{RF}} = \int \mathcal{D}\hat{\Phi}\mathcal{D}\mathscr{A}\,\delta(\chi)\Delta^{\mathrm{red}}e^{iS_{\mathrm{YM}}[\mathscr{A}]+iS_{\mathrm{kin}}[\mathscr{A},\Phi]}, \tag{26}$$

$$= \int \mathcal{D}\hat{\Phi}\mathcal{D}c\mathcal{D}\mathscr{W}\,J\delta(\tilde{\chi})\tilde{\Delta}^{\mathrm{red}}e^{iS_{\mathrm{YM}}[\mathscr{V}+\mathscr{W}]+iS_{\mathrm{m}}[\mathscr{W}]},\ J = 1. \tag{27}$$

This is a gauge-invariant description. We can reproduce the well-known case by taking the special limit (gauge). For instance,

$$\boldsymbol{\phi}^{A}(x) = v\hat{\boldsymbol{\phi}}^{A}(x),\ \hat{\boldsymbol{\phi}}^{A}(x) \to \delta^{A3} \tag{28}$$

$$Z_{\mathrm{RF}} \to \int \mathcal{D}A^{3}\mathcal{D}A^{a}\delta\left(\mathscr{D}^{\mu}[A^{3}]A^{a}_{\mu}\right)\Delta_{\mathrm{FP}}e^{iS_{\mathrm{YM}}[\mathscr{A}]+iS_{\mathrm{m}}[A^{a}]} = Z_{\mathrm{MAG}}. \tag{29}$$

In the limit, the gauge-adjoint scalar model with the radially fixed scalar field is reduced to the Yang-Mills theory with the gauge-fixing term of the Maximal Abelian gauge and the associated Faddeev-Popov ghost term supplemented with a mass term for the off-diagonal gluons.

According to the decomposition $\mathscr{A}_\mu(x) = \mathscr{V}_\mu(x) + \mathscr{W}_\mu(x)$,
the field strength $\mathscr{F}_{\mu\nu}(x)$ of the gauge field $\mathscr{A}_\mu(x)$ is decomposed as

$$\begin{aligned} \mathscr{F}_{\mu\nu}[\mathscr{A}] :=& \partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu - ig[\mathscr{A}_\mu, \mathscr{A}_\nu] \\ =& \mathscr{F}_{\mu\nu}[\mathscr{V}] + \mathscr{D}_\mu[\mathscr{V}]\mathscr{W}_\nu - \mathscr{D}_\nu[\mathscr{V}]\mathscr{W}_\mu - ig[\mathscr{W}_\mu, \mathscr{W}_\nu]. \end{aligned} \tag{30}$$

By substituting this decomposition into the gauge-scalar Lagrangian, we obtain

$$\begin{aligned} \mathscr{L}_{\rm RF} =& -\frac{1}{4}\mathscr{F}_{\mu\nu}[\mathscr{V}] \cdot \mathscr{F}^{\mu\nu}[\mathscr{V}] \\ & -\frac{1}{4}(\mathscr{D}_\mu[\mathscr{V}]\mathscr{W}_\nu - \mathscr{D}_\nu[\mathscr{V}]\mathscr{W}_\mu)^2 \\ & +\frac{1}{2}\mathscr{F}_{\mu\nu}[\mathscr{V}] \cdot ig[\mathscr{W}^\mu, \mathscr{W}^\nu] - \frac{1}{4}(ig[\mathscr{W}_\mu, \mathscr{W}_\nu])^2 \\ & +\frac{1}{2}M_W^2 \mathscr{W}^\mu \cdot \mathscr{W}_\mu, \quad \mathscr{D}_\mu[\mathscr{V}] := \partial_\mu - ig[\mathscr{V}_\mu, \cdot]. \end{aligned} \tag{31}$$

The field $\mathscr{W}_\mu$ has the ordinary kinetic term and the mass term. Therefore, there is a massive vector pole in the propagator of $\mathscr{W}_\mu$ (after a certain gauge fixing). Thus, $\mathscr{W}_\mu$ is not an auxiliary field, but is a propagating field with the mass M_W (up to possible quantum corrections).

⊙ Confined massive phase

Finally, we discuss the implications for quark confinement.
The field strength $\mathscr{F}_{\mu\nu}[\mathscr{V}](x)$ of $\mathscr{V}_\mu(x)$ is shown to be proportional to $\hat{\boldsymbol{\phi}}(x)$:

$$\mathscr{F}_{\mu\nu}[\mathscr{V}](x) = \hat{\boldsymbol{\phi}}(x)\{\partial_\mu c_\nu(x) - \partial_\nu c_\mu(x) + H_{\mu\nu}(x)\},$$
$$H_{\mu\nu}(x) := ig^{-1}\hat{\boldsymbol{\phi}}(x) \cdot [\partial_\mu \hat{\boldsymbol{\phi}}(x), \partial_\nu \hat{\boldsymbol{\phi}}(x)], \quad (32)$$

We can introduce the Abelian-like $SU(2)$ gauge-invariant field strength $f_{\mu\nu}$ by

$$f_{\mu\nu}(x) := \hat{\boldsymbol{\phi}}(x) \cdot \mathscr{F}_{\mu\nu}[\mathscr{V}](x) = \partial_\mu c_\nu(x) - \partial_\nu c_\mu(x) + H_{\mu\nu}(x). \quad (33)$$

In the low-energy $E \ll M_W$ or the long-distance $r \gg M_W^{-1}$ region, we can neglect $\mathscr{W}_\mu$. Then the dominant low-energy modes are described by the restricted Yang-Mills theory:

$$\mathscr{L}_{\rm YM}^{\rm rest} = -\frac{1}{4}\mathscr{F}^{\mu\nu}[\mathscr{V}] \cdot \mathscr{F}_{\mu\nu}[\mathscr{V}] = -\frac{1}{4} f^{\mu\nu} f_{\mu\nu}. \quad (34)$$

In the low-energy $E \ll M_W$ or the long-distance $r \gg M_W^{-1}$ region, the massive components $\mathscr{W}_\mu(x)$ become negligible and the other restricted (residual) fields become dominant.
This is a phenomenon called the "Abelian" dominance in quark confinement. ['tHooft 81, Ezawa-Iwazaki 82]

The "Abelian" dominance in quark confinement is understood as a consequence of the BEH mechanism for the "complementary" gauge-scalar model in the gauge-invariant way.

