Recent Developments in Lattice Studies of IR Propagators

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Application to the study of Landau-gauge gluon and ghost propagators

Origin of Confinement in QCD

$$\mathcal{L} \; = \; -\frac{1}{4}\, F^{a}_{\mu\nu}\, F_{a}^{\mu\nu} \; + \sum_{f=1}^{6} \bar{\psi}_{f,i} \left(i\,\gamma^{\mu} D^{ij}_{\mu} - m_f\, \delta_{ij} \right) \psi_{f,j}$$

$a = 1, \ldots, 8$; $i = 1, \ldots, 3$; T^{a}_{ij} = $SU(3)$ generators

$$F^{a}_{\mu\nu} \;\equiv\; \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} + g_0\, f_{abc}\, A^{b}_{\mu}\, A^{c}_{\nu}$$

$$D_{\mu} \;\equiv\; \partial_{\mu} - i\, g_0\, A^{a}_{\mu}\, T_{a}$$

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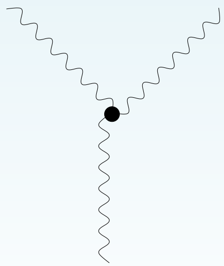
Note: contribution $F^a_{\mu\nu} \sim g_0 \, f^{abc} \, A^b_\mu \, A^c_\nu$ means that in addition to quadratic terms (propagators) and the (quark-quark-gluon vertex)

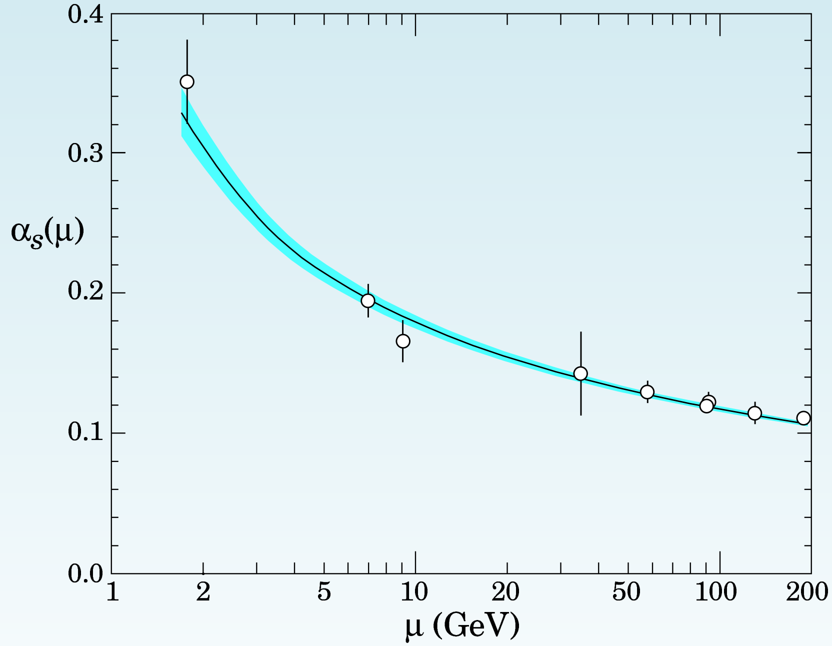
Lagrangian contains terms with 3 and 4 gauge fields, e.g.

$$\mathcal{L}_{AAA} \;=\; g_0 \, f^{abc} \, A^\mu_a \, A^\nu_b \, \partial_\mu A^c_\nu \;\Rightarrow\;$$ three-gluon vertex



How do we perform calculations?

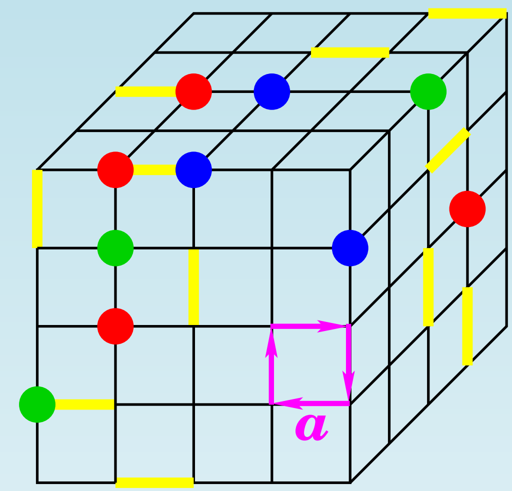
The strength of the interaction α_s increases for larger r (smaller p) and vice-versa (asymptotic freedom).
Perturbation theory breaks down in the limit of small energies.



Lattice QCD Ingredients

Three ingredients

1. Quantization by path integrals $\Rightarrow$ sum over configurations with "weights" $e^{i\,S/\hbar}$
2. Euclidean formulation (analytic continuation to imaginary time) $\Rightarrow$ weight becomes $e^{-S/\hbar}$
3. Discrete space-time $\Rightarrow$ UV cut at momenta $p \lesssim 1/a$ $\Rightarrow$ regularization



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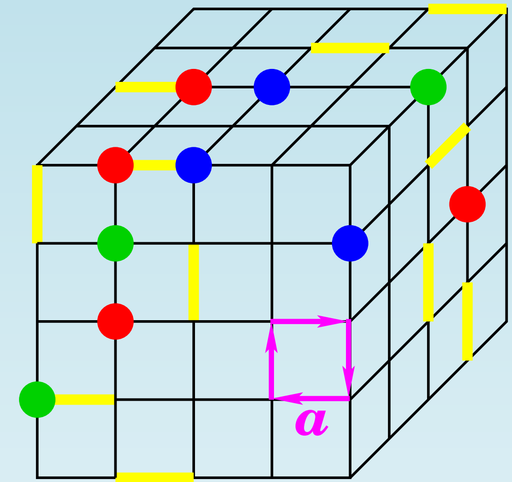

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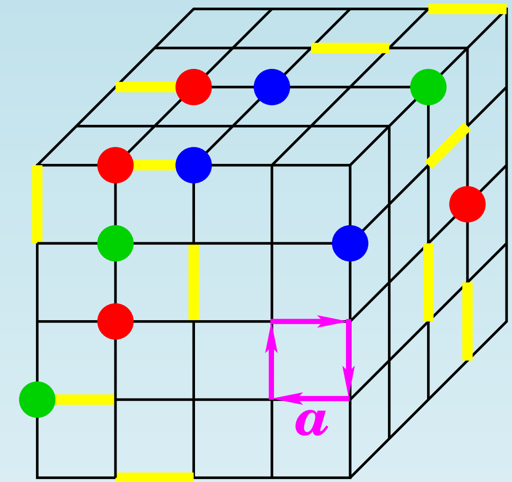

Also: finite-size lattices $\Rightarrow$ IR cut for small momenta $p \approx 1/L$

The Wilson action

- is written for the gauge links $U_{x,\mu} \;\equiv\; e^{i g_0 a A_\mu^b(x) T_b}$
- reduces to the usual action for $a \to 0$
- is gauge-invariant

The Lattice Action

The Wilson action (1974)

$$S = -\frac{\beta}{3} \sum_{\Box} \mathrm{ReTr}\, U_{\Box}\,, \quad U_{x,\mu} \equiv e^{ig_0 a A^b_\mu(x) T_b}\,, \quad \beta = 6/{g_0}^2$$

- written in terms of oriented plaquettes formed by the link variables $U_{x,\mu}$, which are group elements
- under gauge transformations: $U_{x,\mu} \to g(x)\, U_{x,\mu}\, g^\dagger(x+\mu)$, where $g \in SU(3)$ $\Rightarrow$ closed loops are gauge-invariant quantities
- integration volume is finite: no need for gauge-fixing

At small β (i.e. strong coupling) we can perform an expansion analogous to the high-temperature expansion in statistical mechanics. At lowest order, the only surviving terms are represented by diagrams with “double” or “partner” links, i.e. the same link should appear in both orientations, since $\int dU\, U_{x,\mu} = 0$

Confinement and Area Law

Considering a rectangular loop with sides R and T (the Wilson loop) as our observable, the leading contribution to the observable's expectation value is obtained by "tiling" its inside with plaquettes, yielding the area law

$$\langle W(R,T)\rangle \sim \beta^{RT}$$

But this observable is related to the interquark potential for a static quark-antiquark pair

$$\langle W(R,T)\rangle = e^{-V(R)T}$$

We thus have $V(R) \sim \sigma R$, demonstrating confinement at strong coupling (small β)!

Problem: the physical limit is at large β...

