Physics Teams W17/18:

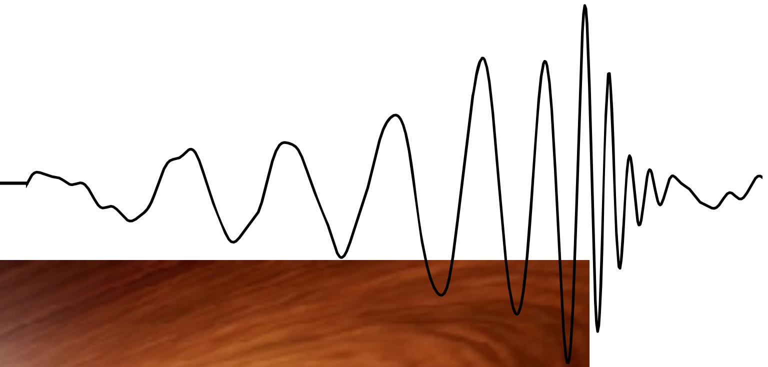
Gravitational Waves

Part I: Peter Reimitz, Patrick Foldenauer and Christian Reichelt

Universität Heidelberg

16. March 2018

Examples of black hole evidence

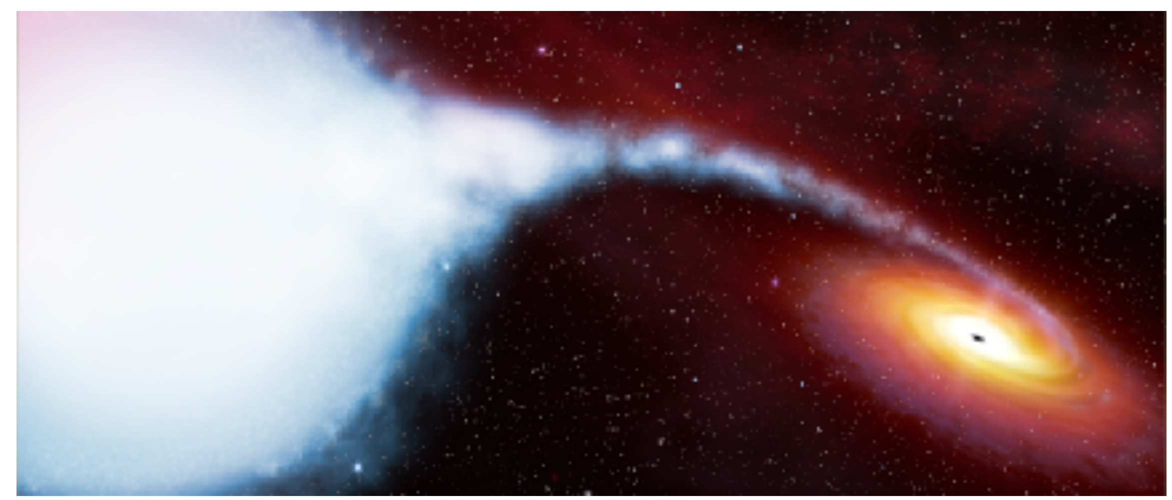


Circumstantial evidence

1969: Donald Lynden-Bell:
Explains that quasars could be powered by supermassive black holes

NASA

1971: Louise Webster, Paul Murdin, and Thomas Bolton:
Discover 15 solar mass invisible companion (Cygnus X-1) to a star



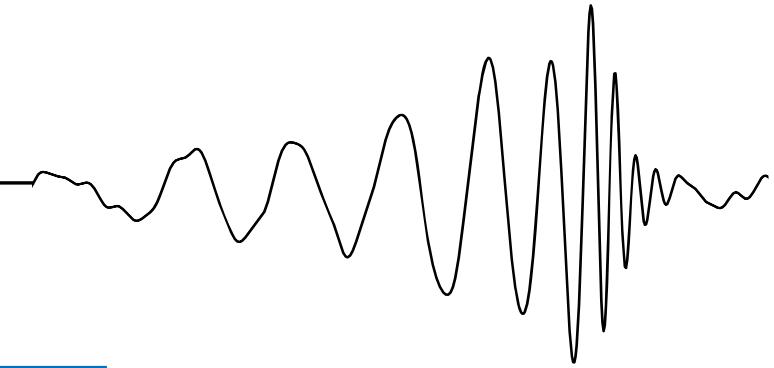
ESA/Hubble

Gravitational waves ➡ Direct evidence for black holes

Soon: Event Horizon Telescope
Shadow of Sagittarius A*

EHT/Hotaka Shiokawa

What are gravitational waves?



Everything starts with the Einstein field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Weak field approximation (linearise) - expanding around flat space

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1$$

Three simple steps:

1. Linearise equation
2. Choose gauge fixing (Lorentz gauge $\Box x^{\mu} = 0$)
3. Define the trace-reversed metric perturbation: $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$

Gives a **relativistic wave equation** for a symmetric two-tensor:

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

Solutions to the wave equation

Consider vacuum solutions $T_{\mu\nu} = 0$, and write a solution

$$\bar{h}_{\mu\nu} = C_{\mu\nu} \mathrm{e}^{ik_\sigma x^\sigma}$$

where k_σ is the wave vector.

What do we learn from the E.O.M. and gauge fixing?

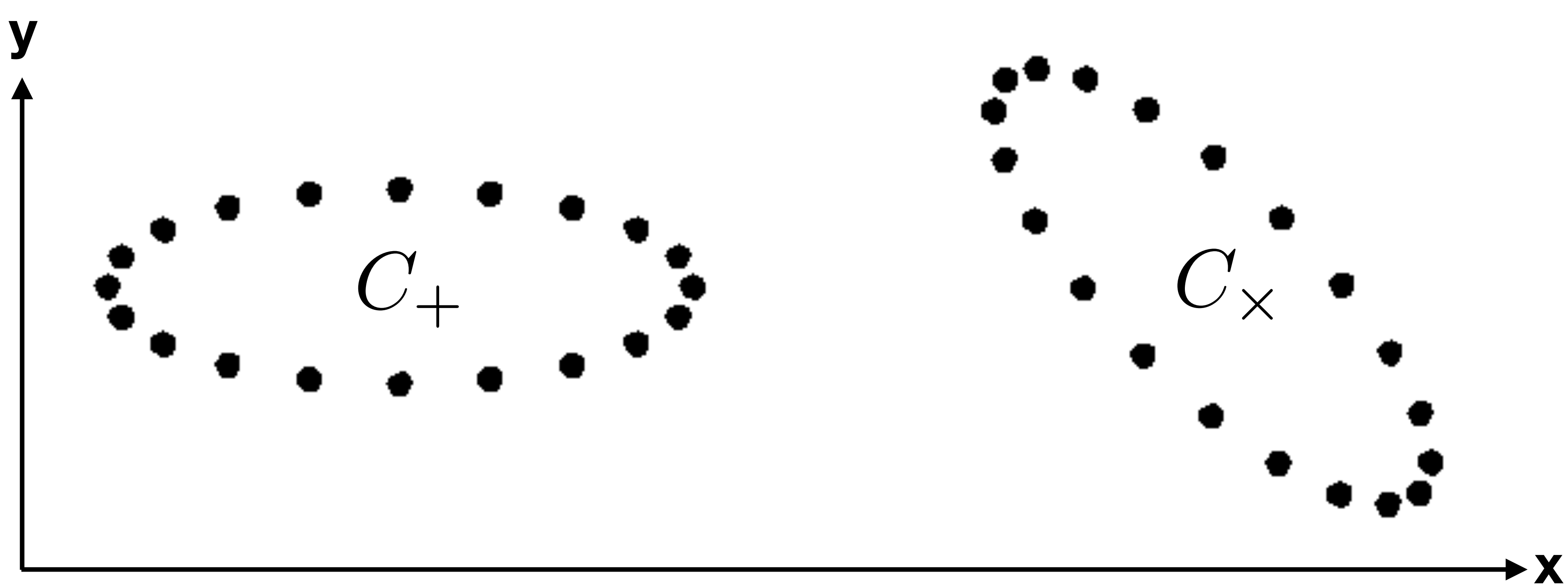
- $k_\mu k^\mu = 0$: Gravitational waves propagate at the **speed of light**
- Parameters of $C_{\mu\nu}$ reduced from 10 to 2: **Two polarisations**

For a wave travelling in x^3 direction ($k^\mu = (\omega, 0, 0, \omega)$) we get:

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_+ & C_\times & 0 \\ 0 & C_\times & -C_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solutions to the wave equation

Effect on separation of particles in the two modes:



Note:
Modes are invariant under 180° rotation

⬇

Spin 2 particle

Generation of gravitational waves? Couple to matter $T_{\mu\nu} \neq 0$

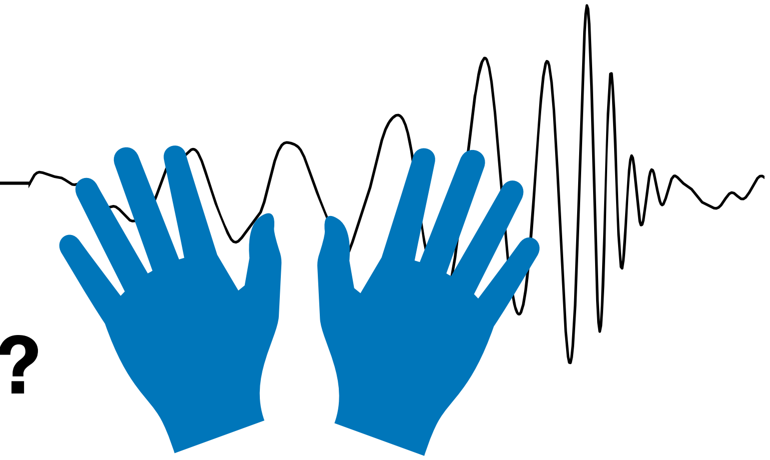
Solution:

$$\bar{h}_{ij}(t, \mathbf{x}) = \frac{2G}{rc^4} \frac{d^2}{dt^2} q_{ij}(t - r/c)$$

Lesson: Gravitational waves generated by the quadrupole moment

$$q_{ij} = \int \rho \left(x^i x^j - \frac{1}{3} \delta^{ij} r^2 \right) d^3x$$

The quadrupole



A moment of hand-waving:

Why is the leading contribution the quadrupole?