⊙ **If the fields $\mathscr{A}$ and ϕ are a set of solutions of the field equations for the gauge-scalar model with a radially fixed scalar field, they automatically satisfy the reduction condition for the pure Yang-Mills theory.**

We introduce a Lagrange multiplier field $u(x)$ to incorporate the constraint

$$\mathscr{L}'_{\rm RF}(x) = \mathscr{L}_{\rm GS}(x) + u(x)\left(\boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x) - v^2\right). \tag{35}$$

Then the field equations are obtained as

$$\frac{\delta S'_{\rm RF}}{\delta u(x)} = \boldsymbol{\phi}(x)\cdot\boldsymbol{\phi}(x) - v^2 = 0, \tag{36}$$

$$\frac{\delta S'_{\rm YMHRF}}{\delta \mathscr{A}^\mu(x)} = \mathscr{D}^\nu[\mathscr{A}]\mathscr{F}_{\nu\mu}(x) - ig[\boldsymbol{\phi}(x), \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}(x)] = 0, \tag{37}$$

$$\frac{\delta S'_{\rm RF}}{\delta \boldsymbol{\phi}(x)} = -\mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}(x) + 2u(x)\boldsymbol{\phi}(x) = 0. \tag{38}$$

The reduction condition is automatically satisfied:

$\mathscr{D}^\mu(37) \Longrightarrow 0 = \mathscr{D}^\mu[\mathscr{A}]\mathscr{D}^\nu[\mathscr{A}]\mathscr{F}_{\nu\mu} = ig\mathscr{D}^\mu[\mathscr{A}][\boldsymbol{\phi}, \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}] = ig[\boldsymbol{\phi}, \mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}]$

$[\boldsymbol{\phi}, (38)] \Longrightarrow [\boldsymbol{\phi}, \mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}] = [\boldsymbol{\phi}, 2u\boldsymbol{\phi}] = 0$

⊙ We can show that magnetic monopoles (configurations) exist in the massive Yang-Mills theory, see S. Nishino, R. Matsudo, M. Warschinke, K.-I. Kondo, e-Print: arXiv:1803.04339 [hep-th] Magnetic monopoles in pure SU(2) Yang–Mills theory with a gauge-invariant mass,

§ Radially fixed $SU(2)$ gauge-fundamental scalar model

We consider the $SU(2)$ gauge-fundamental scalar model with the **radially fixed scalar field**

$$\mathscr{L}_{\rm RF} = -\frac{1}{4}\mathscr{F}^A_{\mu\nu}(x)\mathscr{F}^{\mu\nu A}(x) + (D_\mu[\mathscr{A}]\Phi(x))^\dagger(D^\mu[\mathscr{A}]\Phi(x)) + u(x)\left(\Phi(x)^\dagger\Phi(x) - \frac{1}{2}v^2\right), \tag{1}$$

where $u(x)$ is the **Lagrange multiplier field** to incorporate the constraint:

$$\Phi(x)^\dagger\Phi(x) - \frac{1}{2}v^2 = 0. \tag{2}$$

Here $\Phi(x)$ is the $SU(2)$ **doublet** formed from two complex scalar fields $\phi_1(x), \phi_2(x)$:

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}, \ \phi_1(x), \phi_2(x) \in \mathbb{C}. \tag{3}$$

The Lagrangian density is invariant under the $SU(2)$ gauge transformation given by

$$\begin{aligned} &\Phi(x) \to \Phi'(x) = U(x)\Phi(x), \ U(x) = e^{ig\omega(x)} \in SU(2), \ \omega(x) = \omega^A(x)T_A, \ T_A = \frac{1}{2}\sigma_A, \\ &\mathscr{A}_\mu(x) \to \mathscr{A}'_\mu(x) = U(x)\mathscr{A}_\mu(x)U(x)^{-1} + ig^{-1}U(x)\partial_\mu U(x)^{-1}, \end{aligned} \tag{4}$$

The Lagrange multiplier field $u(x)$ is supposed to be invariant under the $SU(2)$ gauge transformation.

⊙ Matrix scalar field

We introduce the **matrix-valued scalar field** Θ by adding another $SU(2)$ doublet $\tilde{\Phi} := i\tau_2 \Phi^*$ as

$$\Theta := \begin{pmatrix} \tilde{\Phi} & \Phi \end{pmatrix} = \begin{pmatrix} i\tau_2 \Phi^* & \Phi \end{pmatrix} = \begin{pmatrix} \phi_2^* & \phi_1 \\ -\phi_1^* & \phi_2 \end{pmatrix}, \ i\tau_2 = \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5}$$

$\tilde{\Phi}$ has the same gauge transformation property as Φ. Then Θ has the same gauge transformation as Φ,

$$\Theta(x) \to \Theta'(x) = U(x)\Theta(x), \ U(x) \in SU(2). \tag{6}$$

It is shown that the **normalized matrix-valued scalar field** $\hat{\Theta}$ is an element of $SU(2)$:

$$\hat{\Theta}(x) = \Theta(x) / \left(\frac{v}{\sqrt{2}} \right) \in G = SU(2), \quad v > 0. \tag{7}$$

The original kinetic term of the scalar field is rewritten in terms of the matrix-valued scalar field as

$$(D_\mu[\mathscr{A}]\Phi)^\dagger (D^\mu[\mathscr{A}]\Phi) = \frac{1}{2}\mathrm{tr}((D_\mu[\mathscr{A}]\Theta(x))^\dagger D^\mu[\mathscr{A}]\Theta(x)). \tag{8}$$

This is shown as the $SU(2)$ doublet field Φ is extracted from the matrix-valued field Θ by

$$\Phi(x) = \Theta(x)\Omega = \frac{v}{\sqrt{2}}\hat{\Theta}(x)\Omega, \quad \Omega = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{9}$$

⊙ Higgs mechanism for $SU(2)$ gauge-fundamental scalar model

For the gauge group $SU(2)$, we introduce the vector boson field $\mathscr{W}_\mu$ defined in terms of the normalized scalar field $\hat{\Theta}$ and the original gauge field $\mathscr{A}_\mu$ as

$$\begin{aligned}\mathscr{W}_\mu(x) :=& ig^{-1}(D_\mu[\mathscr{A}]\hat{\Theta}(x))\hat{\Theta}(x)^\dagger = -ig^{-1}\hat{\Theta}(x)(D_\mu[\mathscr{A}]\hat{\Theta}(x))^\dagger \\ =& \frac{1}{2}ig^{-1}[(D_\mu[\mathscr{A}]\hat{\Theta}(x))\hat{\Theta}(x)^\dagger - \hat{\Theta}(x)(D_\mu[\mathscr{A}]\hat{\Theta}(x))^\dagger]. \end{aligned} \tag{10}$$

We find that **the kinetic term of the scalar field Φ or Θ is identical to the mass term of the vector boson field $\mathscr{W}^\mu$ with the mass M_W**:

$$\frac{1}{2}\mathrm{tr}((D_\mu[\mathscr{A}]\Theta(x))^\dagger D^\mu[\mathscr{A}]\Theta(x)) = M_W^2 \mathrm{tr}(\mathscr{W}_\mu(x)\mathscr{W}^\mu(x)),\ M_W := \frac{1}{2}gv. \tag{11}$$

In order to see the relationship between the new description and the conventional explanation for the BEH mechanism, we take the unitary gauge, namely, we can use the freedom of $SU(2)$ rotations to write the expectation value in the form:

$$\Phi(x) \to \langle\Phi(x)\rangle = \Phi_\infty := \frac{1}{\sqrt{2}}\begin{pmatrix}0\\v\end{pmatrix} \Longleftrightarrow \Theta(x) \to \langle\Theta(x)\rangle = \Theta_\infty := \frac{1}{\sqrt{2}}\begin{pmatrix}v & 0\\0 & v\end{pmatrix} = \frac{v}{\sqrt{2}}\mathbf{1}. \tag{12}$$

By this choice of the vacuum expectation value of the scalar field, the original gauge symmetry $SU(2)$ is completely broken with no residual gauge symmetry, which is called the **complete SSB**: $G = SU(2) \to H = \{1\}$. Then all the components of the gauge boson become massive.