(Numerical) Lattice QCD

Classical Statistical-Mechanics model with the partition function

$$Z = \int \mathcal{D}U \, e^{-S_g} \int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \; e^{-\int d^4x \, \overline{\psi}(x) \, K \, \psi(x)} = \int \mathcal{D}U \, e^{-S_g} \, \det K(U)$$

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Evaluate expectation values

$$\langle \mathcal{O} \rangle = \int \mathcal{D}U \, \mathcal{O}(U) \, P(U) = \frac{1}{N} \sum_i \mathcal{O}(U_i)$$

with the weight

$$P(U) = \frac{e^{-S_g(U)} \, \det K(U)}{Z}$$

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$\Rightarrow$ Monte Carlo simulations: sample representative gauge configurations, then compute O and take average

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- Gribov-Zwanziger confinement scenario based on suppressed gluon propagator and enhanced ghost propagator in the infrared

Quantization and Gribov Copies

The invariance of the Lagrangian under local gauge transformations implies that, given a configuration $\{A(x), \psi_f(x)\}$, there are infinitely many gauge-equivalent configurations $\{A^g(x), \psi_f^g(x)\}$ (gauge orbits). In the path integral approach we integrate over all possible configurations

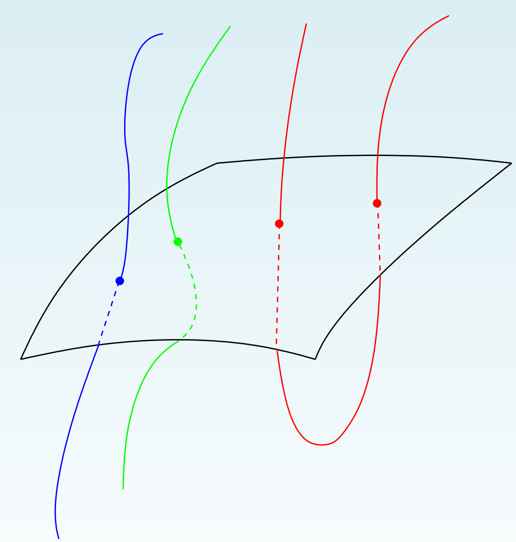
$$Z = \int DA \exp\left[- \int d^4x \, \mathcal{L}(x) \right] .$$

There is an infinite factor coming from gauge invariance: $\int DA = \int D\overline{A}^g J Dg$ and $\int Dg = \infty$.

To solve this problem we can choose a representative $\overline{A}$ on each gauge orbit (gauge fixing) using a gauge-fixing condition $f(\overline{A}) = 0$. The change of variable $A \to \overline{A}$ introduces a Jacobian in the measure.

Question: does the gauge-fixing condition select one and only one representative on each gauge orbit?

Answer: in general this is not true (Gribov copies).



Lattice Landau Gauge (I)

In the continuum: $\partial_\mu A_\mu(x) = 0$. On the lattice the Landau gauge is imposed by minimizing the functional

$$\mathcal{E}[U;g] = -\sum_{x,\mu} Tr\, U_\mu^{(g)}(x) \,,$$

where $g(x) \in SU(N_c)$ and $U_\mu^{(g)}(x) = g(x)\, U_\mu(x)\, g^\dagger(x+\hat{e}_\mu)$ is the lattice gauge transformation.

By considering the relations $U_\mu(x) = e^{i\, A_\mu(x)}$ and $g(x) = e^{i\,\tau\,\gamma(x)}$, we can expand $\mathcal{E}[U;g]$ (for small τ):

$$\mathcal{E}[U;g] = \mathcal{E}[U;\mathbb{1}] + \tau\, \mathcal{E}^{'}[U;\mathbb{1}](b,x)\, \gamma^b(x)$$

$$+ \frac{\tau^2}{2}\, \gamma^b(x)\, \mathcal{E}^{''}[U;\mathbb{1}](b,x;c,y)\, \gamma^c(y) + \ldots \,,$$

where $\mathcal{E}^{''}[U;\mathbb{1}](b,x;c,y) = \mathcal{M}(b,x;c,y)[A]$ is a lattice discretization of the Faddeev-Popov operator $-D\cdot\partial$

Lattice Landau Gauge (II)

At any local minimum (stationary solution)

$$\mathcal{E}^{'}(0) = 0 \quad \forall \;\; \{\gamma^b(x)\} \quad \Rightarrow \quad \left[\left(\nabla \cdot A\right)(x)\right]^b = 0 \quad \forall \;\; x,b \; ,$$

where

$$A_\mu(\vec{x}) = \frac{1}{2\,i} \left[U_\mu(\vec{x}) - U_\mu^\dagger(\vec{x}) \right]_{\text{traceless}}$$

is the gauge field and

$$\left(\nabla \cdot A^b\right)(\vec{x}) = \sum_{\mu=1}^{d} A_\mu^b(\vec{x}) - A_\mu^b(\vec{x} - \hat{e}_\mu)$$

is the (minimal) Landau gauge condition on the lattice

Ghost Propagator

Ghost fields are introduced as one evaluates functional integrals by the Faddeev-Popov method, which restricts the space of configurations through a gauge-fixing condition. The ghosts are unphysical particles, since they correspond to anti-commuting fields with spin zero.

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On the lattice, the (minimal) Landau gauge is imposed as a minimization problem and the ghost propagator is given by

$$G(p) \;=\; \frac{1}{N_c^2 - 1} \sum_{x,\,y,\,a} \frac{e^{-2\pi i\, k\cdot(x-y)}}{V} \,\langle\, \mathcal{M}^{-1}(a,x;a,y)\,\rangle\,,$$

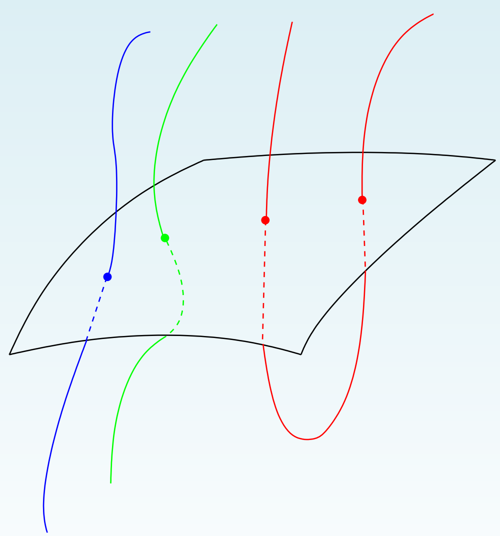
where the Faddeev-Popov (FP) matrix $\mathcal{M}$ is obtained from the second variation of the minimizing functional.

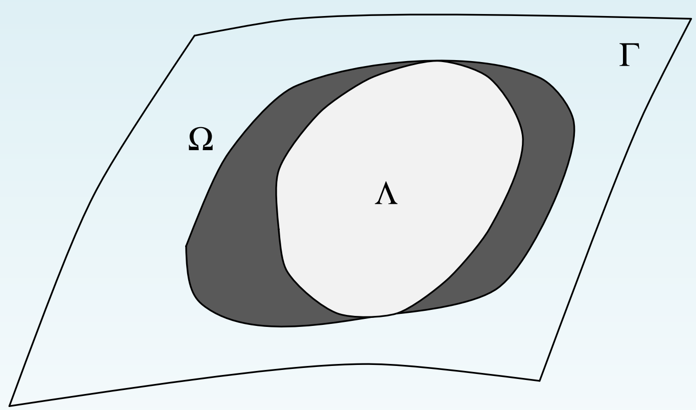
Early simulations: Suman & Schilling, PLB 1996; Cucchieri, NPB 1997

Ghost Enhancement

Gribov's restriction beyond quantization using Faddeev-Popov (FP) method implies taking a minimal gauge, defined by a minimizing functional in terms of gauge fields and gauge transformation

$\Rightarrow$ FP operator (second variation of functional) has non-negative eigenvalues. First Gribov horizon $\partial\Omega$ approached in infinite-volume limit, inducing ghost enhancement





GZ Scenario: Confinement by Ghost

Formulated for Landau gauge, predicts gluon propagator

$$D^{ab}_{\mu\nu}(p) = \sum_x e^{-2i\pi k\cdot x} \langle A^a_\mu(x)\, A^b_\nu(0)\rangle = \delta^{ab} \left(g_{\mu\nu} - \frac{p_\mu\, p_\nu}{p^2} \right) D(p^2)$$

suppressed in the IR limit $\Rightarrow$ gluon confinement

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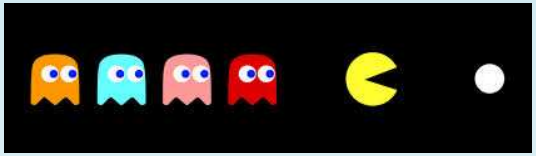

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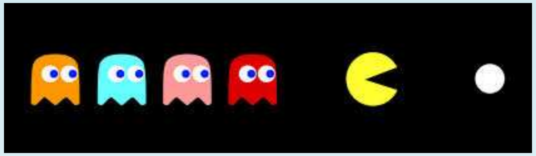

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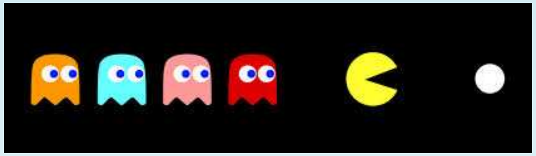

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- Infinite volume favors configurations on the first Gribov horizon, where minimum nonzero eigenvalue λ_{min} of Faddeev-Popov operator $\mathcal{M}$ goes to zero
- In turn, $G(p)$ should be IR enhanced, introducing long-range effects, which are related to the color-confinement mechanism

Gauge-Related Lattice Features

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- get FP matrix without considering ghost fields explicitly
- Lattice momenta given by $\hat{p}_\mu = 2\,\sin\left(\pi\, n_\mu / N\right)$ with $n_\mu = 0, 1, \ldots, N/2$ $\Leftrightarrow$ $p_{min} \sim 2\pi/(a\,N) = 2\pi/L$, $p_{max} = 4/a$ in physical units

