Two dimensional quantum turbulence: evidence of an inverse energy cascade

Theory

Sam Rooney (U. of Otago)
Ewan Wright (U. of Arizona)
Kody Law (U. of Warwick)
Ricardo Carratero-Gonzalez (San Diego State U.)
Panayotis Kevrekedis (U. Mass. at Amherst)
Matthew Davis (U. of Queensland)

Experiment

Tyler Neely (U. of Arizona)
Carlo Samson (U. of Arizona)
Brian Anderson (U. of Arizona)
Turbulence

Da Vinci: “La turbolenza” (1507)

Altamaria (18000yrs)
Turbulence: what is it?

- Extremely complex, chaotic flow.
- Highly *stochastic* flow characteristics.
- Significance in atmospheric mixing, engineering applications, stellar dynamics, plasma dynamics, oceanic flows ...
- Breakdown of regular flow into eddies. Each eddy is then unstable to further decay into even smaller rotating structures (in 3D).

Flow past a cylinder

Flow visualization of a turbulent jet via laser-induced fluorescence
Richardson cascade

- Richardson proposed a model for the cascade of energy down through smaller and smaller vortices
- Famously summarized in the Swiftian verse:

  “Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity.”

![Richardson cascade diagram](image)

Lewis Richardson

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(R. Ecke 2009)
Kolmogorov (1941) found kinetic energy spectrum for Richardson cascade scenario.

Input to the theory: energy injected at large length-scale. Removed at small viscosity scale (by viscosity).

Within a so called inertial range kinetic energy is conservatively and locally (in scale-space) transmitted to successively smaller scales.

Dimensional analysis gives:

\[ E(k) \propto k^{-5/3} \]

- Kinetic energy per unit mass
- Wave-number \( \sim 1/(\text{length scale}) \)
Energy (per unit mass)

Turbulence spectra from a tidal channel

By H. L. Grant, R. W. Stewart† and A. Moilliet
1961

E(k) \sim k^{-5/3}

\sim 1/(\text{length scale of eddy})

Viscosity

Inertial range

Universal

Pyroclastic eruption plume from Mt. St. Helens (from Ecke 2009)

Andrey Kolmogorov

Kolmogorov
• Turn-around time: 14 days.
• Apparently in a state of equilibrium
• How does energy move to large scales?
• Dynamics in flattened systems is completely at odds with the Richardson cascade idea.

• In 1665 Giovanni Cassini described a “permanent red spot” on the Jovian surface. It has no lasted for at least 181 years, and possibly as long as 346 years.

• In 1945 Lars Onsager developed a 2D point-vortex model which exhibited clustering of same-sign vortices in equilibrium.

"...the large compound vortices formed in this manner will remain as the only conspicuous features of the motion; because the weaker vortices, free to roam practically at random, will yield rather erratic and disorganised contributions to the flow."

Lars Onsager, 1945 letter to L. Pauling.
Robert Kraichnan proposed that in two-dimensional classical turbulence, the Richardson cascade is reversed. Energy flows from small to large scales:

“Little whirls meet bigger whirls and merge with great affection, bigger whirls forge greater whirls, and so on to advection.”

Important quantity in 2D:

*Enstrophy* = amount of vorticity

Without external forcing, enstrophy is *conserved* in 2D: vortices cannot bend, tilt, or stretch.
Even more surprisingly, a *double-cascade* can occur in 2D: one for energy (*inverse*), and one for enstrophy (*direct*).

In fact, the two cascades are **mutually exclusive** in their respective wave-number ranges.

\[
\frac{1}{2} \partial_t \langle \eta^2 \rangle = \sum_{r,s} A_{nrs} \langle a_r a_s a_n \rangle + \langle f_n a_n \rangle - \text{diss} \quad (4.11)
\]

where \( f_n \) is the amplitude of \( f \) in the mode \( n \). Note that the enstrophy production is the correlation \( \langle f_n a_n \rangle \) between the vorticity \( \eta \) and its source \( f \), and the corresponding energy production \( k^{-2} \langle f_n a_n \rangle \).

The sums of these quantities over the modes are the total enstrophy production \( \eta \) and energy production \( \epsilon \) respectively.