In terms of the matrix-valued scalar field, the kinetic term in the unitary gauge reduces to the mass term

$$\begin{aligned}&\frac{1}{2}\mathrm{tr}((D_\mu[\mathscr{A}]\Theta(x))^\dagger D^\mu[\mathscr{A}]\Theta(x))\\ \to&\frac{1}{2}\mathrm{tr}(ig\Theta_\infty^\dagger\mathscr{A}_\mu(x)[-ig\mathscr{A}^\mu(x)\Theta_\infty]) = \frac{1}{2}g^2\frac{v^2}{2}\mathrm{tr}(\mathscr{A}_\mu(x)\mathscr{A}^\mu(x)).\end{aligned} \tag{13}$$

In the unitary gauge, indeed, $\mathscr{W}_\mu$ reduces to the original gauge field,

$$\mathscr{W}_\mu(x) \to ig^{-1}(D_\mu[\mathscr{A}(x)]\hat{\Theta}_\infty)\hat{\Theta}_\infty^\dagger = \mathscr{A}_\mu(x), \tag{14}$$

⊙ Field decomposition for $SU(2)$ gauge-fundamental scalar model

The original gauge field $\mathscr{A}_\mu$ is separated into the massive vector field $\mathscr{W}_\mu$ and the residual one $\mathscr{V}_\mu$:

$$\mathscr{A}_\mu(x) = \mathscr{V}_\mu(x) + \mathscr{W}_\mu(x). \tag{15}$$

Under the gauge transformation $U(x) \in SU(2)$, the original fields $\mathscr{A}_\mu$ and Θ transform as

$$\mathscr{A}_\mu(x) \to U(x)\mathscr{A}_\mu(x)U(x)^\dagger + ig^{-1}U(x)\partial_\mu U(x)^\dagger, \ \Theta(x) \to U(x)\Theta(x). \tag{16}$$

We have constructed $\mathscr{W}_\mu$ so that it transform according to the adjoint representation,

$$\mathscr{W}_\mu(x) \to U(x)\mathscr{W}_\mu(x)U(x)^\dagger. \tag{17}$$

Therefore, $\mathscr{V}_\mu$ transform just like the original gauge field,

$$\mathscr{V}_\mu(x) \to U(x)\mathscr{V}_\mu(x)U(x)^\dagger + ig^{-1}U(x)\partial_\mu U(x)^\dagger. \tag{18}$$

To obtain the explicit expression for $\mathscr{V}_\mu$, we observe that $\mathscr{W}_\mu = 0$ yields the following condition for $\mathscr{V}_\mu$

$$D_\mu[\mathscr{V}]\hat{\Theta}(x) = \mathbf{0} \Leftrightarrow \partial_\mu\hat{\Theta}(x) - ig\mathscr{V}_\mu(x)\hat{\Theta}(x) = \mathbf{0}. \tag{19}$$

The **residual field** $\mathscr{V}_\mu$ is obtained by solving this equation using $\hat{\Theta}\hat{\Theta}^\dagger = \mathbf{1}$ as

$$\begin{aligned}\mathscr{V}_\mu(x) =& -ig^{-1}\partial_\mu\hat{\Theta}(x)\hat{\Theta}(x)^\dagger = ig^{-1}\hat{\Theta}(x)\partial_\mu\hat{\Theta}(x)^\dagger \\ =&\frac{1}{2}ig^{-1}[-\partial_\mu\hat{\Theta}(x)\hat{\Theta}(x)^\dagger + \hat{\Theta}(x)\partial_\mu\hat{\Theta}^\dagger],\ \hat{\Theta}(x) \in SU(2).\end{aligned} \tag{20}$$

⊙ Field equations to the reduction condition for $SU(2)$ gauge-fundamental scalar model

We discuss the relationship between the reduction condition and the field equation of the gauge-scalar model described by

$$\tilde{\mathscr{L}}_{\mathrm{RF}} = -\frac{1}{2}\mathrm{tr}(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}) + \frac{1}{2}\mathrm{tr}((D_\mu[\mathscr{A}]\Theta)^\dagger D^\mu[\mathscr{A}]\Theta) + u\,\mathrm{tr}\left(\Theta^\dagger\Theta - \frac{1}{2}v^2\mathbf{1}\right)/\mathrm{tr}(\mathbf{1}) \tag{21}$$

For the $SU(2)$ gauge-scalar model with a radially fixed fundamental scalar field, the field equations are obtained by variation as

$$0 = \frac{\delta \tilde{S}_{\rm RF}}{\delta u(x)} = {\rm tr}\left(\Theta(x)^\dagger \Theta(x) - \frac{1}{2} v^2 \mathbf{1} \right) / {\rm tr}(\mathbf{1}), \tag{22}$$

$$0 = \frac{\delta \tilde{S}_{\rm RF}}{\delta \Theta^\dagger(x)} = - D_\mu[\mathscr{A}] D^\mu[\mathscr{A}] \Theta(x) + \Theta(x) u(x), \tag{23}$$

$$0 = \frac{\delta \tilde{S}_{\rm RF}}{\delta \Theta(x)} = - (D^\mu[\mathscr{A}]\Theta(x))^\dagger \overleftarrow{D_\mu[\mathscr{A}]}^\dagger + u(x) \Theta(x)^\dagger, \tag{24}$$

$$0 = \frac{\delta \tilde{S}_{\rm RF}}{\delta \mathscr{A}^\mu(x)} = \mathscr{D}^\nu[\mathscr{A}] \mathscr{F}_{\nu\mu}[\mathscr{A}](x) + \frac{1}{2} ig [(D_\mu[\mathscr{A}]\Theta(x))\Theta(x)^\dagger - \Theta(x)(D_\mu[\mathscr{A}]\Theta(x))^\dagger], \tag{25}$$

$$= \mathscr{D}^\nu[\mathscr{A}] \mathscr{F}_{\nu\mu}[\mathscr{A}](x) + M_W^2 \mathscr{W}_\mu(x) = 0. \tag{26}$$

Due to (22), the scalar field Θ is normalized $\hat{\Theta}$. Multiplying (23) by $\hat{\Theta}^\dagger$ and (24) by $\hat{\Theta}$ yields

$$0 = \{-D_\mu[\mathscr{A}](D^\mu[\mathscr{A}]\hat{\Theta}) + \hat{\Theta} u\} \hat{\Theta}^\dagger - \hat{\Theta}\{-(D^\mu[\mathscr{A}]\hat{\Theta})^\dagger \overleftarrow{D_\mu[\mathscr{A}]}^\dagger + u \hat{\Theta}^\dagger\} = 2ig \mathscr{D}_\mu[\mathscr{A}] \mathscr{W}^\mu. \tag{27}$$

Applying the covariant derivative to (25) or (26) yields

$$0 = \mathscr{D}_\mu[\mathscr{A}](\mathscr{D}_\nu[\mathscr{A}] \mathscr{F}^{\nu\mu}[\mathscr{A}] + M^2 \mathscr{W}^\mu) = M^2 \mathscr{D}_\mu[\mathscr{A}] \mathscr{W}^\mu. \tag{28}$$

If the fields $\mathscr{A}$ and Θ are a set of solutions of the field equations for the gauge-scalar model with a radially fixed fundamental scalar field, they automatically satisfy the reduction condition for the pure Yang-Mills theory.