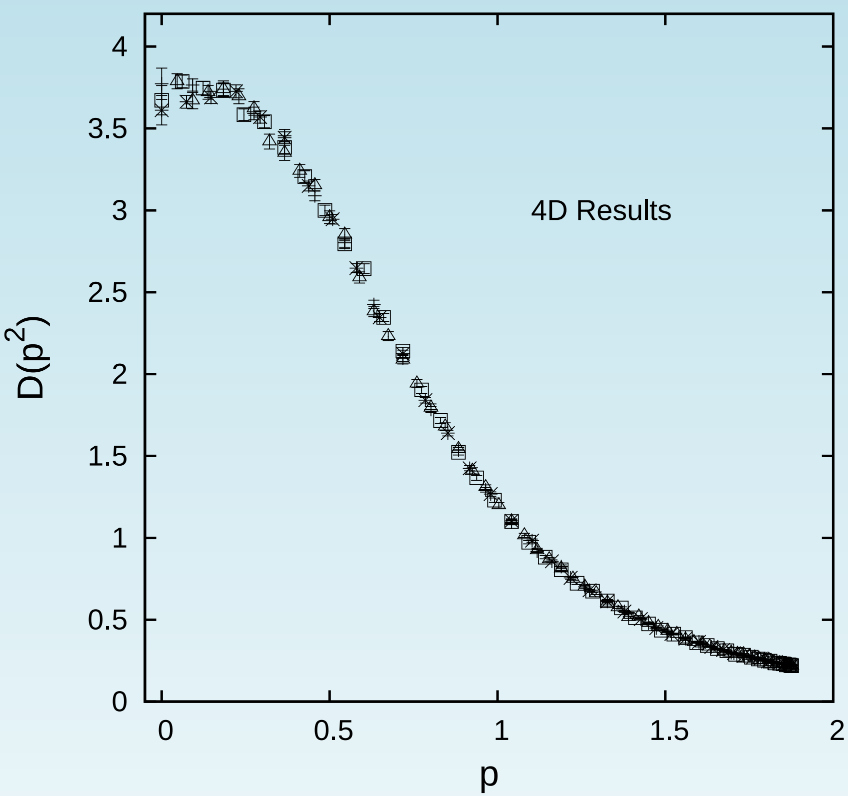
3-Step Code

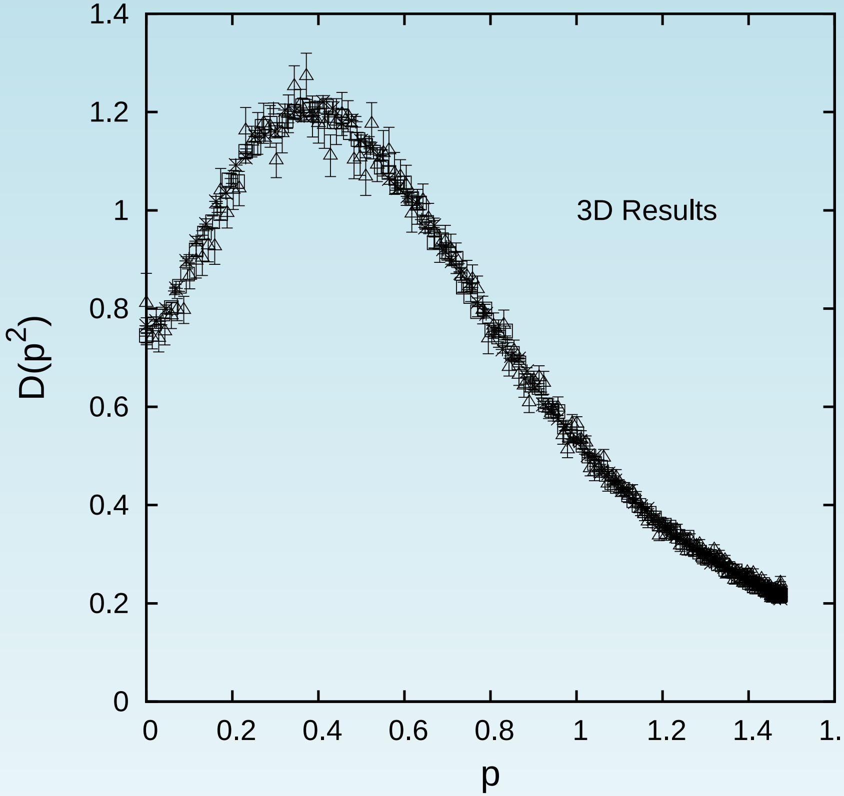
```
main() {
/* set parameters: beta, number of configurations NC,
                   number of thermalization sweeps NT */
     read_parameters();
/* {U} is the link configuration */
     set_initial_configuration(U);
/* cycle over NC configurations */
     for (int c=0; c < NC; c++) {
          thermalize(U,beta,NT);
          gauge_fix(U,g);
          evaluate_propagators(U[g]);
     }
}
```

Algorithms: Heat-Bath and Micro-canonical (thermalization), overrelaxation and simulated annealing (gauge fixing), conjugate gradient and Fourier transform (propagators, etc.).

Gluon Propagator at “Infinite” Volume

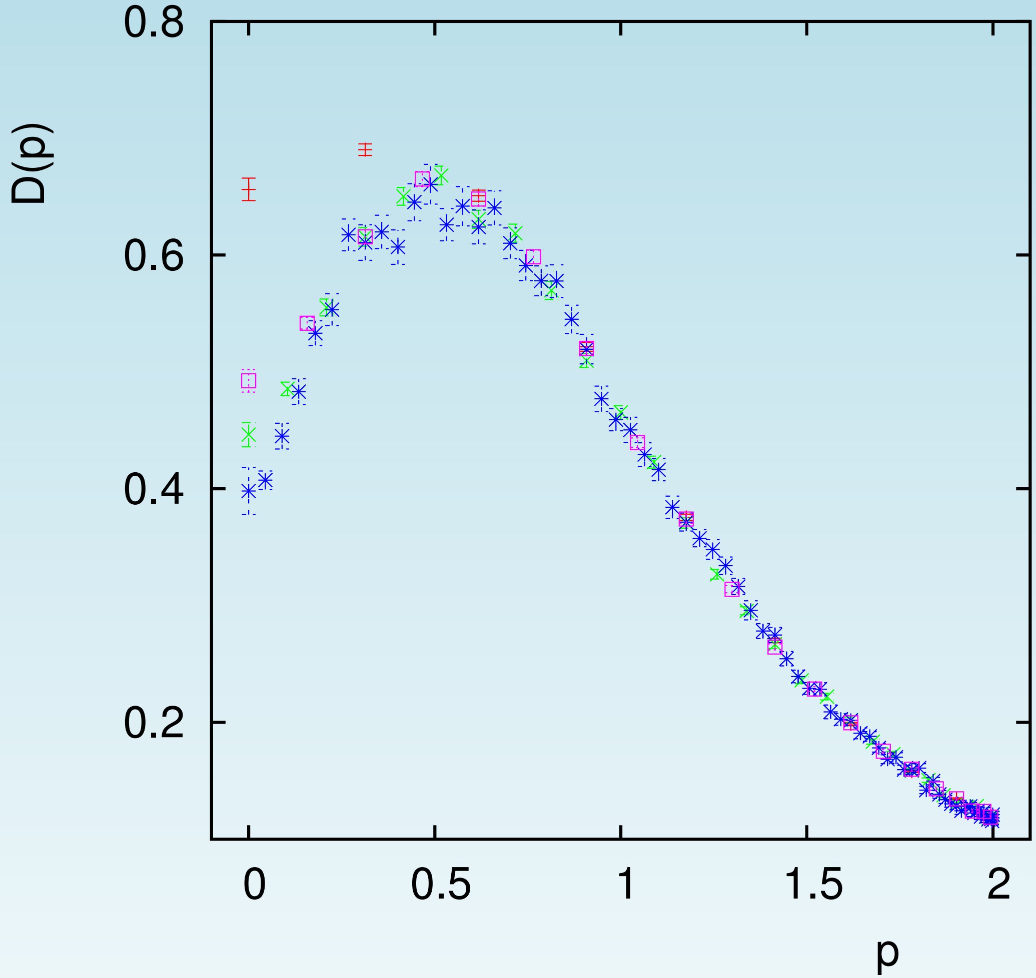
Attilio Cucchieri & T.M. (2008)





Gluon propagator $D(k)$ as a function of the lattice momenta k (both in physical units) for the pure-$SU(2)$ case in $d = 4$ (left), considering volumes of up to 128^4 (lattice extent ~ 27 fm) and $d = 3$ (right), considering volumes of up to 320^3 (lattice extent ~ 85 fm)

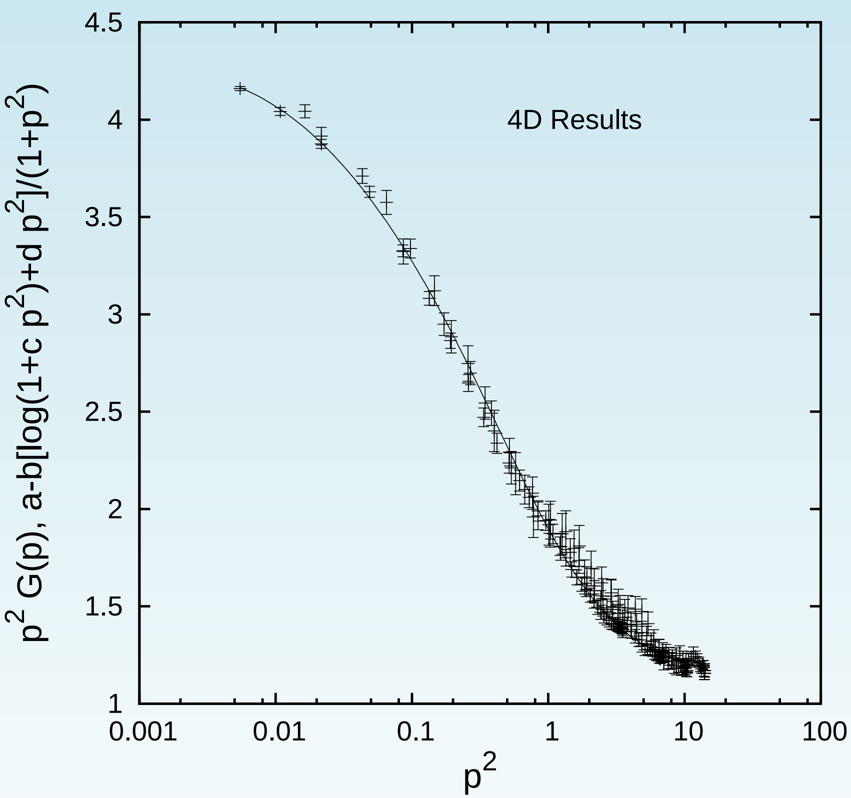
Gluon Propagator: Volume Effects



Gluon propagator as a function of the lattice momentum p for lattice volumes $V = 20^3$, 40^3, 60^3 and 140^3 at $\beta = 3.0$. About 100 days using a 13 Gflops PC cluster (2003)