For the considered homogeneous isotropic turbulence, the discrete mode amplitudes \( a_n \) is replaced by the Fourier transform \( \hat{\omega}(k) \), and the energy equation (4.11) takes the form of a conservation equation for the energy spectrum \( E(k) \),

\[
\frac{\partial E}{\partial t} = -\frac{\partial \Pi}{\partial k} + \text{forcing} - \text{diss} \quad (4.12)
\]

\( \Pi \) can be viewed as the energy flux due to nonlinear interactions and \( \partial \Pi / \partial k \) its divergence. The same equation can be also written in a form displaying enstrophy conservation

\[
\frac{1}{2} k^2 \frac{\partial E}{\partial t} = -\frac{\partial Z}{\partial k} + \text{forcing} - \text{diss} \quad (4.13)
\]

with an enstrophy flux \( Z \).

These fluxes are explicitly obtained \([56, 64]\) by Fourier transform of the Euler equations and integration over wavevector.

(From Sommeria, 2001)

\[ k_i = 1/\text{[length scale of the forcing (i.e. stirring)]} \]

(From Sommeria, 2001)
Two additional properties are often considered in defining turbulence (see e.g. Tennekes and Lumley [103]): the existence of strong vorticity fluctuations and strong energy dissipation. Although vorticity dynamics is also essential in 2D turbulence, there is no mechanism of vorticity amplification. We shall see in Section 2 that as a consequence energy dissipation is forbidden in the limit of small viscosity: this is the main dynamical signature of 2D turbulence. In that sense, 2D turbulence is different from usual turbulence, but still the defining properties listed above can be satisfied in two dimensions.

Fig. 1. Grid turbulence in a soap film (from Rutgers, 1996 (http://www.physics.ohio-state.edu/~maarten)). The fluid is moving from left to right, at velocity 2 m/s, across the comb with mesh 0.3 cm, while the total width is 4 cm. Visualization is provided by interference fringes due to small fluctuations of the film thickness (this is like the color patterns in usual soap bubbles). The increase of turbulent scale with distance to the grid is clearly visible, and it has been measured by laser Doppler velocimetry [71]. Note that this technique for producing 2D turbulence has been first developed by Couder [30].

The very existence of 2D turbulence has been questioned in the past. It has been considered as “a statistical extension of XIX century fluid dynamics”, limited to ideal 2D flow problems remote from the real physical world. Indeed the two cases of 2D turbulence considered above may seem at first sight equally unrealistic: z-independent flows are (by definition) unstable.
Quantum Turbulence

- A lot of turbulence theory in 3D BEC (Barenghi, Tsubota, Adams groups)
- Recent experiments by Bagnato group observed first signatures of 3DQT in BEC. (Henn et al, PRL 2009).
- 2DQT difficult to establish in superfluid Helium: thin films always have some 3D behavior that is hard to completely eliminate.
- Dilute gas Bose-Einstein condensates are highly controllable, and 2D regimes can be achieved.
- The major question is: **do the classical concepts of 2D turbulence still apply for a quantum fluid?**

**Can we observe an inverse cascade in 2D quantum turbulence?**
Stir and hold in toroidal trap

Highly oblate optical + magnetic trap: 8 Hz (radial) x 90 Hz (axial).

Form a BEC (2x10^6 atoms, ^{87}\text{Rb}), stir it with blue-detuned laser, then watch what happens.
Experiment: stir, then hold (+evaporate)

External forcing + Bose-Einstein condensation

Equilibrium without rotation

Persistent current

~ Great Red Spot??
• For *quantitative* calculations of persistent current formation using *Stochastic projected Gross-Pitaevskii equation* (with no fitting parameters) see poster by Sam Rooney. [correct timescale and number of pinned quanta].

• Full SPGPE sims are difficult to interpret in terms of turbulence measures.

• In this work we simulate the stir (333 ms)+ hold (of order 20s) using:

  - 3D damped GPE (SPGPE with noise disabled, i.e. grand canonical).
  - Damping calculated for expt. parameters
    [ASB, C. W. Gardiner, M. J. Davis, PRA 77, 033616 (2008)]
  - N, μ, trap, timescales correspond to expt. parameters
    [S. J. Rooney, ASB, P. B. Blakie, PRA 81, 023630 (2010)]
Figure 4: Numerically obtained BEC column density is shown for $t_h = 30$ ms on a $\mu m^2$ region. The red circle indicates a vortex aggregate with the same sense of rotation as the stir. Blue shapes indicate aggregates with the opposite rotational sense. A vortex dipole and its direction of propagation is indicated by green arrows. See also Fig. S for data showing BECs at $t_h = 5$ s and $t_h = 10$ s.