Imposing the reduction condition $\chi^A(x) = 0$ ($A = 1, 2, 3$) eliminates three extra degrees of freedom introduced through the radially fixed scalar field $\hat{\Phi} \in SU(2)$ ($\mathrm{dim} SU(2) = 3$), which is necessary to convert the $SU(2)$ gauge-scalar theory to the pure $SU(2)$ Yang-Mills theory.

$$\chi^A(x) := (\mathscr{D}^\mu[\mathscr{A}]\mathscr{W}_\mu)^A(x) = 0 \Longleftrightarrow \chi^A(x) := (\mathscr{D}^\mu[\mathscr{V}]\mathscr{W}_\mu)^A(x) = 0 \ (A = 1, 2, 3). \tag{29}$$

The reduction condition $\chi(x) = \chi^A(x) T_A$ is rewritten in terms of the scalar field $\hat{\Theta}$ and the original gauge field $\mathscr{A}_\mu$ as

$$\chi(x) := \mathscr{D}_\mu[\mathscr{A}][(D^\mu[\mathscr{A}]\hat{\Theta}(x))\hat{\Theta}(x)^\dagger] = \mathbf{0}. \tag{30}$$

The reduction condition is gauge covariant equation, $\chi(x) \to U(x)\chi(x)U(x)^\dagger$. This implies that the reduction condition retains the same form under the gauge transformation, namely, it is form-invariant.

In order to obtain the expressions in terms of the original scalar field Φ, it is sufficient to impose the condition:

$$D_\mu[\mathscr{V}]\hat{\Phi}(x) = 0. \tag{31}$$

In fact, (19) follows from $D_\mu[\mathscr{V}]\hat{\Phi}(x) = 0 \Longrightarrow D_\mu[\mathscr{V}]\hat{\Theta}(x) = 0$. For $G = SU(2)$, thus, the gauge field is decomposed as $\mathscr{A}_\mu^A(x) = \mathscr{W}_\mu^A(x) + \mathscr{V}_\mu^A(x)$

$$\begin{aligned} \mathscr{V}_\mu^A(x) &= -ig^{-1}[\hat{\Phi}^\dagger(x)\sigma_A\partial_\mu\hat{\Phi}(x) - \partial_\mu\hat{\Phi}^\dagger(x)\sigma_A\hat{\Phi}(x)], \\ \mathscr{W}_\mu^A(x) &= ig^{-1}[\hat{\Phi}^\dagger(x)\sigma_A D_\mu[\mathscr{A}]\hat{\Phi}(x) - (D_\mu[\mathscr{A}]\hat{\Phi}(x))^\dagger\sigma_A\hat{\Phi}(x)]. \end{aligned} \tag{32}$$

$SU(2)$ gauge-scalar model ("complementary") to the massive $SU(2)$ Yang-Mills theory

$G = SU(2)$	Fundamental Φ, $\Theta = (\tilde{\Phi}, \Phi)$	Adjoint $\boldsymbol{\phi}$
SSB pattern $G \to H$	complete: $SU(2) \to \{1\}$	partial: $SU(2) \to U(1)$
field decomposition	$\mathscr{A}_\mu = \mathscr{W}_\mu + \mathscr{V}_\mu$	$\mathscr{A}_\mu = \mathscr{W}_\mu + \mathscr{V}_\mu$
massive mode $\mathscr{W}_\mu$ $\mathscr{W}_\mu = \mathscr{W}_\mu^A T_A \in su(2)$	$\mathscr{W}_\mu = -ig^{-1}\hat{\Theta}(D_\mu[\mathscr{A}]\hat{\Theta})^\dagger$ $\mathscr{W}_\mu^A = ig^{-1}[\hat{\Phi}^\dagger \sigma_A D_\mu[\mathscr{A}]\hat{\Phi}$ $-(D_\mu[\mathscr{A}]\hat{\Phi})^\dagger \sigma_A \hat{\Phi}]$	$\mathscr{W}_\mu = -ig^{-1}[\hat{\boldsymbol{\phi}}, \mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}]$ $\mathscr{W}_\mu^A = g^{-1}\epsilon^{ABC}\hat{\phi}^B(\mathscr{D}_\mu[\mathscr{A}]\hat{\phi})^C$
gauge transformation	$\mathscr{W}_\mu \to U\mathscr{W}_\mu U^\dagger$	same as on the left
residual mode $\mathscr{V}_\mu$ $\mathscr{V}_\mu = \mathscr{V}_\mu^A T_A \in su(2)$	$\mathscr{V}_\mu = ig^{-1}\hat{\Theta}\partial_\mu\hat{\Theta}^\dagger$ $\mathscr{V}_\mu^A = -ig^{-1}[\hat{\Phi}^\dagger\sigma_A\partial_\mu\hat{\Phi}$ $-\partial_\mu\hat{\Phi}^\dagger\sigma_A\hat{\Phi}]$	$\mathscr{V}_\mu = c_\mu\hat{\boldsymbol{\phi}} + ig^{-1}[\hat{\boldsymbol{\phi}}, \partial_\mu\hat{\boldsymbol{\phi}}]$ $c_\mu = \mathscr{A}_\mu \cdot \hat{\boldsymbol{\phi}}$ $\mathscr{V}_\mu^A = c_\mu\hat{\phi}^A - g^{-1}\epsilon^{ABC}\hat{\phi}^B\partial_\mu\hat{\phi}^C$
gauge transformation	$\mathscr{V}_\mu \to U\mathscr{V}_\mu U^\dagger + ig^{-1}U\partial_\mu U^\dagger$	same as on the left
Defining equation	$D_\mu[\mathscr{V}]\hat{\Phi} = 0$, $D_\mu[\mathscr{V}]\hat{\Theta} = 0$ $(\mathscr{W} \cdot \hat{\Phi} \neq 0)$	$\mathscr{D}_\mu[\mathscr{V}]\hat{\boldsymbol{\phi}} = 0$ $\mathscr{W}_\mu \cdot \hat{\boldsymbol{\phi}} = 0$
field equation 1	$\mathrm{tr}\left(\Theta^\dagger\Theta - \frac{1}{2}v^2\mathbf{1}\right)/\mathrm{tr}(\mathbf{1}) = 0$	$\boldsymbol{\phi}\cdot\boldsymbol{\phi} - v^2 = 0$
field equation 2	$-D_\mu[\mathscr{A}]D^\mu[\mathscr{A}]\Theta + \Theta u = 0$	$-\mathscr{D}^\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi} + 2u\boldsymbol{\phi} = 0$
field equation 3	$\mathscr{D}^\nu[\mathscr{A}]\mathscr{F}_{\nu\mu}[\mathscr{A}] + M_W^2\mathscr{W}_\mu = 0$	$\mathscr{D}^\nu[\mathscr{A}]\mathscr{F}_{\nu\mu}[\mathscr{A}] + M_W^2\mathscr{W}_\mu = 0$
reduction condition χ	$\mathscr{D}^\mu[\mathscr{V}]\mathscr{W}_\mu = 0$	$\mathscr{D}^\mu[\mathscr{V}]\mathscr{W}_\mu = 0$
color field $\boldsymbol{n} = n^A\sigma_A$	$\boldsymbol{n} = \hat{\Theta}\sigma_3\hat{\Theta}^\dagger$, $n^A = -\hat{\Phi}^\dagger\sigma_A\hat{\Phi}$	$\boldsymbol{n} = \hat{\boldsymbol{\phi}}$, $n^A = \hat{\phi}^A$

§ $SU(2)$ Yang-Mills theory

Kondo, Kato, Shibata and Shinohara, Phys. Report 579, 1–226 (2015), arXiv:1409.1599 [hep-th]