Ghost Propagator Results

Fit of the ghost dressing function $p^2 G(p^2)$ as a function of p^2 (in GeV) for the 4d case ($\beta = 2.2$ with volume 80^4). We find that $p^2 G(p^2)$ is best fitted by the form $p^2 G(p^2) = a - b[\log(1 + cp^2) + dp^2]/(1 + p^2)$, with



$$a = 4.32(2),$$
$$b = 0.38(1)\, GeV^2,$$
$$c = 80(10)\, GeV^{-2},$$
$$d = 8.2(3)\, GeV^{-2}.$$

In IR limit $p^2 G(p^2) \sim a$.

Attilio Cucchieri & T.M. (2008)

Issues

Simulation on large lattices (IR limit) is very costly. How to be more efficient?

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How to disentangle approach to Gribov horizon (as lattice volume increases) and behavior of $G(p^2)$ (or $\lambda_{\min}$)?

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Get insight from features of the lattice simulations themselves:

1) Explore Gribov horizon by visiting neighboring (unsampled) configurations, get info about $\lambda_{\min}$

2) Simulate on effectively large lattices by “faking” periodic crystal and invoking Bloch’s theorem

Upper and Lower Bounds for $G(p)$

On the lattice, the ghost propagator is given by

$$G(p) \;=\; \frac{1}{N_c^2 - 1} \sum_{x,\, y,\, a} \frac{e^{-2\pi i\, k\cdot(x-y)}}{V}\, \mathcal{M}^{-1}(a, x; a, y)$$

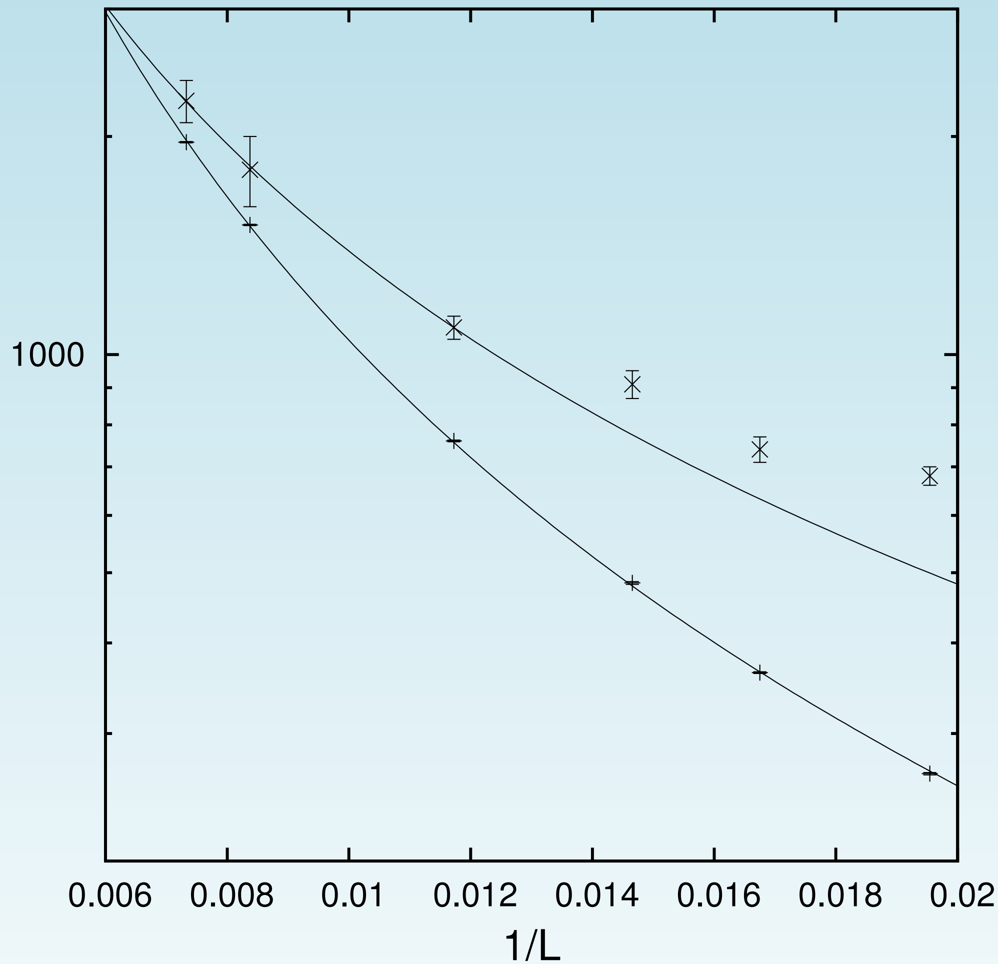
$$= \frac{1}{N_c^2 - 1} \sum_{i,\lambda_i \neq 0} \frac{1}{\lambda_i} \sum_a |\widetilde{\psi}_i(a, p)|^2 \;,$$

where $\psi_i(a, x)$ and λ_i are the eigenvectors and eigenvalues of the FP matrix. Then, one can prove (A.Cucchieri, TM, PRD 78, 2008) that

$$\frac{1}{N_c^2 - 1} \frac{1}{\lambda_{min}} \sum_a |\widetilde{\psi}_1(a, p)|^2 \;\leq\; G(p) \;\leq\; \frac{1}{\lambda_{min}} \;.$$

If λ_{min} behaves as $L^{-\alpha}$ in the infinite-volume limit, $\alpha > 2$ is a necessary condition to obtain an IR-enhanced ghost propagator $G(p)$.

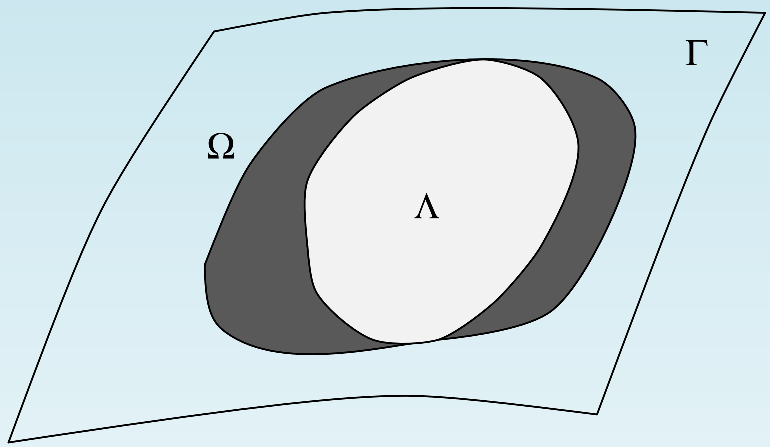
Upper bound for $G(p_{min})$



$$2\kappa = 0.043(8),\ \alpha = 1.53(2)$$

The Infinite-Volume Limit

We thus see that, as the infinite-volume limit is approached, the sampled configurations (inside Ω = region for which $\mathcal{M}$ is positive semi-definite) are closer and closer to the first Gribov horizon $\partial\Omega$



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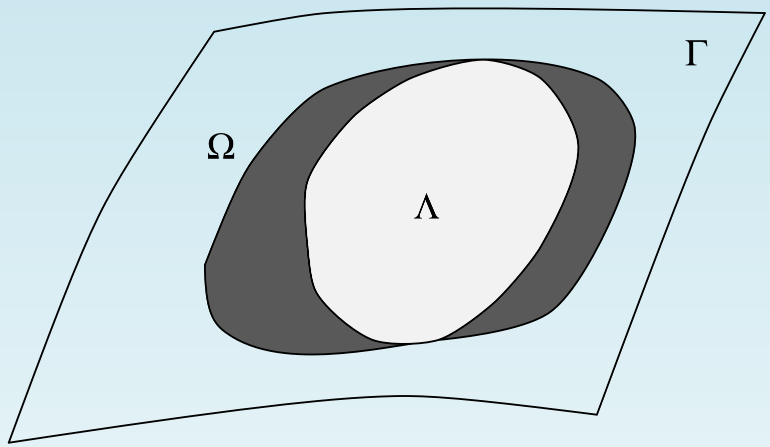

Can we learn more about the geometry of this region?