Experiment: column density after barrier removal and expansion

Simulations: column density after barrier removal
Simulation: stir+hold

Column Density

Phase

$\rho(y, t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \delta(x - y) \, dx$

$\varphi(y, t) = \int_{-\infty}^{\infty} \delta(x - y) \, dx$

$\int_{-\infty}^{\infty} \rho(y, t) \, dy = 1$

$\int_{-\infty}^{\infty} \varphi(y, t) \, dy = \frac{2\pi}{\omega}$
Simulation: persistent current

Column Density

Phase

t = 13.756s
Simulation: vortex injection
Vortex cluster with clockwise rotation

Vortex cluster with anticlockwise rotation

Linear momentum of vortex dipole

Right after injection
Figure S2: (A) Numerically obtained BEC column density is shown for three hold times indicated in the figure. For ease of comparison, the leftmost figure reproduces Fig. 4 of the main text. In each image, red shapes indicate a vortex aggregate with the same sense of rotation as the stir, blue shapes indicate aggregates with the opposite rotational sense. Vortex dipoles and their directions of propagation are indicated by green arrows. The two-vortex aggregates seen at $t_h = 30\text{ ms}$ also exist at $t_h = 160\text{ ms}$ (middle image); each shape represents the same aggregate in the different images. At $t_h = 648\text{ ms}$, one of the aggregates remains, after having traveled clockwise halfway around the BEC. The vortex dipole indicated in this image subsequently interacts with the aggregate and causes its dissociation; no vortex-antivortex annihilation occurs in this interaction.

(B) Vortex aggregation also occurs after the simulated ramp-down of the laser barrier; one set of examples is shown here for a hold time of $t_h = 0\text{ ms}$ and a subsequent $\Delta t = 5\text{ ms}$ barrier ramp-down. The three examples use $\Delta t = 3\text{ ms}$, $\Delta t = 3\text{ s}$, and $\Delta t = 3\text{ s}$ of additional hold time in the trap after the barrier is removed. From left to right, in the leftmost image, five two-vortex aggregates are seen; red shapes indicate vortex aggregates with the same sense of rotation as the stir, blue shapes indicate aggregates with the opposite rotational sense. Only two or three of these aggregates existed at $t_h = 0\text{ ms}$ prior to the beam ramp (red circle, blue triangle, and square). While the remainder formed from vortices that were pinned to the beam and then released into the quantum fluid. By $\Delta t = 3\text{ ms}$ after the laser barrier has been ramped completely off, one of the two-vortex aggregates has acquired a third vortex (red circle). and these three vortices orbit each other. This structure eventually grows to form a loose five-vortex aggregate shown $\Delta t = 3\text{ s}$ after the barrier ramp. This aggregate persists for about $\Delta t = 50\text{ ms}$. The remaining aggregates have dissociated by this time, but three new two-vortex aggregates have formed (rectangles).

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**Timescales**

Cluster turn-over time 10-50ms

Cluster lifetime ~600ms

**Long-lived, spontaneously formed clusters of vortices with same sign!**
Kinetic energy spectra

- What is the mechanism and length scale of forcing?
At the moment of vortex injection:

Mechanism appears to be breakdown of short wavelength density modulations into vortices.

Agrees with predictions of classical double-cascade theory based on energy and enstrophy injection rates extracted from spectra.
Conclusions

- In 2D classical turbulence energy moves to large length-scales through formation of increasingly large eddies.

- In 2DQT this also occurs: long lived vortex clusters spontaneously form.

- Can we find a Kraichnan double-cascade kinetic energy spectrum in a quantum fluid? Yes! (but interpret with caution).

- The quantum analogue of the Great Red Spot appears to be the persistent current: the end-state of spectral condensation.

Quantitative correspondence between classical and quantum fluid turbulence in 2D, linking spectra and dynamics.
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