GW should fall of as $h_{ij} \sim 1/r$ (energy conservation)

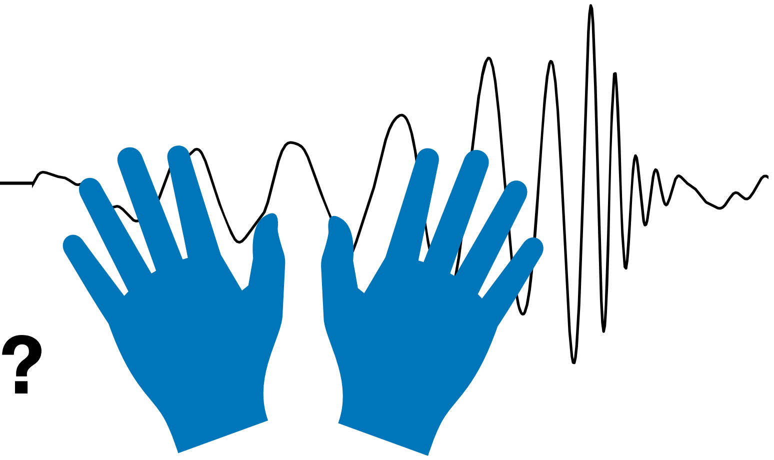
Multipole expansion of gravitational potential

$$\Phi = -\frac{Gm}{r} + \frac{Gq_i}{r^2} + \frac{Gq_{ij}}{r^3} + \cdots$$

So by dimensional analysis

$$h_{ij} \sim \frac{Gm}{c^2 r} + \frac{G}{c^3 r}\frac{\partial q_i}{\partial t} + \frac{G}{c^4 r}\frac{\partial^2 q_{ij}}{\partial t^2} + \cdots$$

The quadrupole



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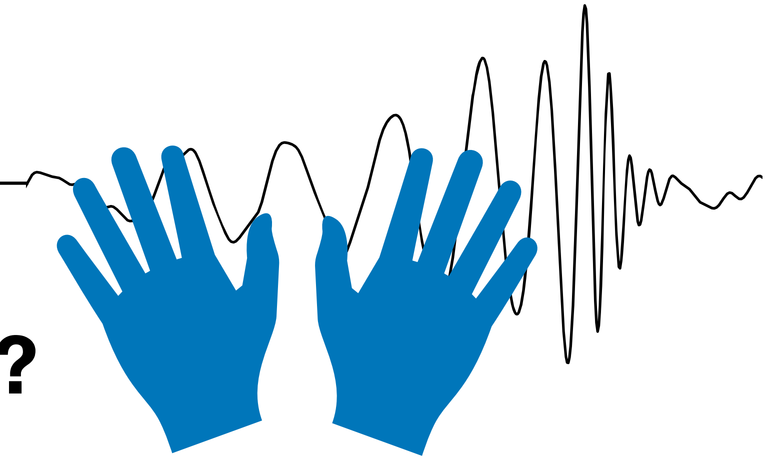
No fluctuations:
Mass conservation

$\frac{\partial q_i}{\partial t} \sim p$

No fluctuations:
Momentum conservation

The quadrupole



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No fluctuations:
Momentum conservation

What about the equivalent "magnetic dipole"?
Forbidden by conservation of angular momentum

In general quanta with spin s has multipoles $l \geq s$

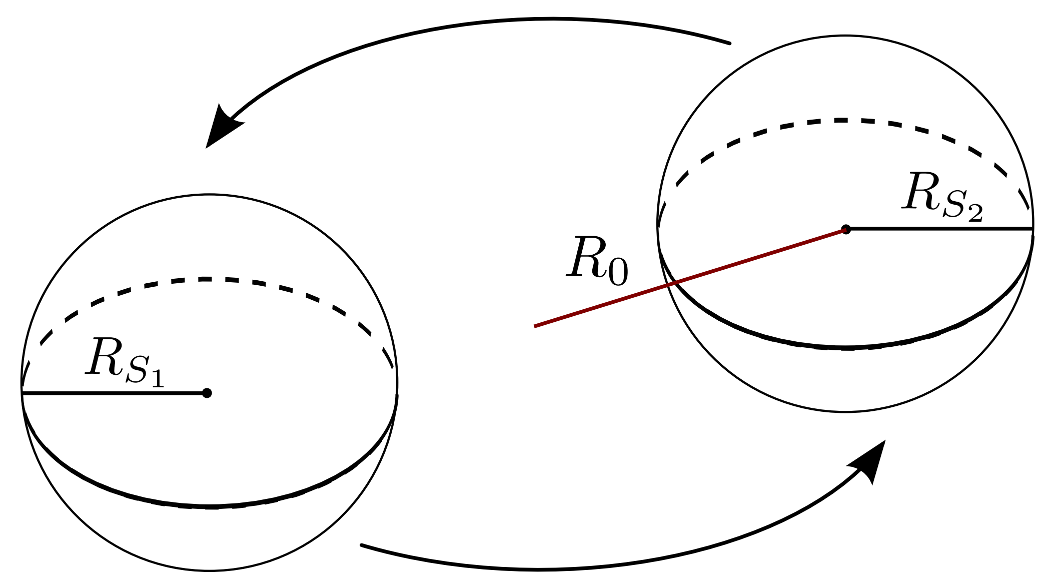
Estimating the amplitude of GW

Going back to the solution

$$\bar{h}_{ij}(t, \mathbf{x}) = \frac{2G}{rc^4}\frac{d^2}{dt^2}q_{ij}(t - r/c)$$

These perturbations are tiny: $2G/c^4 \sim 10^{-44}$

Crude estimate for a black hole binary:



$$|h| \sim \frac{R_{S_1}R_{S_2}}{R_0 r}$$

Assuming close by BHs, masses $M_\odot$
and located in the Virgo cluster ($r \approx 15\,\mathrm{Mpc}$)

$$|h| \sim 10^{-21}$$

How can these tiny fluctuations be measured?