⊙ Static quark potential and string tension

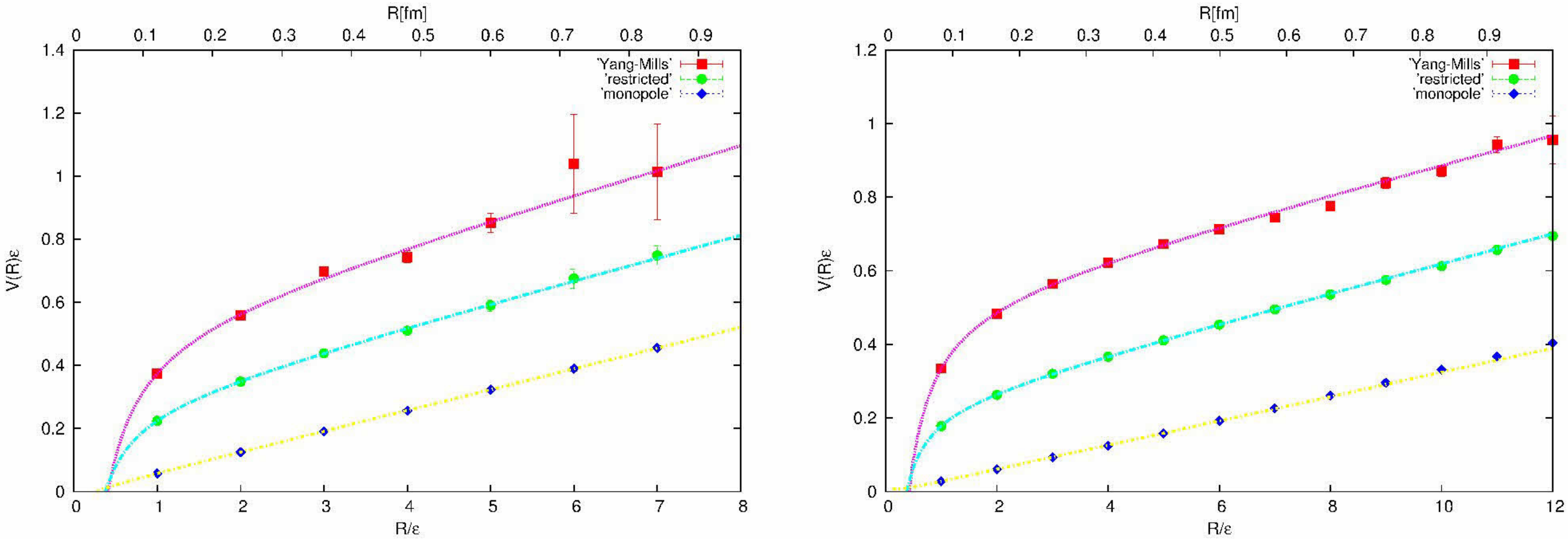


Figure 1: [Kato, Kondo and Shibata, 2014] The static quark-antiquark potentials as functions of the quark-antiquark distance R: (from above to below) (i) full potential $V_{\rm full}(R)$, (ii) restricted part $V_{\rm rest}(R)$ and (iii) magnetic–monopole part $V_{\rm mono}(R)$. (Left) on 16^4 lattice at $\beta = 2.4$, (Right) on 24^4 lattice at $\beta = 2.5$ where the Wilson loop with $T = 12$ was used for obtaining $V_{\rm full}(R)$ and $V_{\rm rest}(R)$, and $T = 8$ for $V_{\rm mono}(R)$.

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⊙ Chromoelectric flux

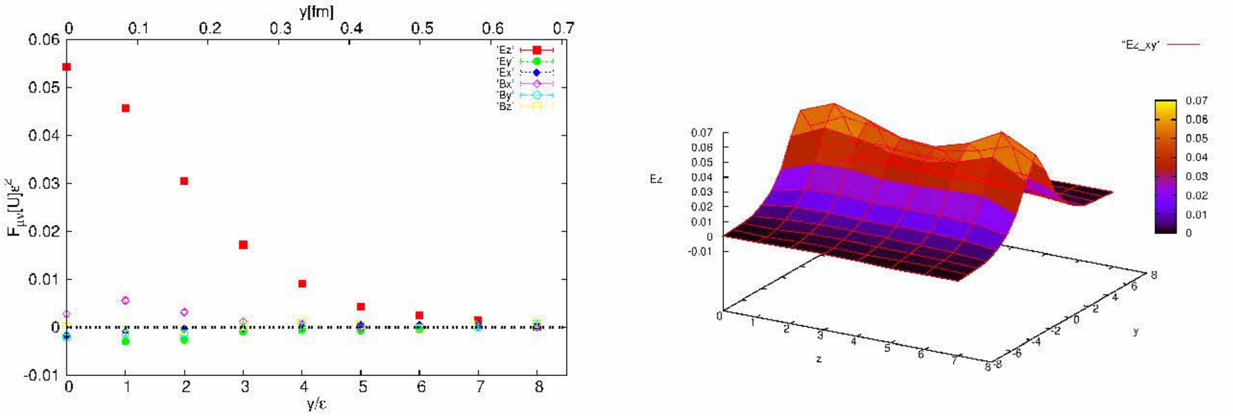

Figure 2: [Kato, Kondo and Shibata, 2014] The chromoelectric and chromomagnetic fields obtained from the full field U on 24^4 lattice at $\beta = 2.5$. (Left panel) y dependence of the chromoelectric field $E_i(y) = F_{4i}(y)$ $(i = x, y, z)$ at fixed $z = 4$ (mid-point of $q\bar{q}$). (Right panel) The distribution of $E_z(y, z)$ obtained for the 8×8 Wilson loop with $\bar{q}$ at $(y, z) = (0, 0)$ and q at $(y, z) = (0, 8)$.

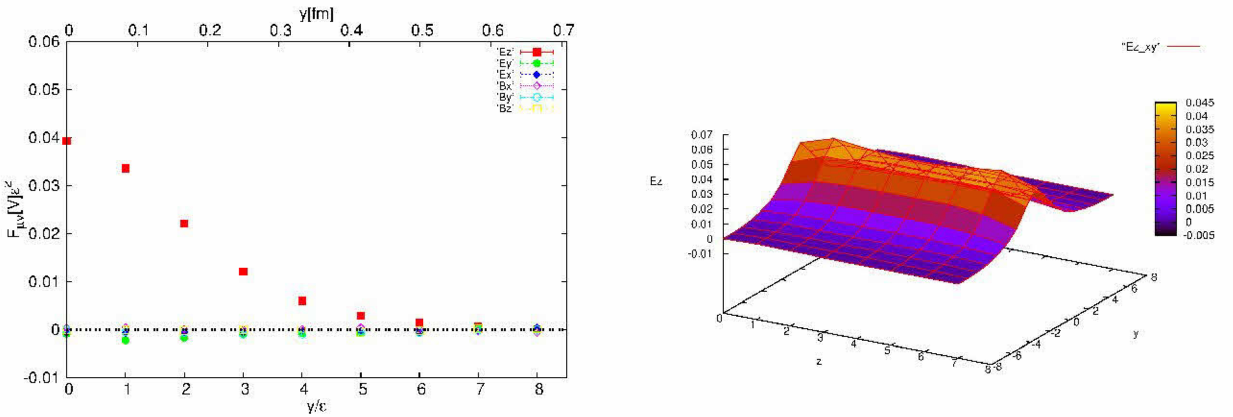

Figure 3: [Kato, Kondo and Shibata, 2014] The chromoelectric field obtained from the restricted field V on 24^4 lattice at $\beta = 2.5$.