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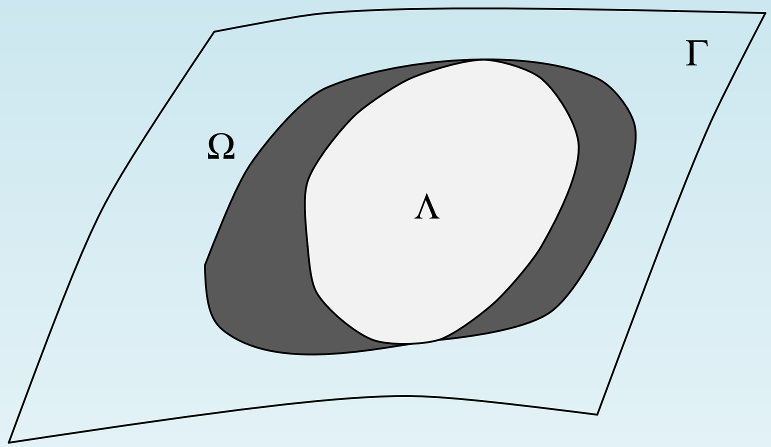
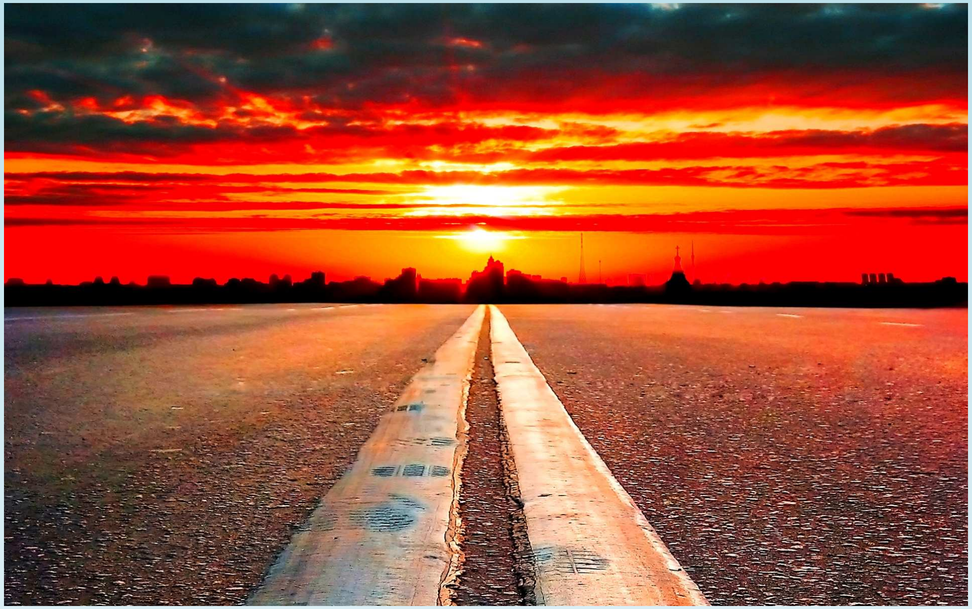


Can we learn more about the geometry of this region?

Lattice simulation produces thermalized gauge configurations, but we can also “visit” nearby configs and extract info from them!

Reaching (and Crossing!) the Horizon

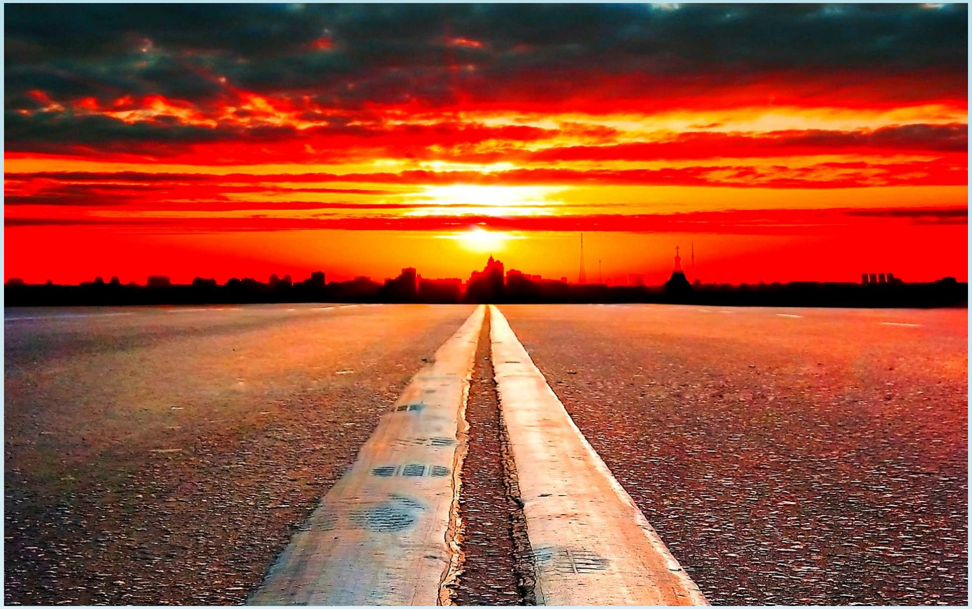


How many roads have I wondered?
None, and each my own
Behind me the bridges have crumbled
No question of return

Nowhere to go but the horizon
where, then, will I call my home?

The Same Song, Susheela Raman

Reaching (and Crossing!) the Horizon



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— They say that communism is just over the horizon. What's a horizon?
— A horizon is an imaginary line which continues to recede as you approach it.

Russian joke from Khrushchev's time

The Region Ω: Properties

Three important properties have been proven (D. Zwanziger, 1982) for the Gribov region Ω:

1. the trivial vacuum $A_\mu = 0$ belongs to Ω;
2. the region Ω is convex;
3. the region Ω is bounded in every direction.

(The same properties can be proven also for the fundamental modular region Λ.)

The first property is trivial, since $A_\mu = 0$ implies that $\mathcal{M}(b, x; c, y)[0]$ is (minus) the Laplacian $-\partial^2$ (which is a semi-positive-definite operator).

Convexity of Ω

The gauge condition $\partial \cdot A = 0$ and the operators $D^{bc}(x, y)[A]$, $\mathcal{M}(b, x; c, y)[A] = -\partial^2 + \mathcal{K}[A]$ and $\mathcal{K}[A]$ are linear in the gauge field A_μ:

$$\mathcal{M}[\gamma A_1 + (1-\gamma) A_2] = -\partial^2 + \mathcal{K}[\gamma A_1 + (1-\gamma) A_2]$$

$$= \gamma \left(-\partial^2 + \mathcal{K}[A_1]\right) + (1-\gamma) \left(-\partial^2 + \mathcal{K}[A_2]\right)$$

$$= \gamma \mathcal{M}[A_1] + (1-\gamma) \mathcal{M}[A_2]$$

and, for $\gamma \in [0, 1]$, $\mathcal{M}[\gamma A_1 + (1-\gamma) A_2]$ is semi-positive definite if $\mathcal{M}[A_1]$ and $\mathcal{M}[A_2]$ are semi-positive definite. Also

$$\gamma \, \partial \cdot A_1 + (1-\gamma) \, \partial \cdot A_2 = 0$$

if $\partial \cdot A_1 = \partial \cdot A_2 = 0$. $\Longrightarrow$ The convex combination $\gamma A_1 + (1-\gamma) A_2$ belongs to Ω, for any value of $\gamma \in [0, 1]$, if $A_1, A_2 \in \Omega$.

Boundary of Ω

Using properties 1 and 2 and with $A_1 = 0$, $A_2 = A$, $1 - \gamma = \rho$ we have

$$\mathcal{M}[\rho A] = -\partial^2 + \mathcal{K}[\rho A] = (1-\rho)\,(-\partial^2) + \rho\,\mathcal{M}[A]$$

and, if $A \in \Omega$, then $\rho A \in \Omega$ for any value of $\rho \in [0, 1]$.

Since the color indices of $\mathcal{K}[A]$ are given by $\mathcal{K}^{bc}[A] \sim f^{bce} A^e_\mu$, we have that all the diagonal elements of $\mathcal{K}[A]$ are zero $\Longrightarrow$ the trace of the operator $\mathcal{K}[A]$ is zero.

The operator $\mathcal{K}^{bc}_{xy}[A]$ is real and symmetric (under simultaneous interchange of x with y and b with c) and its eigenvalues are real $\Longrightarrow$ at least one of the eigenvalues of $\mathcal{K}[A]$ is (real and) negative. If ϕ_{neg} is the corresponding eigenvector, than for a sufficiently large (but finite) value of $\rho > 1$ the scalar product $(\phi_{neg}, \mathcal{M}[\rho A]\phi_{neg})$ must be negative $\Longrightarrow$ $\mathcal{M}[\rho A]$ is not semi-positive definite and $\rho A \notin \Omega$.

Relating $\lambda_{\min}$ and Geometry

For the infrared behavior of the ghost propagator $G(p^2)$,

the key point seems to be the rate at which $\lambda_{\min}$ goes to zero, which, in turn, should be related to the rate at which a thermalized and gauge-fixed configuration approaches $\partial\Omega$.

How do we relate $\lambda_{\min}$

to the geometry of the Gribov region Ω ?

Lower Bound for $\lambda_{\min}$ (I)

Consider a configuration A' belonging to the boundary $\partial\Omega$ of Ω and write

$$\lambda_{\min}\left[\,\mathcal{M}[\rho A']\,\right] \,=\, \lambda_{\min}\left[(1-\rho)\,(-\partial^2)\,+\,\rho\,\mathcal{M}[A']\right]\;.$$

From the second property, $\rho A' \in \Omega$ for $\rho \in [0,1]$. Since

$$\lambda_{\min}\left[(1-\rho)\,(-\partial^2)\,+\,\rho\,\mathcal{M}[A']\right]$$

$$=\,\min_{\chi}\,\left(\chi\,,\left[\,(1-\rho)\,(-\partial^2)\,+\,\rho\,\mathcal{M}[A']\,\right]\,\chi\right)\;,$$

with $(\chi\,,\chi) = 1$ and $\chi \neq$ constant, we can use the concavity of the minimum function

$$\boxed{\min_{\chi}\,(\chi, [M_1 + M_2]\,\chi) \,\geq\, \min_{\chi}\,(\chi, M_1\chi) + \min_{\chi}\,(\chi, M_2\chi)\;.}$$

Note: we have an equality if the eigenvector χ_1 corresponding to the smallest eigen-value $\lambda_{\min}$ of M_1 and of M_2 is the same.