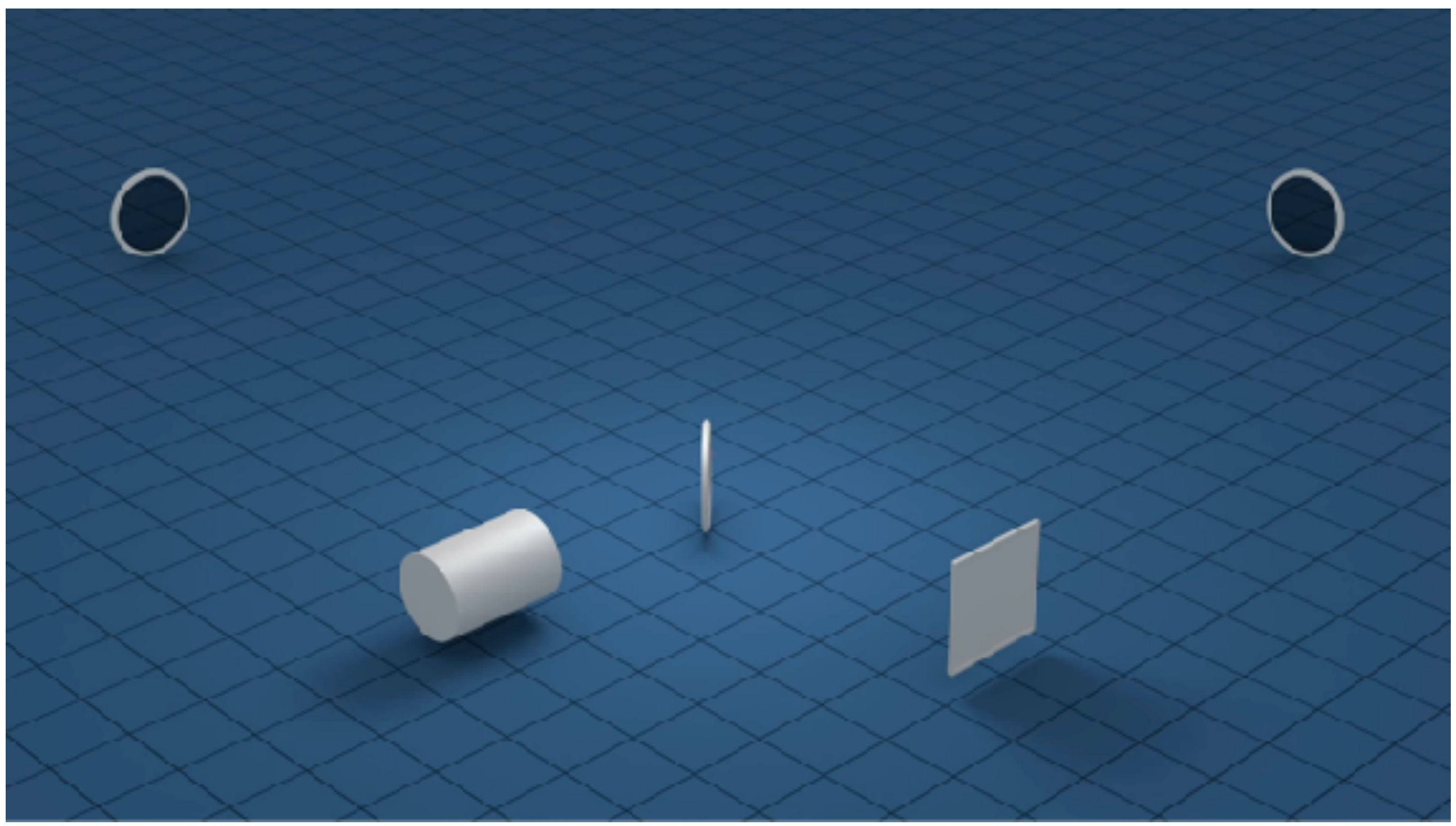
Measurement principle

Proposition: Time difference of light travelling in one arm?

In principle possible, but less attractive

➡ **Laser interferometry:**

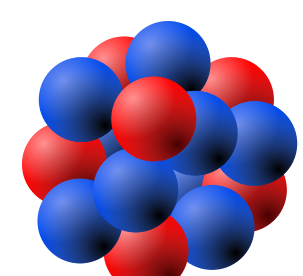
- Quadrupole nature of waves makes Michelson interferometer ideal



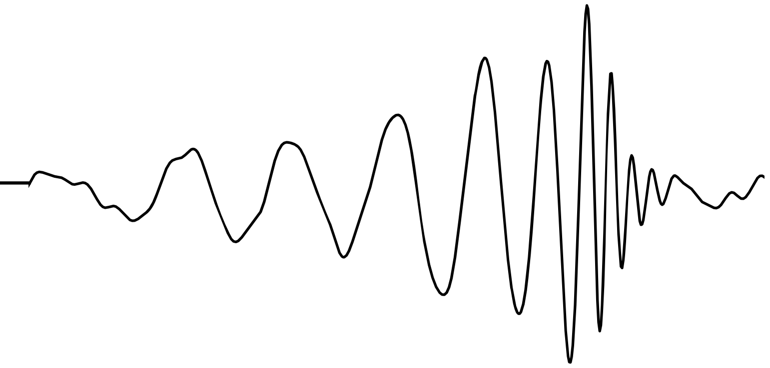
LIGO/T. Pyle

Need long distances and high precision since

$$\Delta L = L \cdot h \approx L \cdot 10^{-21}$$

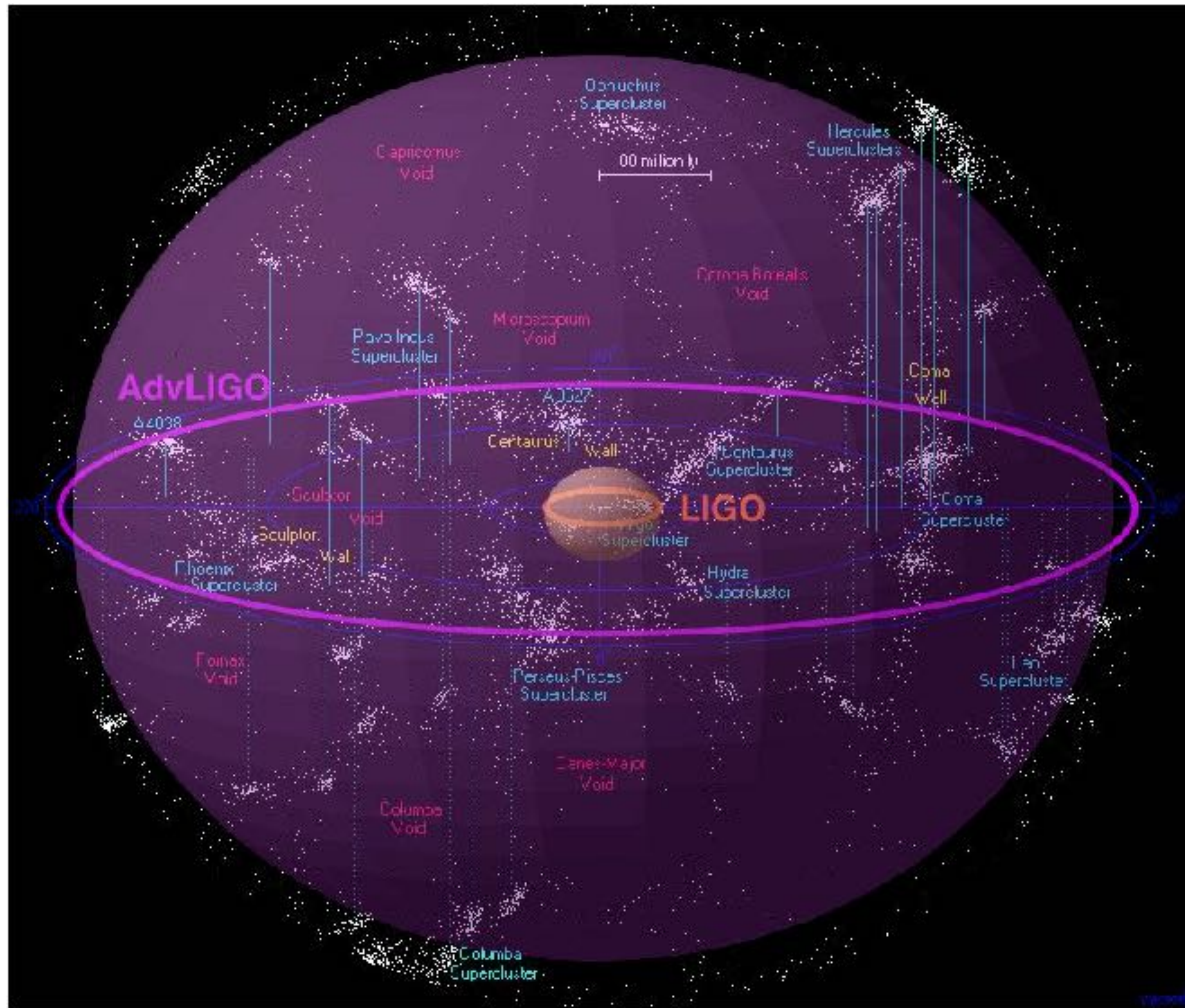
 $\sim 10^{-15}\,\mathrm{m}$

How are gravitational waves produced



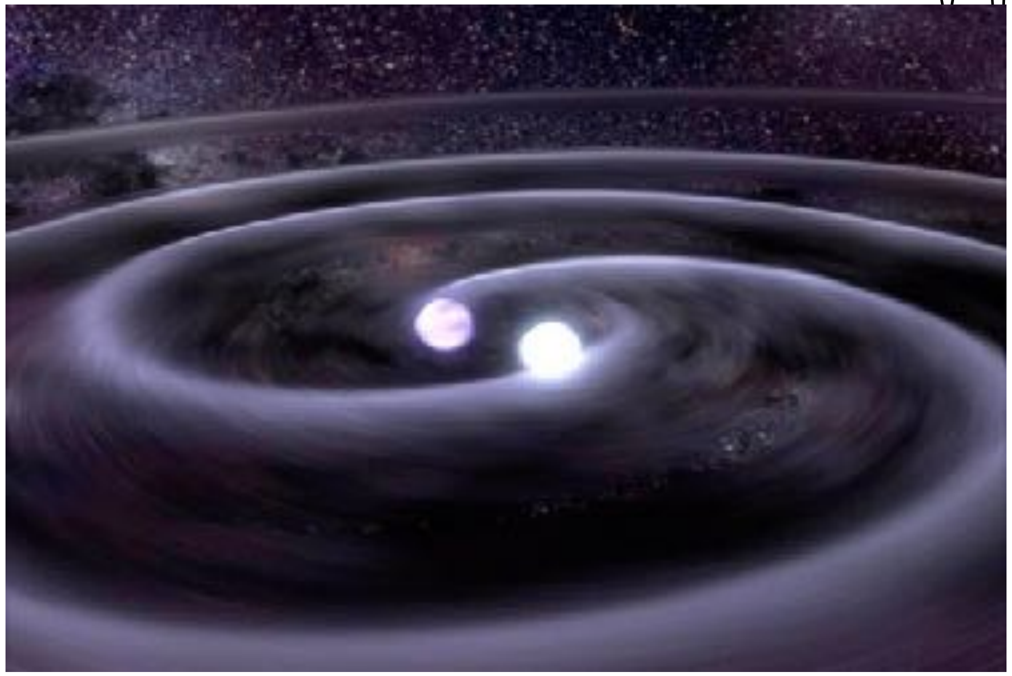
Need the following ingredients:

- Very compact
- Relativistic
- Non spherically symmetric mass distribution
- Strongly time-varying quadrupole moment
- Close enough for observation



Sources of Gravitational Waves

Well-defined



Compact binary coalescene:
Short-lived signal form inspiraling of two compact objects like e.g. neutron stars or black holes (first GW observation)

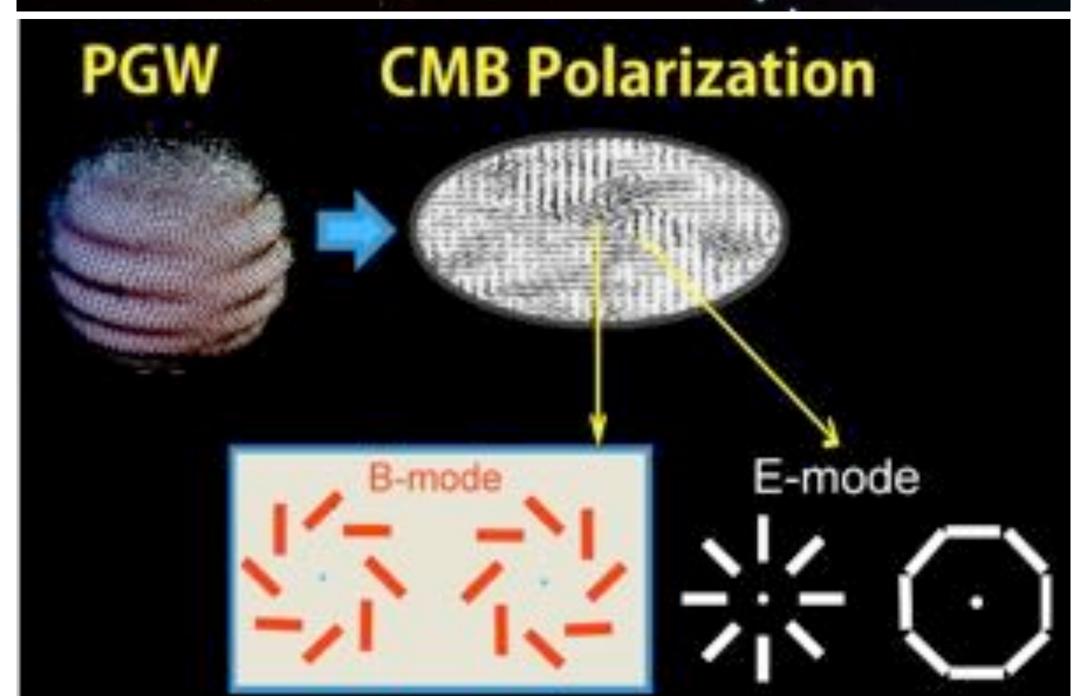
Continuous wave:
Long-lived signal from e.g. spinning (non-axisymmetric) neutron stars

Sources of Gravitational Waves

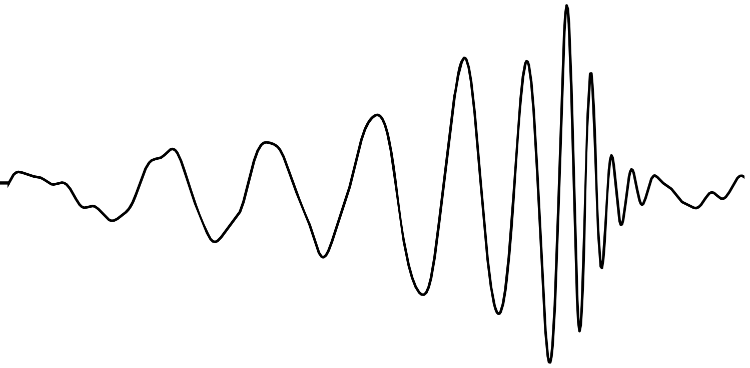
Not so well-defined …

Bursts:
Very short-lived, poorly known transients as e.g. produced from supernovae

Stochastic background:
Long-lived superposition of incoherent sources, e.g. primordial gravitational waves from CMB (see second part)



First discovery: Black hole merger



To see how a coalescence signal is produced we have to look at the total energy luminosity in the radiation zone

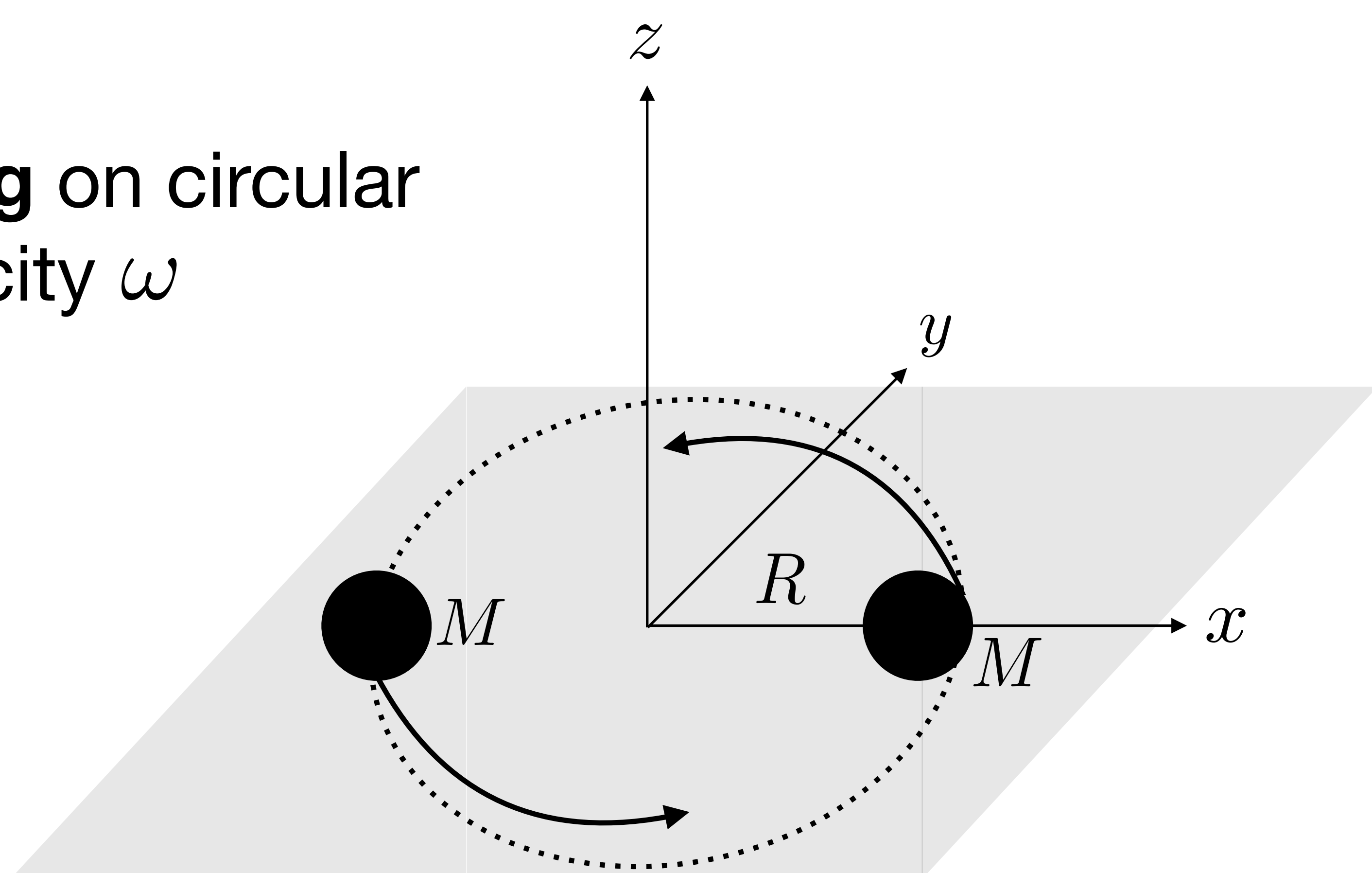
$$\mathcal{L} = \frac{G}{5\,c^5}\langle \dddot{q}_{ij}\dddot{q}^{ij}\rangle \qquad \left(\mathcal{L} = \frac{\mathrm{d}E}{\mathrm{d}t}\right)$$