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⊙ magnetic monopole

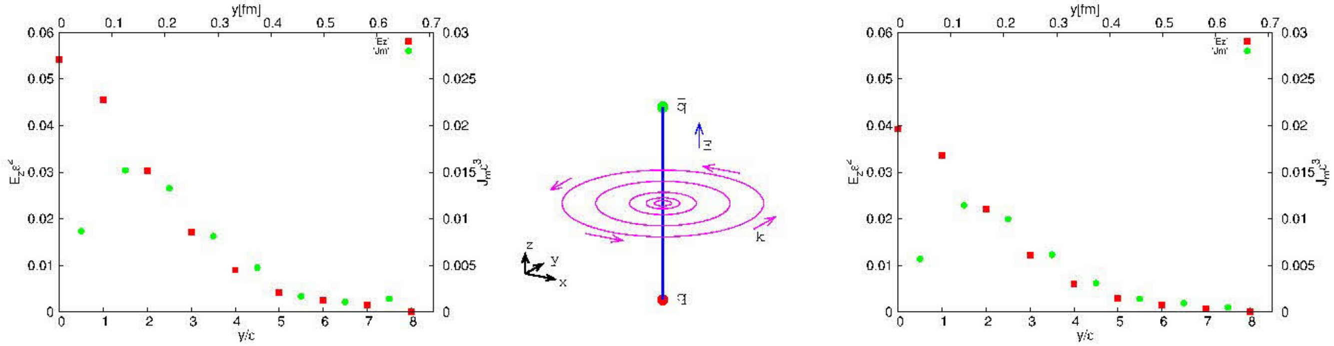

Figure 4: [Kato, Kondo and Shibata, 2014] The magnetic-monopole current $\mathbf{k}$ induced around the chromoelectric flux along the z axis connecting a pair of quark and antiquark. (Center panel) The positional relationship between the chromoelectric field E_z and the magnetic current $\mathbf{k}$. (Left panel) The magnitude of the chromoelectric field E_z and the magnetic current $J_m = |\mathbf{k}|$ as functions of the distance y from the z axis calculated from the original full variables. (Right panel) The counterparts of the left graph calculated from the restricted variables.

Notice that $H_{\mu\nu}(x)$ is *locally* closed $(dH = 0)$ and hence it can be *locally* exact $(H = dh)$ due to the Poincaré lemma. Then $H_{\mu\nu}(x)$ has the Abelian potential $h_\mu(x)$:

$$H_{\mu\nu}(x) =\partial_\mu h_\nu(x) - \partial_\nu h_\mu(x). \tag{1}$$

Therefore, the $SU(2)$ gauge-invariant Abelian-like field strength $f_{\mu\nu}$ is rewritten as

$$f_{\mu\nu}(x) =\partial_\mu G_\nu(x) - \partial_\nu G_\mu(x),\ G_\mu(x) := c_\mu(x) + h_\mu(x). \tag{2}$$

We call c_μ the **electric potential** and h_μ the **magnetic potential**. Indeed, h_μ agrees with the Dirac magnetic potential, see section 6.10 of the review[Physics Reports]. The magnetic current $k^\mu(x)$ is not identically zero, since the Bianchi identity valid for c_μ is violated by h_μ.

⊙ In the Yang-Mills theory, indeed, the mass M_W can be generated in a dynamical way, e.g., by a gauge-invariant vacuum condensation $\langle \mathscr{W}^\mu \cdot \mathscr{W}_\mu \rangle$ so that $M_W^2 \simeq \langle \mathscr{W}^\mu \cdot \mathscr{W}_\mu \rangle$ due to the quartic self-interactions $-\frac{1}{4}(ig[\mathscr{W}_\mu(x), \mathscr{W}_\nu(x)])^2$ among $\mathscr{W}_\mu(x)$ field, in sharp contrast to the ordinary gauge-scalar model. The analytical calculation for such a condensate was done in [Dudal et al, 2004].

$$M_W \simeq 2.25\Lambda_{\rm phys} \simeq 0.524\text{GeV} \tag{3}$$

The numerical simulations on the lattice in [Shibata et al, 2007] gives

$$M_W \simeq 2.69\sqrt{\sigma_{\rm phys}} \simeq 1.19\text{GeV}, \tag{4}$$

where $\sigma_{\rm phys}$ is the string tension of the linear potential in the quark-antiquark potential.

⊙ The mass M_W is used to show the existence of confinement-deconfinement phase transition at a finite critical temperature T_c, separating confinement phase with vanishing Polyakov loop average at low temperature and deconfinement phase with non-vanishing Polyakov loop average at high temperature [**?**]. The critical temperature T_c is obtained from the calculated ratio T_c/M_W for a given value of M_W, which provides a reasonable estimate.