Lower bound for $\lambda_{\min}$ (II)

We find

$$
\begin{aligned}
\lambda_{\min}\left[\,\mathcal{M}[\rho A']\,\right] \;&=\; \lambda_{\min}\left[(1-\rho)\,(-\partial^2)\;+\;\rho\,\mathcal{M}[A']\right] \\
&\geq (1-\rho)\min_{\chi}\big(\chi,(-\partial^2)\,\chi\big) + \rho\min_{\chi}\,\big(\chi\,,\mathcal{M}[A']\,\chi\big) \\
&=\; (1-\rho)\,p^2_{min}\;.
\end{aligned}
$$

Recall that $A' \in \partial\Omega \Longrightarrow$ the smallest non-trivial eigenvalue of the FP matrix $\mathcal{M}[A']$ is null, and that the smallest non-trivial eigenvalue of (minus) the Laplacian $-\partial^2$ is p^2_{min}.

In the Abelian case one has $\mathcal{M} = -\partial^2$ and $\lambda_{\min} = p^2_{min}$. $\Longrightarrow$ All non-Abelian effects are included in the $(1-\rho)$ factor (and in the inequality).

Lower bound for $\lambda_{\min}$ (III)

As the lattice side L goes to infinity, $\lambda_{\min}\,[\,\mathcal{M}[\rho A']\,]$ cannot go to zero faster than $(1-\rho)\,p^2_{min}$. Since $p^2_{min} \sim 1/L^2$ at large $L \Longrightarrow \lambda_{\min}$ behaves as $L^{-2-\alpha}$ in the same limit, with $\alpha > 0$, only if $1-\rho$ goes to zero at least as fast as $L^{-\alpha}$.

With $\rho A' = A$ the above inequality may also be written as

$$\boxed{\lambda_{\min}\,[\,\mathcal{M}[A]\,]\;\geq\;[1-\rho(A)]\,p^2_{min}\;.}$$

Here $1-\rho(A) \leq 1$ measures the distance of a configuration $A \in \Omega$ from the boundary $\partial\Omega$ (in such a way that $\rho^{-1}A \in \partial\Omega$). This result applies to any Gribov copy belonging to Ω.

Summarizing

Using properties of Ω and the concavity of the minimum function, one can show (A.C. & T.Mendes, 2013)

$$\lambda_{\min}\left[\,\mathcal{M}[A]\,\right] \;\geq\; \left[1-\rho(A)\right] p^2_{min}$$

Here $1-\rho(A) \leq 1$ measures the distance of a configuration $A \in \Omega$ from the boundary $\partial\Omega$ (in such a way that $\rho^{-1}A \equiv A' \in \partial\Omega$). This result applies to any Gribov copy belonging to Ω.

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$\Longrightarrow$ non-Abelian effects are included in the $(1-\rho)$ factor

Simulating the Math

We used 70 configurations, for the SU(2) case at $\beta = 2.2$, for $V = 16^4$, $24^4, 32^4, 40^4$ and 50 configurations for $V = 48^4, 56^4, 64^4, 72^4, 80^4$.

In order to cross the first Gribov horizon we applied scale transformations $\widehat{A}_\mu^{(i)}(x) = \tau_i A_\mu^{(i-1)}(x)$ to the gauge configuration A with

- $\tau_0 = 1$,
- $\tau_i = \delta\, \tau_{i-1}$,
- $\delta = 1.001$ if $\lambda_{\min} \geq 5 \times 10^{-3}$,
- $\delta = 1.0005$ if $\lambda_{\min} \in [5 \times 10^{-4}, 5 \times 10^{-3})$
- and $\delta = 1.0001$ if $\lambda_{\min} < 5 \times 10^{-4}$,

where $\lambda_{\min}$ is evaluated at the step $i-1$. After n steps, the modified gauge field $\widehat{A}_\mu^{(n)}(x)$ does not belong anymore to the region Ω, i.e. the eigenvalue $\lambda_{\min}$ of $\mathcal{M}[\widehat{A}^{(n)}]$ is negative (while λ_2 is still positive).

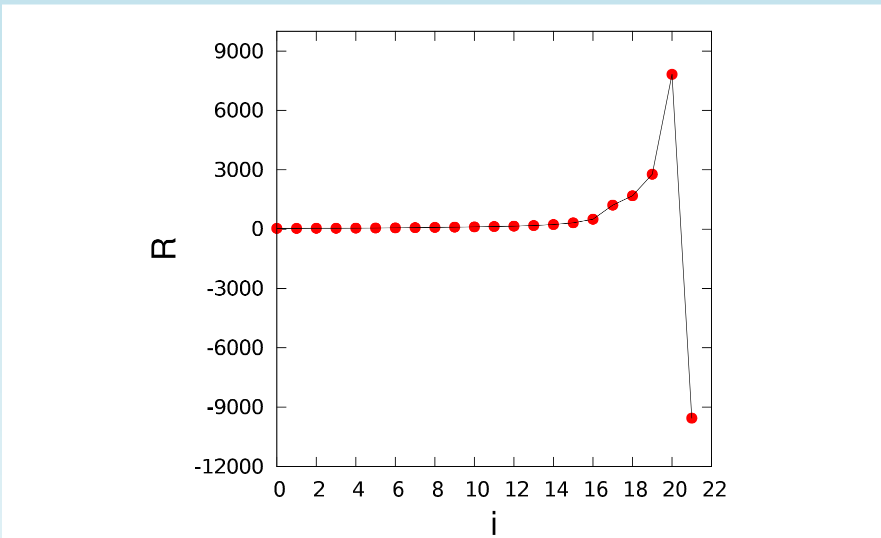
Crossing the Horizon (I)

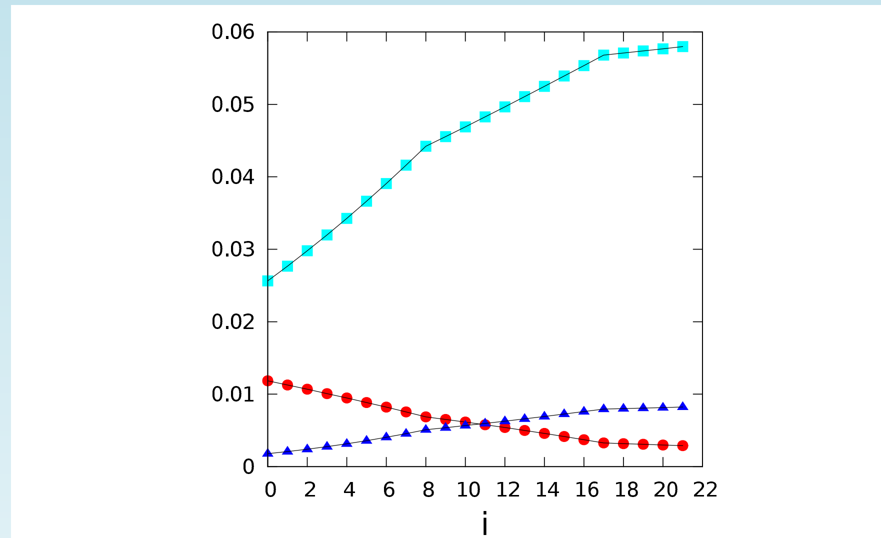
N	$\max(n)$	$\min(n)$	$\langle n \rangle$	R_{before}	R_{after}
16	30	6	17.2	15(3)	-30(12)
24	27	4	15.1	20(7)	-26(6)
32	19	5	11.7	26(9)	-51(20)
40	18	4	9.4	155(143)	-21(6)
48	13	2	7.8	21(5)	-21(5)
56	12	3	7.6	16(4)	-21(7)
64	11	2	6.8	20(7)	-42(18)
72	11	2	6.1	129(96)	-42(13)
80	12	3	6.1	15(4)	-24(4)

The maximum, minimum and average number of steps n, necessary to "cross the Gribov horizon" along the direction $A_\mu^b(x)$, as a function of the lattice size N. We also show the ratio $R[A] = (S''')^2/(S''\, S'''')$, divided by 1000, for the modified gauge fields $\tau_{n-1} A_\mu^b(x)$ and $\tau_n A_\mu^b(x)$, i.e. for the configurations immediately before and after crossing $\partial\Omega$.