Information encoded in third time derivative of quadrupole moment q^{ij}

(Over)simple Example:
Two objects of mass M **inspiraling** on circular orbit of radius R and angular velocity ω

Total energy $E_{\mathrm{tot}} = -\frac{GM^2}{4\,R(r)}$

Energy loss $\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{GM^2}{4R^2}\frac{\mathrm{d}R}{\mathrm{d}t}$



First discovery: Black hole merger

One finds for the quadrupole moment:

$$q^{ij} = M \sum_{n=1}^{2} x_n^i x_n^j = MR^2 \begin{pmatrix} 1+\cos(2\omega t) & \sin(2\omega t) & 0 \\ \sin(2\omega t) & 1-\cos(2\omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Using Kepler's third law $\omega^2 = \frac{GM}{4R^3}$ one can find a differential equation for the radius $R(t)$ using luminosity and energy loss:

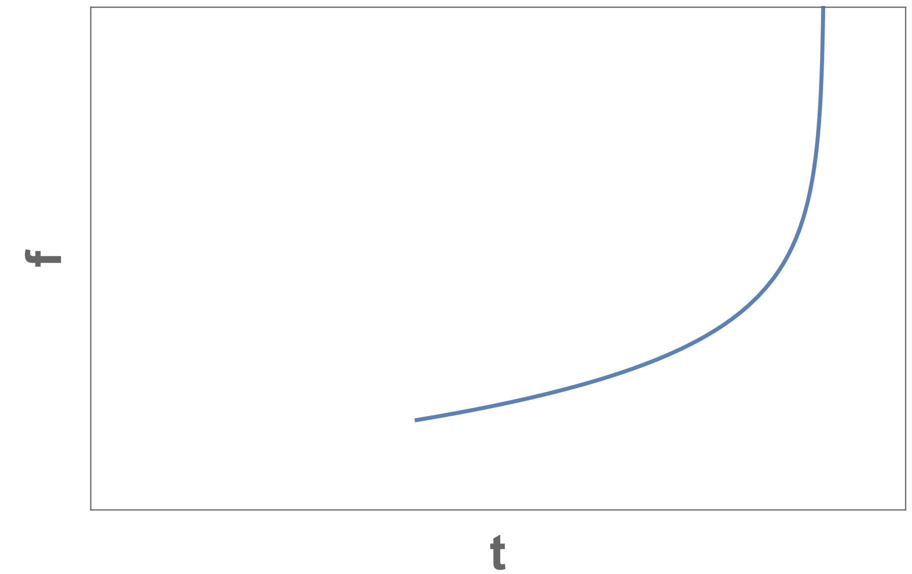
$$\mathcal{L} = \frac{\mathrm{d}E}{\mathrm{d}t} \qquad \Rightarrow R^3 \frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{8}{5} G^3 M^3$$

$$\boxed{R(t) = \left[\frac{32 G^3 M^3}{5} (t_{\mathrm{coal}} - t) \right]^{\frac{1}{4}}}$$

First discovery: Black hole merger

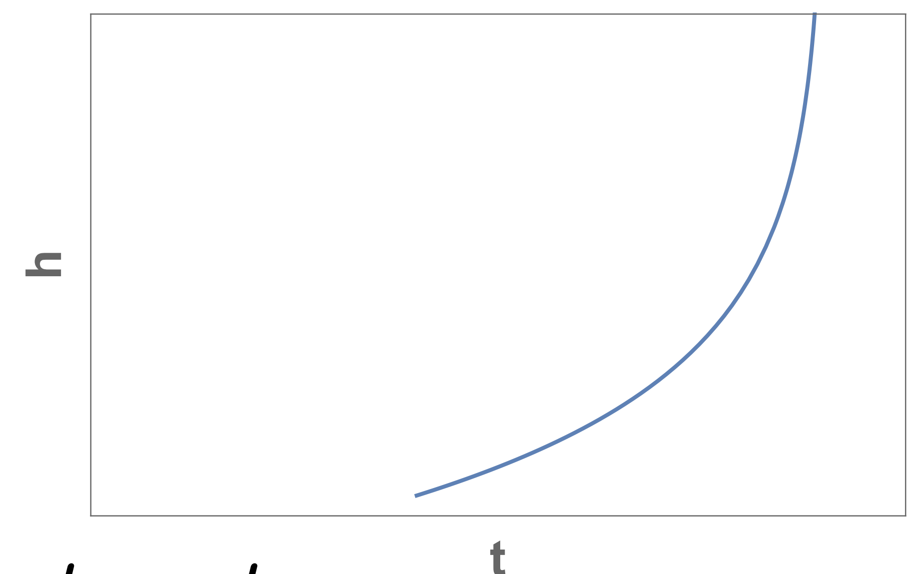
With the time dependent radius we can obtain the frequency

$$f_{\rm GW} = \frac{2\omega}{2\pi} = \frac{[2 \cdot 5^3]^{\frac{1}{8}}}{8\pi(GM)^{\frac{5}{8}}(t_{\rm coal} - t)^{\frac{3}{8}}}$$



and the amplitude

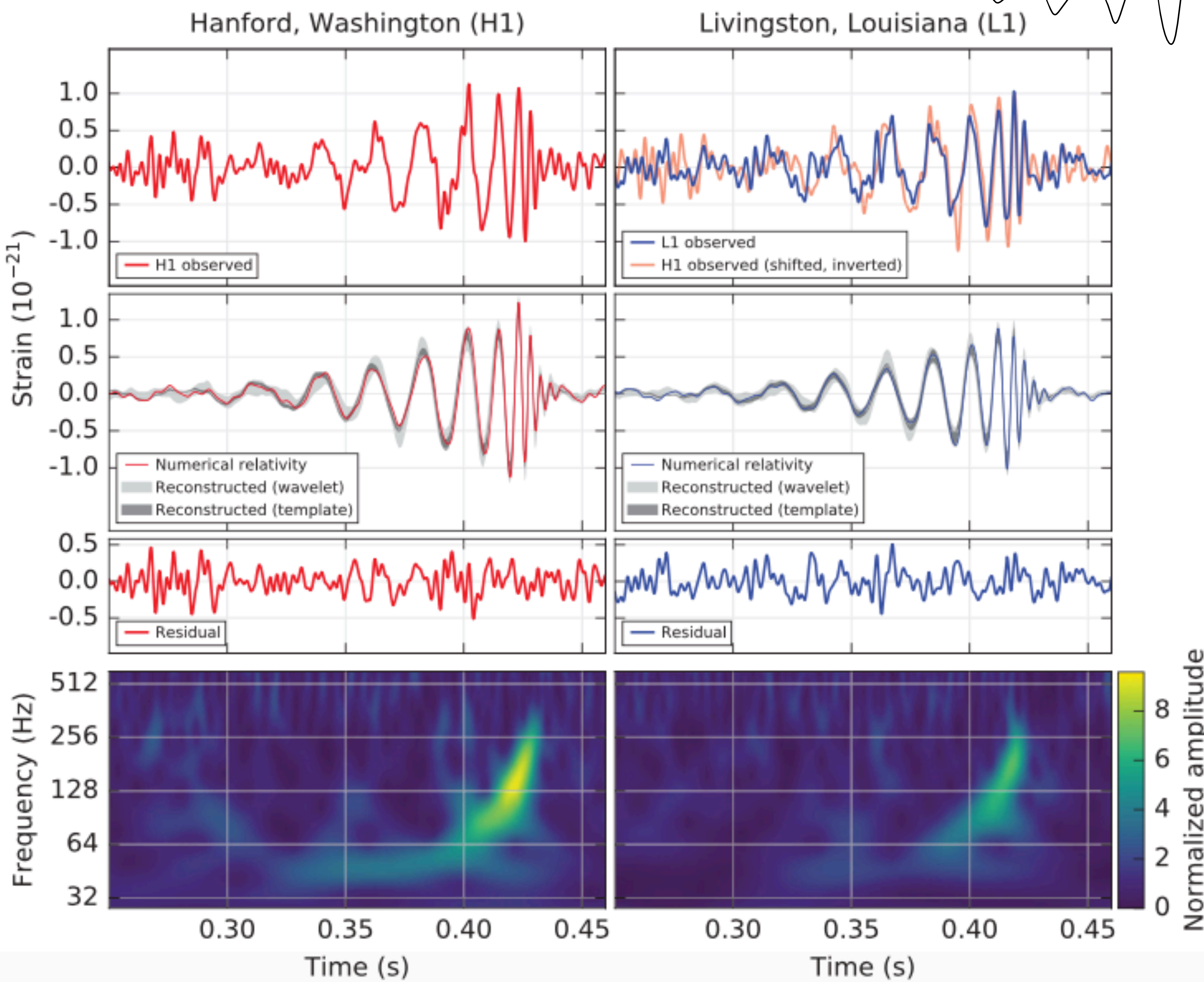
$$h_0 = \frac{1}{r}\left[\frac{5G^5M^5}{2}\right]^{\frac{1}{4}} \frac{1}{(t_{\rm coal} - t)^{\frac{1}{4}}}$$