⊙ Correlation function

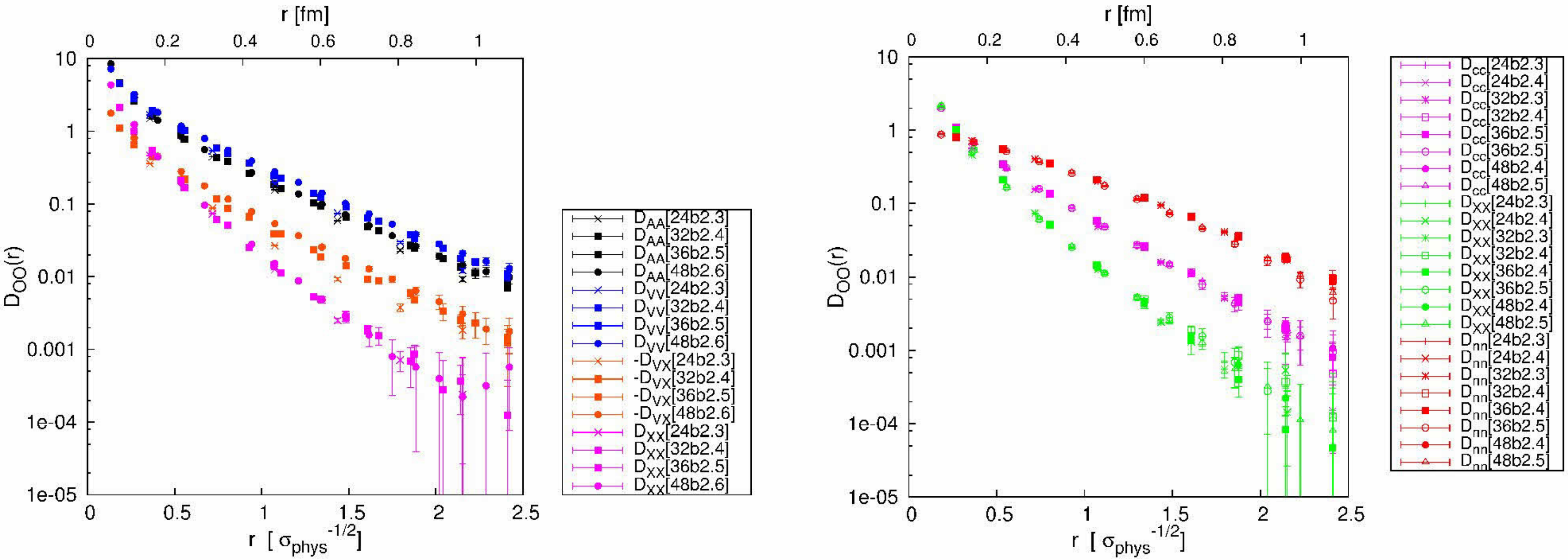

Figure 5: [Shibata, Kato, Kondo, Murakami, Shinohara, and Ito, 2007] Logarithmic plots of scalar-type two-point correlation functions $D_{\mathcal{O}\mathcal{O}'}(r) := \langle \mathcal{O}(x)\mathcal{O}'(y) \rangle$ as a function of the Euclidean distance $r := \sqrt{(x-y)^2}$ for $\mathcal{O}$ and $\mathcal{O}'$. (Left panel) $\mathcal{O}(x)\mathcal{O}'(y) = \mathbb{V}_\mu^A(x)\mathbb{V}_\mu^A(y)$, $\mathbb{A}_\mu^A(x)\mathbb{A}_\mu^A(y)$, $-\mathbb{V}_\mu^A(x)\mathbb{X}_\mu^A(y)$, $\mathbb{X}_\mu^A(x)\mathbb{X}_\mu^A(y)$, (Right panel) $\mathcal{O}(x)\mathcal{O}'(y) = \mathbf{n}^A(x)\mathbf{n}^A(y)$, $c_\mu(x)c_\mu(y)$, $\mathbb{X}_\mu^A(x)\mathbb{X}_\mu^A(y)$, from above to below using data on the 24^4 lattice ($\beta = 2.3, 2.4$), 32^4 lattice ($\beta = 2.3, 2.4$), 36^4 lattice ($\beta = 2.4, 2.5$), and 48^4 lattice ($\beta = 2.4, 2.5, 2.6$). Here plots are given in the physical unit [fm] or in unit of square root of the string tension $\sqrt{\sigma_{\rm phys}}$.

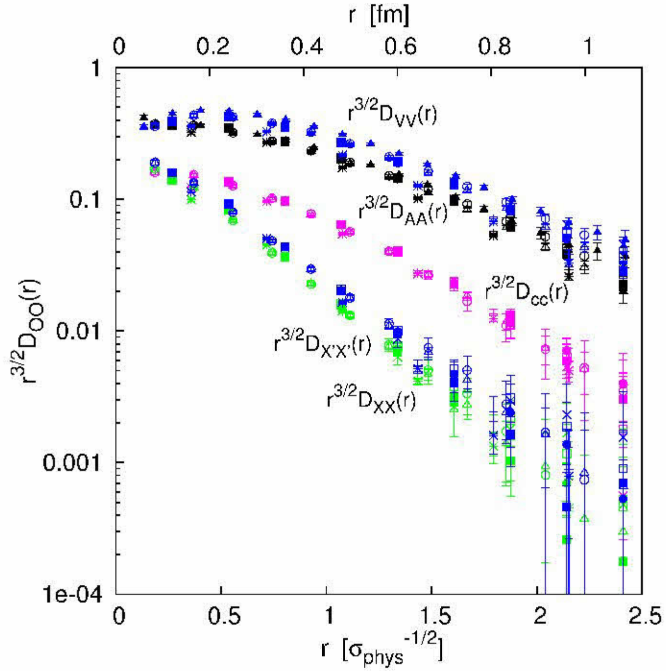

Figure 6: [Shibata, Kato, Kondo, Murakami, Shinohara, and Ito, 2007] Logarithmic plots of the rescaled correlation function $r^{3/2}D_{OO}(r)$ as a function of r for $O = \mathbb{V}_\mu^A, \mathbb{A}_\mu^A, c_\mu, \mathbb{X}_\mu^A$ (and $\mathbb{X}'^A_\mu$) from above to below, using the same colors and symbols as those in Fig. 5. Here two sets of data for the correlation function $D_{XX}(x-y)$ are plotted according to the two definitions of the $\mathbb{X}_\mu^A$ field on a lattice.

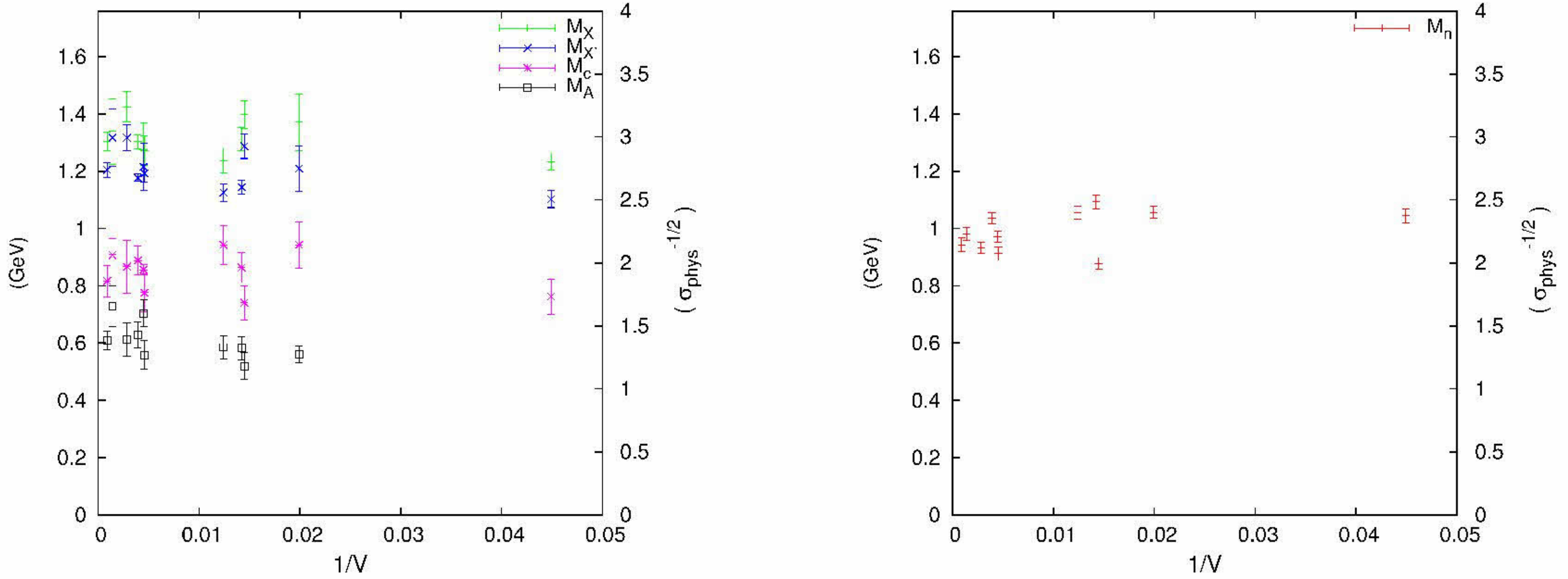

Figure 7: [Shibata, Kato, Kondo, Murakami, Shinohara, and Ito, 2007] Gluon "mass" and decay rates (in units of GeV and $\sqrt{\sigma_{\rm phys}}$) as the function of the inverse lattice volume $1/V$ in the physical unit. (Left panel) for $\mathcal{O} = \mathbb{X}^A_\mu, (\mathbb{X}'^A_\mu), c_\mu, \mathbb{A}^A_\mu$ from above to below extracted according to the fitting: $\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \sim r^{-3/2}\exp(-M_{\mathcal{O}} r)$, (Right panel) for $\mathbf{n}^A(x)$ extracted according to the fitting: $\left\langle \mathbf{n}^A(x)\mathbf{n}^A(y)\right\rangle \sim \exp(-M_n r)$.