Crossing the Horizon (II)

typical configuration



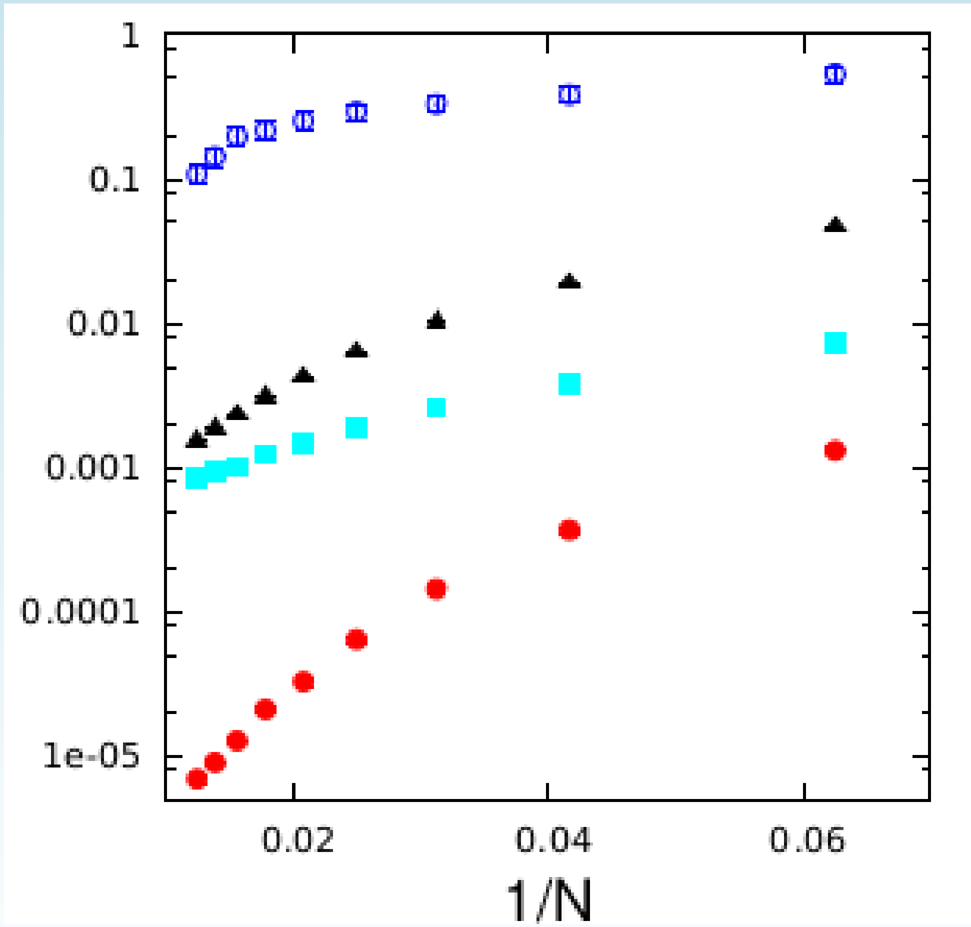


Plot of the ratio R, as a function of the iteration step i, for a configuration with lattice volume 16^4.

Plot of λ_2 (full circes), $|\mathcal{E}'''|$ (full squares) and $\mathcal{E}''''$ (full triangles) as a function of the iteration step i, for the same configuration.

How far from Equality? Very far...

Using $A' = \widetilde{\tau}\, A \equiv A(\tau_{n-1} + \tau_n)/2 \in \partial\Omega$ and $\rho = 1/\widetilde{\tau} < 1$: plot inverse of the lower bound for $G(p)$, $1/G(p_{min})$, λ_{min} and the quantity $(1-\rho)\, p_{min}^2$ as functions of the inverse lattice size $1/N$.



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Now notice that:

The inequality $\lambda_{min}\,[\,\mathcal{M}[A]\,] \geq [1-\rho(A)]\,p^2_{min}$ becomes an equality if and only if the eigenvectors corresponding to the smallest nonzero eigenvalues of $\mathcal{M}[A]$ and $-\partial^2$ coincide

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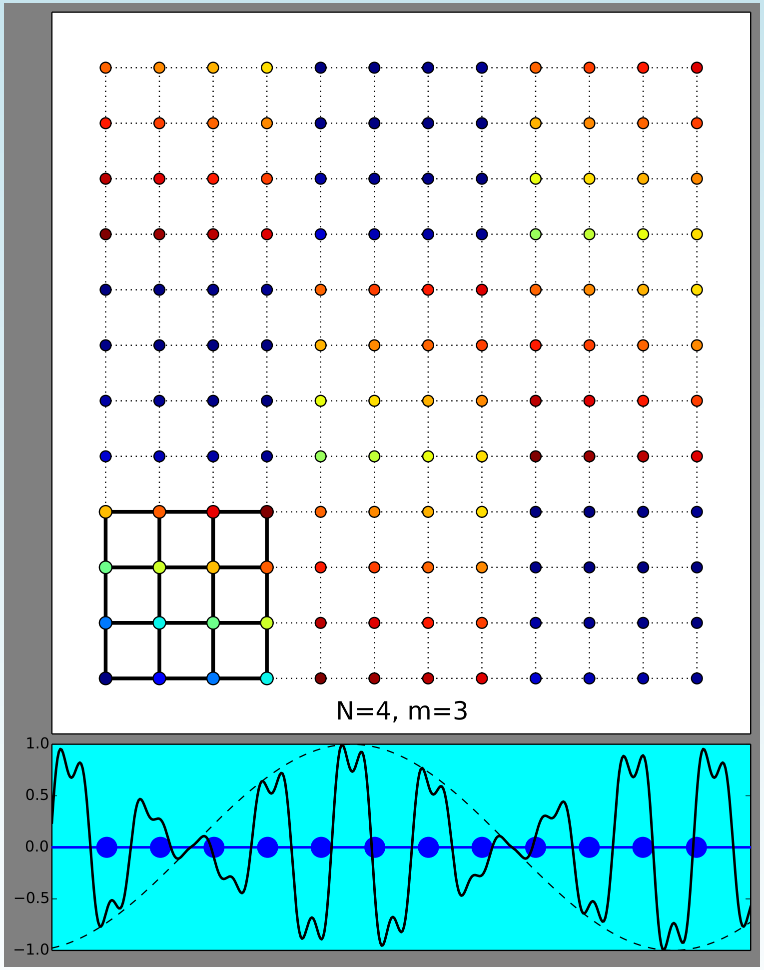
Our results show that the eigenvector ψ_{min} is very different from the plane waves corresponding to p_{min}

This should serve to illustrate the (nontrivial) non-enhancement of $G(p)$ in the IR

Large Lattices via Bloch's Theorem

Perform thermalization step on small lattice, then replicate it and use Bloch's theorem from condensed-matter physics to obtain gauge-fixing step for much larger lattice (A. Cucchieri, TM, PRL 2017)



Two-step Infinite-Volume Limit

In the paper D. Zwanziger, NPB 412 (1994) 657, the infinite-volume limit is taken in two steps:

1) first, considering the $V \to +\infty$ limit for the gauge transformation $g(x)$

2) then, taking the same limit for the gluon field [i.e. the link variables $\{U_\mu(x)\}$]

How can one do that?

The Extended Lattice

Setup:

1. Consider a d-dimensional link configuration $\{U_\mu(\vec{x})\} \in$ SU(N_c), defined on a lattice Λ_x with volume $V = N^d$ and periodic boundary conditions (PBC)

2. Replicate this configuration m times along each direction, yielding an extended lattice Λ_z with volume $m^d V$ and PBC

3. Indicate the points of Λ_z with $\vec{z} = \vec{x} + \vec{y}N$, where $\vec{x} \in \Lambda_x$ and $\vec{y}$ is a point on the replica lattice Λ_y

4. By construction, $\{U_\mu(\vec{z})\}$ in Λ_z is invariant under translations by N (in any direction)

The Extended Gauge Transformation (I)

Impose the minimal-Landau-gauge condition on Λ_z, i.e. consider the minimizing functional

$$\mathcal{E}_U[g] = -\frac{\Re\,\mathrm{Tr}}{d\,N_c\,m^d V} \sum_{\mu=1}^{d} \sum_{\vec{z}\in\Lambda_z} g(\vec{z})\,U_\mu(\vec{z})\,g(\vec{z}+\hat{e}_\mu)^\dagger$$

where $g(\vec{z}) \in \mathrm{SU}(N_c)$ has periodicity mN, i.e. $g(\vec{z}+mN\hat{e}_\mu) = g(\vec{z})$ (PBC in Λ_z)

The two limits: first take $m \to +\infty$ and then $N \to +\infty$

Bloch's Theorem (I)

For an ideal crystalline solid in d dimensions, one considers an electrostatic potential $U(\vec{r})$ with the periodicity of the Bravais lattice, i.e. $U(\vec{r}) = U(\vec{r} + \vec{R})$ for any vector $\vec{R} = n_\mu \vec{a}_\mu$.

Ingredients:

1. The Hamiltonian $\mathcal{H}$ for a single electron is invariant under translations by $\vec{R}$
2. Translation operators $\mathcal{T}(\vec{R})$ commute, i.e.

$$\mathcal{T}(\vec{R})\,\mathcal{T}(\vec{R}') = \mathcal{T}(\vec{R}')\,\mathcal{T}(\vec{R}) = \mathcal{T}(\vec{R} + \vec{R}')$$

3. We can choose the eigenstates $\psi(\vec{r})$ of $\mathcal{H}$ to be also eigenstates of $\mathcal{T}(\vec{R})$

Bloch's Theorem (II)

4. The eigenvalues $c(\vec{R})$ of $\mathcal{T}(\vec{R})$ are $\exp(i\vec{k}\cdot\vec{R}) = \exp(2\pi i\, k_\nu\, n_\nu)$, where $\vec{k} = k_\nu \vec{b}_\nu$ is a vector of the reciprocal lattice (i.e. $\vec{a}_\mu \cdot \vec{b}_\nu = 2\pi\delta_{\mu\nu}$)

5. Since

$$\mathcal{T}(\vec{R})\,\psi(\vec{r}) = \psi(\vec{r}+\vec{R}) = \exp(i\vec{k}\cdot\vec{R})\,\psi(\vec{r})\ ,$$

   the eigenstates $\psi(\vec{r})$ can be written as Bloch waves

$$\psi_{\vec{k}}(\vec{r}) \;=\; \exp(i\vec{k}\cdot\vec{r})\,h_{\vec{k}}(\vec{r})\ ,$$

   where the functions $h_{\vec{k}}(\vec{r})$ have the periodicity of the Bravais lattice, i.e. $h_{\vec{k}}(\vec{r}+\vec{R}) = h_{\vec{k}}(\vec{r})$