Modulation of both frequency and amplitude as $t \rightarrow t_{\rm coal}$

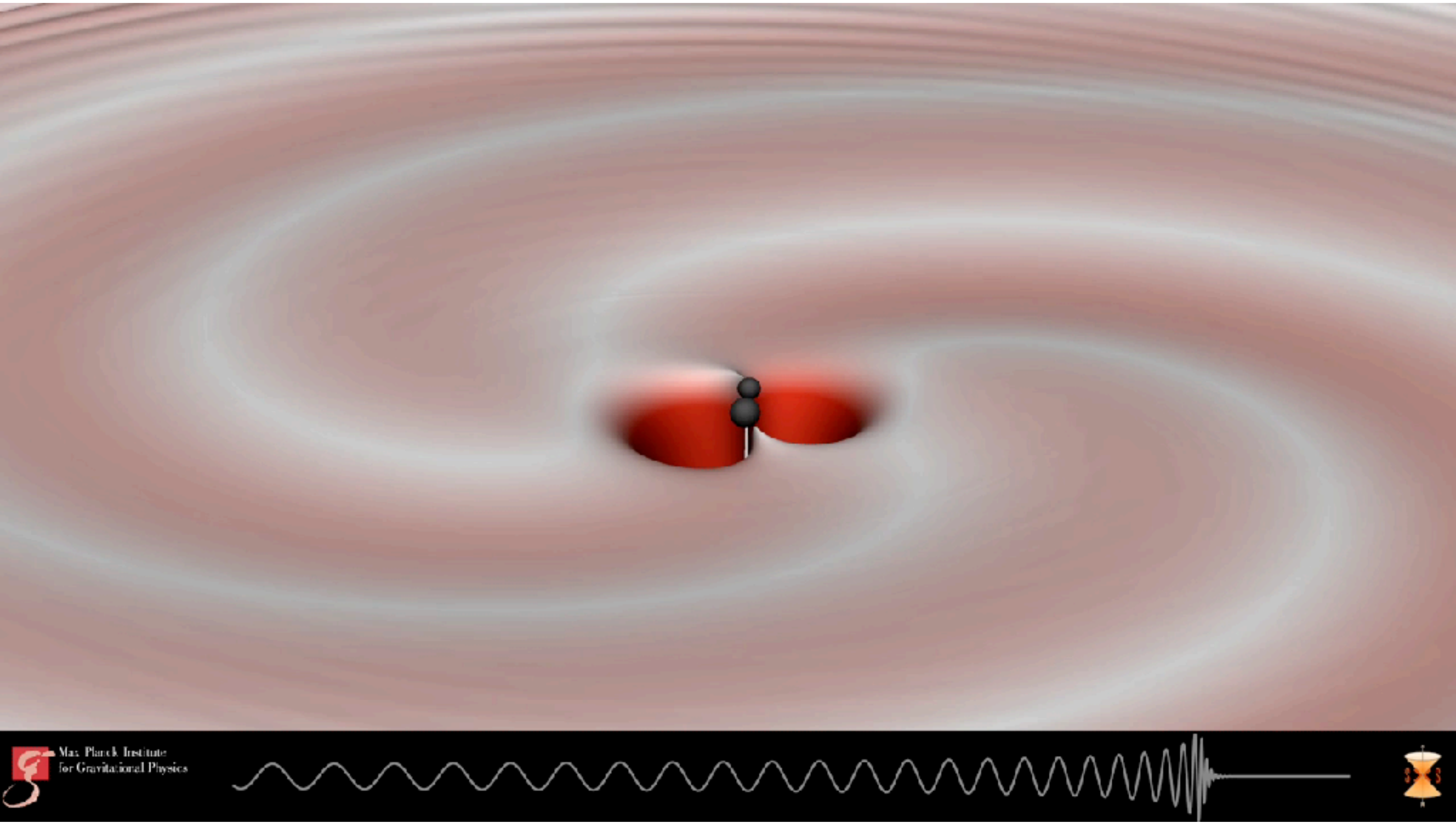
Can extract M from $f_{\rm GW}$ and the distance r to us from h_0

First discovery: The Signal



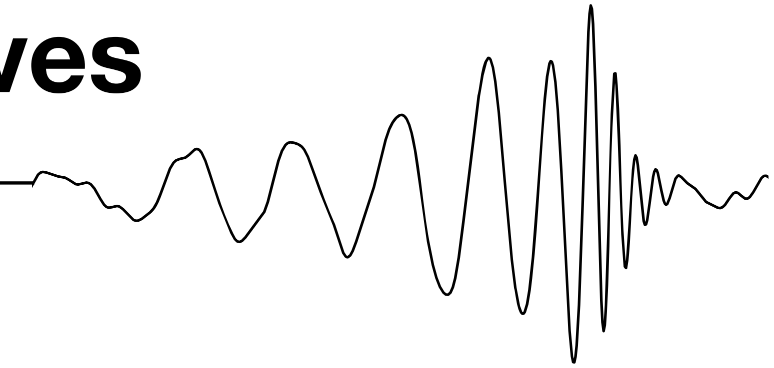
Simulation of GW150914

First black hole merger event detected



S. Ossokine , A. Buonanno, R. Haas (Max Planck Institute for Gravitational Physics)

Detectable effects of gravitational waves



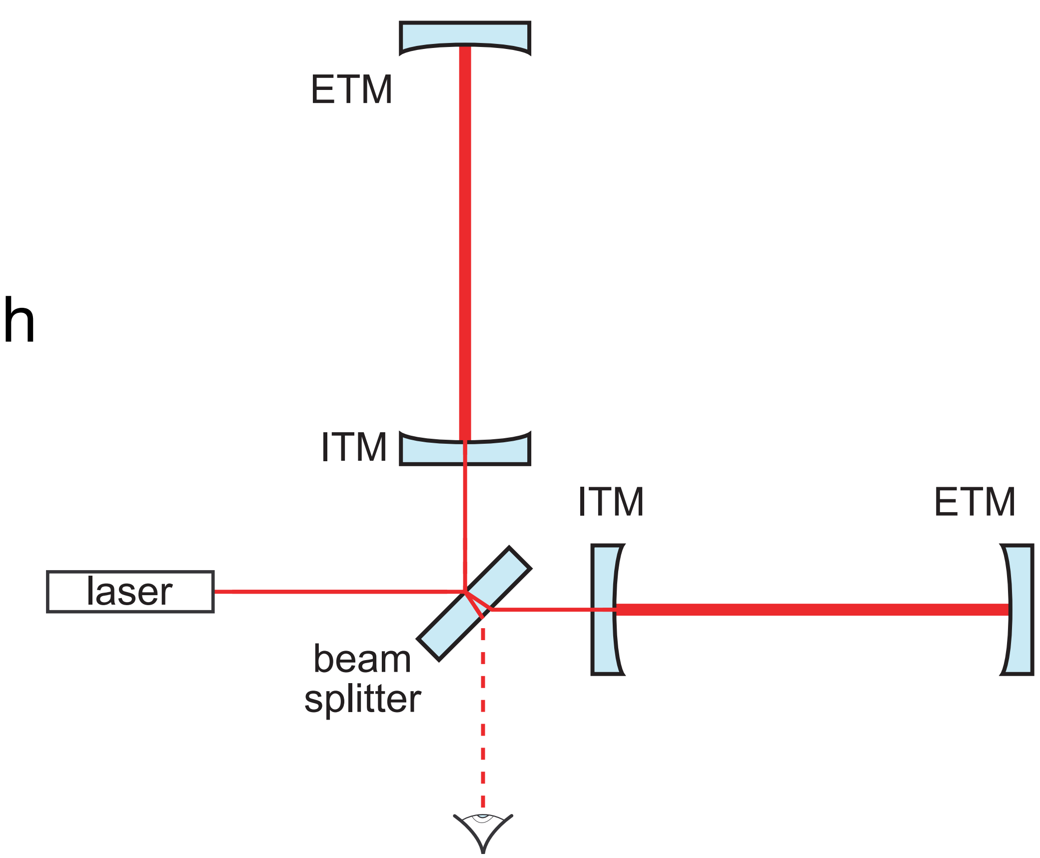
Effect of gravitational waves:
Change of separation of two masses in space, i.e. the arm length of our detector

$$\Delta L \sim \frac{hL}{2} \qquad \Rightarrow \qquad h \sim \frac{2\Delta L}{L} \lesssim 10^{-21}$$

⇨ longer detectors imply larger measurable separations ΔL

⇨ very small GW amplitudes h can only be resolved if separation of masses is long enough

⇨ effective arm length is increased by bounces between mirrors



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Signal-to-noise ratio:

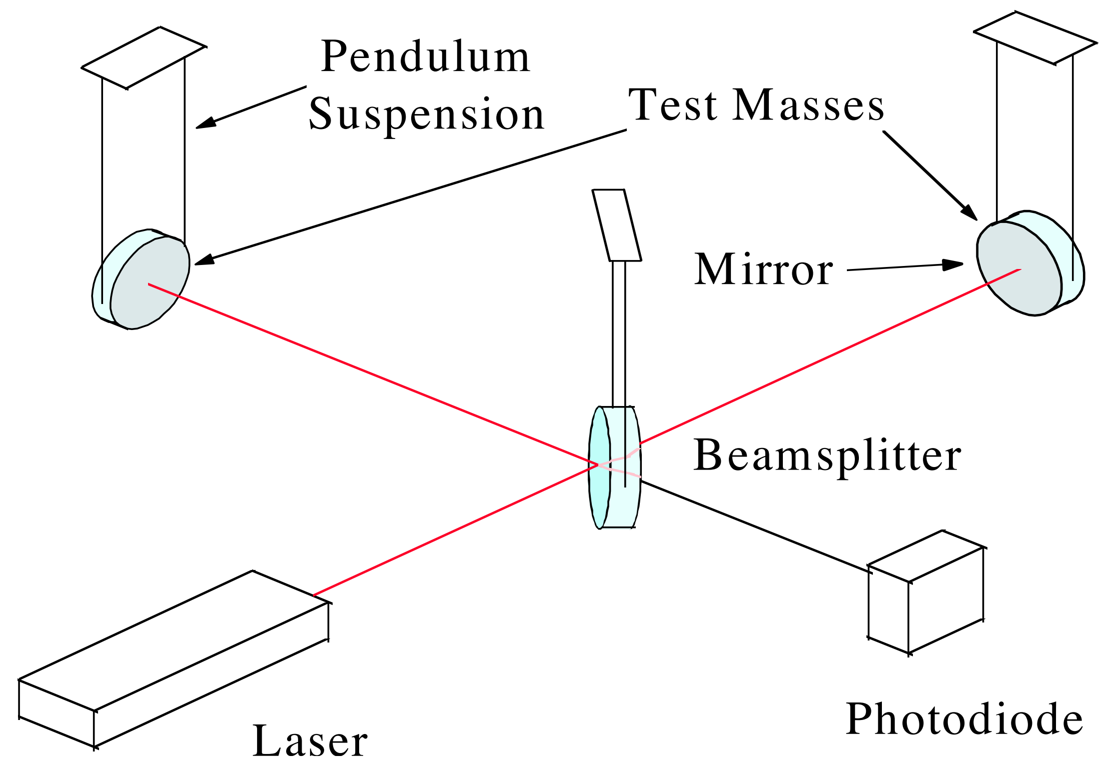
$$\left(\frac{S}{N}\right)^2 \sim \int \frac{|\tilde{h}(f)|^2}{S_n(f)} df$$

For $h \sim 10^{-21}$ the noise level must have an amplitude spectral density

$$\sqrt{S_n(f)} \simeq 10^{-23}\ \mathrm{Hz}^{-1/2}$$

in frequency range of interest

⇨ sensitivity of gravitational wave detectors limited by noise



- System of **freely** suspended masses
- Resonant frequencies of pendulums should be smaller than frequencies of waves

Main noice sources

- **Residual gas noise** ➡ eliminated by vacuum pipes
- **Photoelectron shot noise** (PSN) ≳ 200 Hz

$$h_{\text{min}} \sim \frac{1}{bL}\left(\frac{\hbar c\tilde{\lambda}}{\tau I_0}\right)^{1/2}$$

Minimize w.r.t. I_0 **!**

- **Radiation pressure noise** (RPN) ~ 10-50 Hz

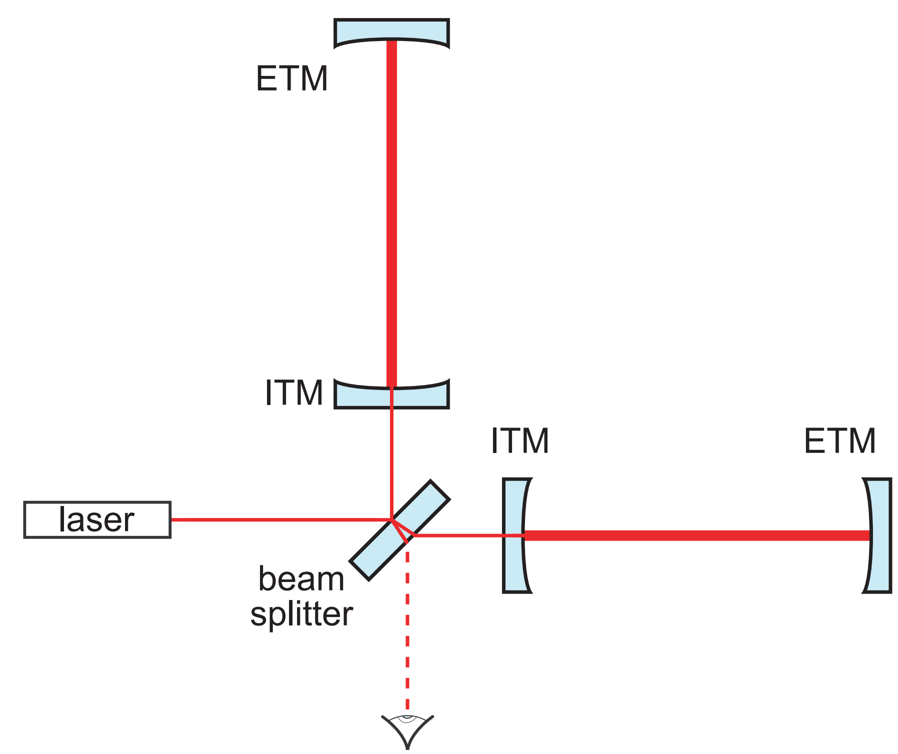
$$h_{\text{min}} \sim \frac{\tau}{m}\frac{b}{L}\left(\frac{\tau\hbar I_0}{c\tilde{\lambda}}\right)^{1/2}$$

High-power Laser needed:

1. Use of Fabry-Pérot resonant cavities

⇒ required laser power reduced to kW

⇒ still very large!



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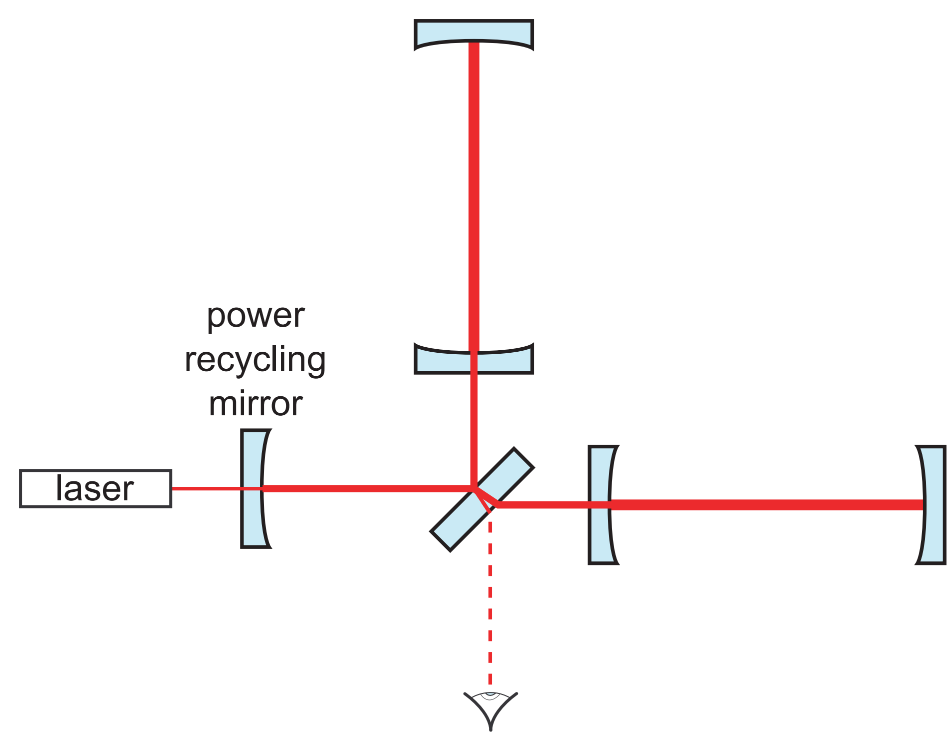
High-power Laser needed:

1. Use of Fabry-Pérot resonant cavities
2. Power recycling:

   Power recycling mirror in front of laser
   ⇒ power built-up inside interferometer

   ⇒ required power reduced to 10 W



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→ Minimize w.r.t. I_0

$$h_{\min} \sim \frac{1}{L}\left(\frac{\tau\hbar}{m}\right)^{1/2}$$

Standard Quantum Limit!

Note: Operation with optimal intensity for minimal PSN and RPN only depends on test mass!