$$M_X \simeq 2.98\sqrt{\sigma_{\rm phys}} \simeq 1.31{\rm GeV},\ M_{X'} \simeq 2.69\sqrt{\sigma_{\rm phys}} \simeq 1.19{\rm GeV}. \quad (5)$$

$$M_n \simeq 2.24\sqrt{\sigma_{\rm phys}} \simeq 0.986{\rm GeV}, M_c \simeq 1.94\sqrt{\sigma_{\rm phys}} \simeq 0.856{\rm GeV},$$
$$M_A \simeq 1.35\sqrt{\sigma_{\rm phys}} \simeq 0.596{\rm GeV}. \quad (6)$$

⊙ Fundamental scalar case: A perturbative approach

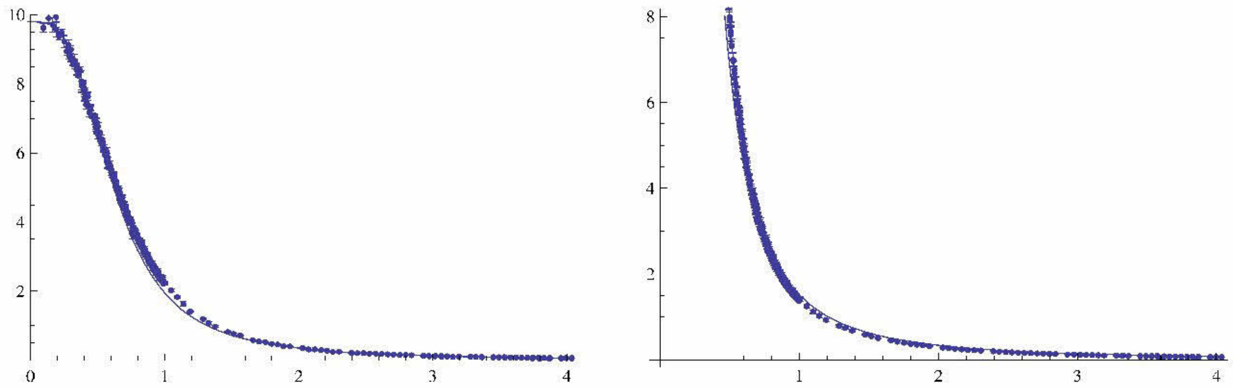

Figure 8: (Left) gluon propagator $\tilde{D}_T(k)$, (Right) ghost propagator $\Delta(k)$.
Fit of the analytical 1-loop calculation of the massive Yang-Mills theory to the numerical simulations on the lattice in the covariant Landau gauge for $SU(3)$ Yang-Mills theory [A.G. Duarte, O. Oliveira, and P.J. Silva, Phys. Rev. D**94** (2016) 014502.]

$$\tilde{D}_T\left(k=\mu\right)\Big|_{\alpha\to 0} = \frac{1}{\mu^2},\ \Delta\left(k=\mu\right)\big|_{\alpha\to 0} = \frac{1}{\mu^2}\ ,\quad \text{at}\ \ \mu = 4\ \text{GeV} \tag{7}$$

$$M = 0.261\ \text{GeV},\ g = 2.36 \tag{8}$$

This should be compared with [M. Tissier and N. Wschebor, Phys.Rev. D**82**, 101701 (2010).]

$$M = 0.54\ \text{GeV},\ g = 4.9 \tag{9}$$

These propagators leads to positivity violation which is consistent with gluon confinement.

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§ Conclusion and discussion

Conclusion:

⊙ We propose a gauge-independent description of the BEH or Higgs mechanism by which massless gauge bosons acquire their mass.

The conventional description of the BEH mechanism states that massless gauge bosons become massive vector bosons by absorbing the Nambu-Goldstone particles associated with the spontaneous breaking of the gauge symmetry.
This description requires a non-vanishing vacuum expectation value of the scalar field $\langle 0|\phi(x)|0\rangle = v$, which is clearly gauge dependent and impossible to be realized without fixing the gauge.

In the new description, instead, the scalar field is supposed to obey a gauge-invariant condition which forces the radial length of the scalar field to have a certain fixed value $||\phi(x)|| = v$ without breaking the gauge symmetry.

This result is regarded as an explicit realization in the framework of the continuum field theory of the proposition derived by Fradkin and Shenker, and Osterwalder-Seiler in the gauge-invariant framework of lattice gauge theory for the gauge-scalar model with a radially fixed fundamental scalar field.

⊙ We can include a gauge-invariant mass term in the pure Yang-Mills theory.

This extension enables one to study quak confinement and mass gap in the pure Yang-Mills theory as the implications of the BEH mechanism in the "complementary" gauge-scalar model.

⊙ This allows one to decompose the original gauge field $\mathscr{A}$ into the massive vector mode $\mathscr{W}$ and the residual gauge mode $\mathscr{V}$, $\mathscr{A} = \mathscr{W} + \mathscr{V}$.

$\mathscr{W}$=massive vector mode, $\mathscr{V}$=residual gauge mode

The massive vector modes $\mathscr{W}$ mediate only the short-range force between quark sources. Therefore, the long-range force giving a linear piece of the static quark potential responsible for quark confinement must be mediated by the residual gauge mode $\mathscr{V}$.

In the case of the adjoint scalar field, the residual gauge mode include massless gauge boson which is able to mediate the long-range force.

In the case of the fundamental scalar field, there are no massless gauge bosons in the residual mode $\mathscr{V}$ once the BEH mechanism occurs. In fact, the residual gauge mode $\mathscr{V}$ has exactly the same form as the pure gauge $\mathscr{V} = ig^{-1}UdU^{-1}$ with the group element U which is written in terms of the scalar field Φ. Therefore, solitons and defects converging to the pure gauge in the long distance can be good candidates for the dominant components in the residual mode.

In both cases, we can extract the magnetic monopoles and/or a pair of magnetic monopole and antimonopole using the color direction field $\boldsymbol{n}(x)$ defined in terms of the scalar field $\phi(x)$ in the gauge invariant way which are promising candidate of field configurations responsible for quark confinement.

Discussion:

For $SU(2)$, the two cases are different for quark confinement.

In the adjoint scalar field case $(SU(2) \to U(1))$,
the external quark source in the fundamental representation cannot be screened by the adjoint scalar and the chromoelectric flux connecting a pair of quark and antiquark is formed for $r > r_0$ where $r < r_0$ the Coulomb-like perturbative part becomes dominant.

In the fundamental scalar field case $(SU(2) \to \{0\})$,
the external quark source in the fundamental representation can be screened by the fundamental scalar and the chromoelectric flux connecting a pair of quark and antiquark will break at certain distance $r = r_c \simeq 1/(2m)$ with the mass m of the scalar particle. The static quark potential exhibits the linear potential in the intermediate region $r_0 < r < r_c$, and flattens in the long-distance region $r > r_c$. This situation is similar to the realistic QCD in which light dynamical quarks are included into the theory.

Gluon confinement will be realized by ...

Thank you very much for your attention.