The Extended Gauge Transformation (II)

The analogy:

1. $\Lambda_y \Longleftrightarrow$ finite Bravais lattice with PBC
2. $\{U_\mu(\vec{z})\} \Longleftrightarrow$ periodic electrostatic potential $U(\vec{r})$

One can prove that:

- $g(\vec{z})$ can be written as $g(\vec{z}) = \exp\left(i\Theta_\mu\, z_\mu/N\right) h(\vec{z})$
- $h(\vec{z})$ has periodicity N, i.e. $h(\vec{z} + N\hat{e}_\mu) = h(\vec{z}) \;\Rightarrow\; h(\vec{x})$
- The matrices $\Theta_\mu = \tau^a \theta^a_\mu$ (with $a = 1, \ldots, N_c^2 - 1$) have eigenvalues $2\pi n_\mu/m$, with $n_\mu \in \mathcal{Z}$
- The matrices θ^a_μ are elements of a Cartan sub-algebra of the SU(N_c) Lie algebra

The New Minimizing Functional

Due to the expression for $g(\vec{z})$ and to the cyclicity of the trace, the minimizing functional becomes

$$\mathcal{E}_U[h, \Theta_\mu] \;=\; -\frac{\Re\,\mathrm{Tr}}{d\, N_c\, V} \sum_{\mu=1}^{d} e^{-i\Theta_\mu/N}\, Q_\mu \;,$$

$$Q_\mu \;=\; \sum_{\vec{x}\in\Lambda_x} h(\vec{x})\, U_\mu(\vec{x})\, h(\vec{x}+\hat{e}_\mu)^\dagger \;,$$

i.e. the numerical minimization is still carried out on the original lattice Λ_x

The Proof: Ingredients (I)

1. The original minimizing problem is invariant under translations $\mathcal{T}(N\hat{e}_\mu)$
2. Due to the cyclicity of the trace, the minimizing functional $\mathcal{E}_U[g]$ is invariant under global (left) gauge transformations, i.e. $g(\vec{z}) \to v\, g(\vec{z})$, with $v \in$ SU(N_c)
3. If the sought gauge transformation $\{g(\vec{z})\}$ is unique, then $g(\vec{z})$ and $g(\vec{z} + N\hat{e}_\mu)$ can differ only by a global transformation, i.e.

$$\mathcal{T}(N\hat{e}_\mu)\, g(\vec{z}) \,=\, g(\vec{z} + N\hat{e}_\mu) \,=\, v_\mu\, g(\vec{z}) \;,$$

where $v_\mu \in$ SU(N_c) is a $\vec{z}$-independent matrix

The Proof: Ingredients (II)

4. Since the translation operators commute, the v_μ matrices are commuting matrices, i.e. they can be written as $\exp(i\Theta_\mu) = \exp(i\tau^a\theta^a_\mu)$, where the τ^a matrices are Cartan generators

5. Then

$$g(\vec{z}) = g(\vec{x} + \vec{y}N) = \mathcal{T}(N\vec{y})\, g(\vec{x}) = \exp(i\Theta_\mu y_\mu)\, g(\vec{x})$$

   and the proof is complete if we define

$$g(\vec{x}) \equiv \exp(i\Theta_\mu x_\mu / N)\, h(\vec{x})$$

6. Due to the PBC for Λ_z, we need to impose the condition $[\exp(i\Theta_\mu)]^m = \mathbb{1} \Longrightarrow$ the eigenvalues of the matrices Θ_μ are of the type $2\pi n_\mu/m$, with $n_\mu \in \mathcal{Z}$

Numerical Simulations

In the SU(N_c) case:

1. generate a thermalized d-dimensional link configuration $U_\mu(x)$ with periodicity N, i.e. $V = N^d$ with PBC
2. minimize $\mathcal{E}_U[h, \Theta_\mu]$ with respect to $h(x)$ and Θ_μ using two alternating steps:
   a) the matrices Θ_μ are kept fixed and one updates the matrices $h(\vec{x})$ by sweeping through the lattice
   b) the matrices Q_μ are kept fixed and one minimizes $\mathcal{E}_U[h, \Theta_\mu]$ with respect to the matrices Θ_μ, belonging to the corresponding Cartan sub-algebra
3. evaluate the gluon propagator using the extended gauge-fixed link variables $U_\mu^{(g)}(\vec{z}) = g(\vec{z})\, U_\mu(\vec{z})\, g(\vec{z} + \hat{e}_\mu)^\dagger$

The SU(2) Case

In the $SU(2)$ case (one-dimensional Cartan sub-algebra) we can write

$$\Theta_\mu = 2\pi(v^\dagger \sigma_3 v)\, n_\mu / m$$

with $v \in$ SU(2) and eigenvalues $\pm 2\pi n_\mu / m$

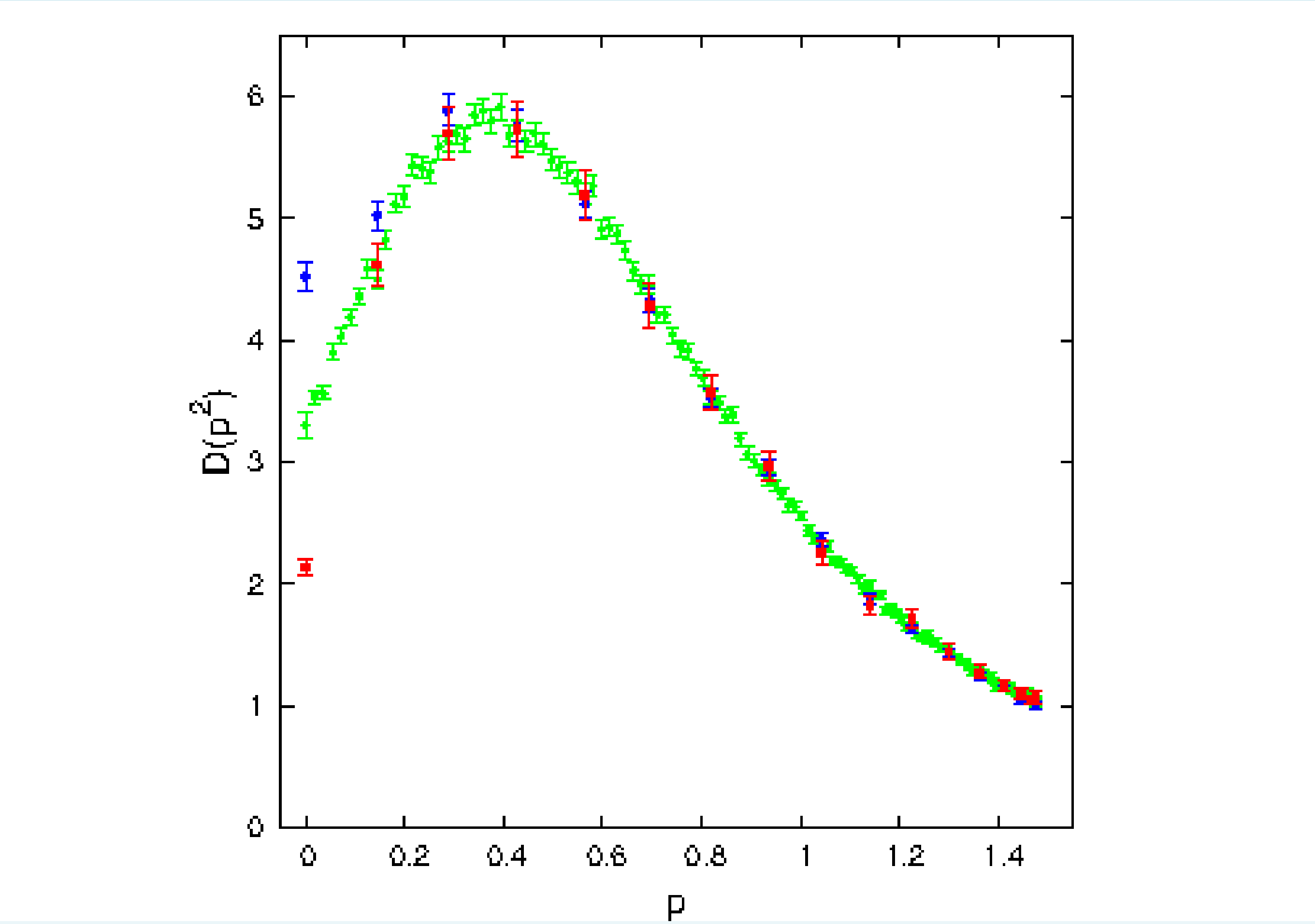
Then, in the new minimizing functional

$$\exp\left(-i\Theta_\mu/N\right) \,=\, \cos(\theta_\mu)\,\mathbb{1} \,-\, i\,\sin(\theta_\mu)\,v^\dagger \sigma_3 v$$

and $\theta_\mu = 2\pi n_\mu/(mN)$

Also, the matrices Q_μ are proportional to SU(2) matrices

Results: $3D$ Gluon Propagator



The gluon propagator $D(p^2)$ as a function of the lattice momentum p at $\beta = 3.0$ for the Λ_x lattice volumes $V = 32^3$ and 256^3 and for the Λ_z lattice volume $V = 32^3 \times 8^3 = 256^3$

Conclusions

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2) Lattice used as a (periodic) crystalline structure allowed large-lattice numerical results (in the gluon sector) to be obtained using small lattice volumes with extended gauge transformations. Notice:

i) the information encoded in a thermalized configuration does not depend much on the lattice volume V

ii) the properties of the Landau-gauge Green's functions are essentially set by the gauge-fixing procedure and the size of V matters!