Heisenberg Uncertainty: $$\Delta L \Delta p = \Delta L m \frac{\Delta L}{\tau} = \hbar$$

$$\Rightarrow \Delta L^2 = \frac{\tau\hbar}{m}$$

Main noice sources

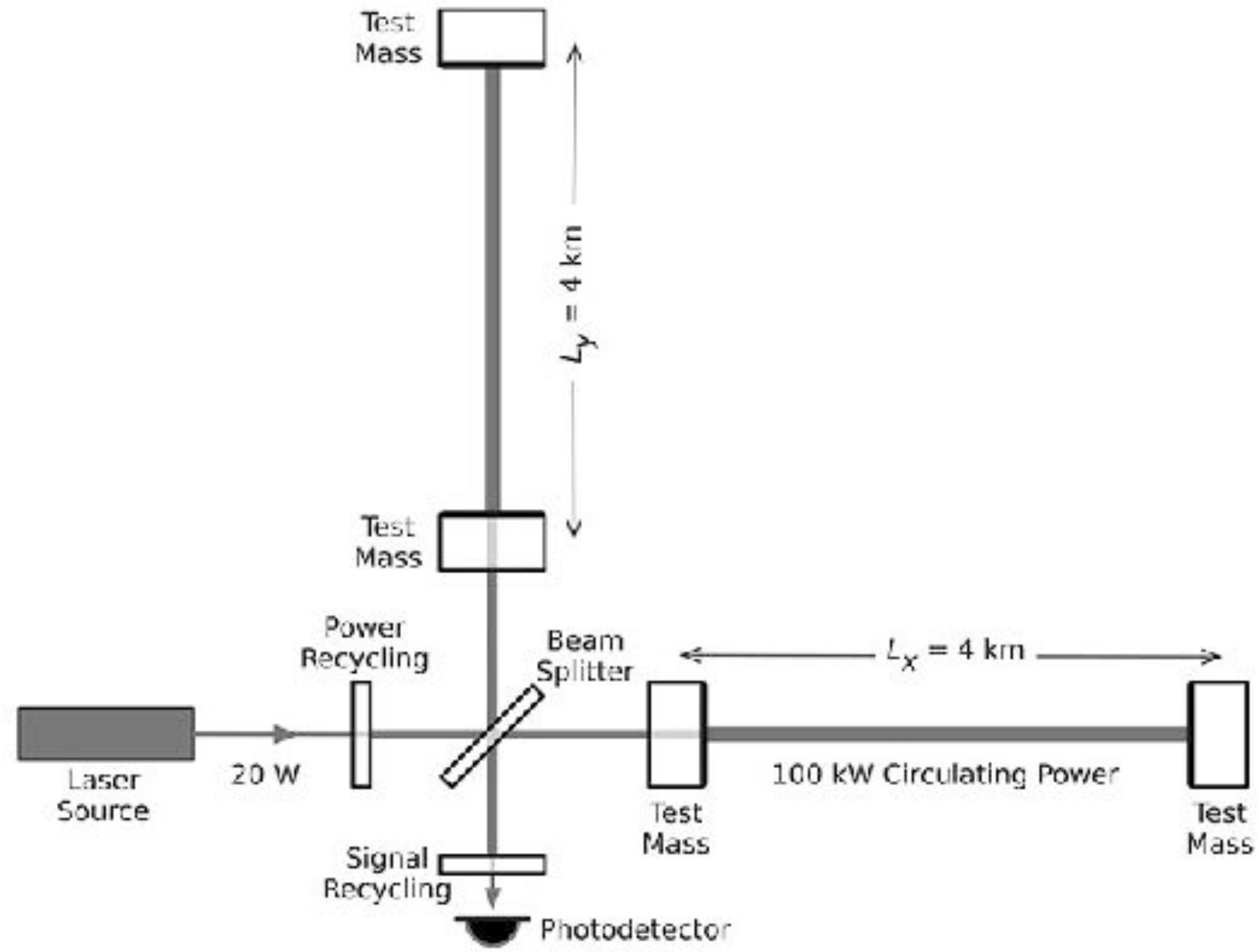
$$h_{\rm min} \sim \frac{1}{L}\left(\frac{\tau\hbar}{m}\right)^{1/2}$$

E.g. for Ligo:

$$m = 100 \text{ kg}, L = 4 \text{ km}, \tau = 1 \text{ ms}$$

⇨ $h_{\rm min} \sim 10^{-23}$

Controlled very well!



Phys. Rev. Lett. **116**, 061102

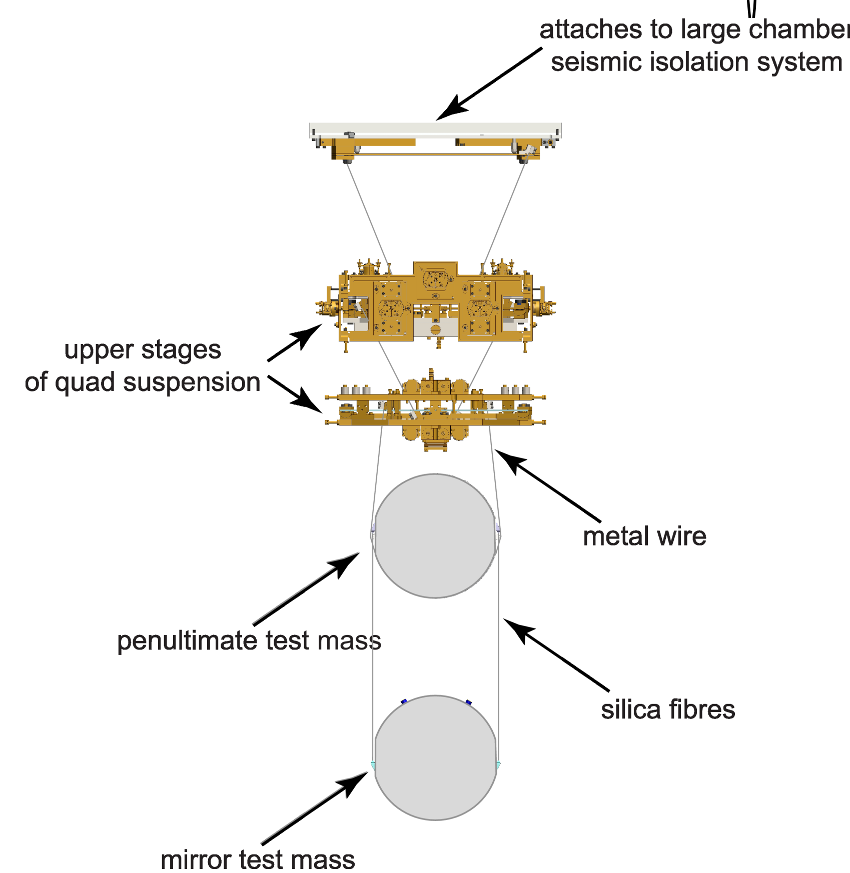
Signal recyling:

- Further enhances sensitivity of detector
- “resonating” signal in optical cavities
- Narrowing detection bandwidth by choosing suitable reflexivity
- center of frequency band set by length of formed cavity

Main noice sources

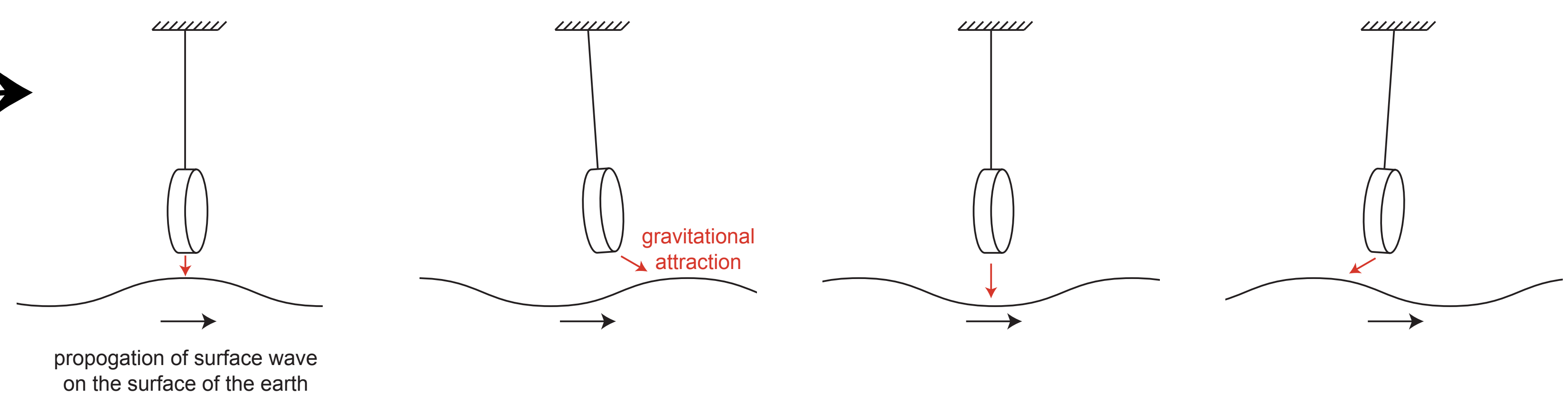
- **Thermal noise:**

1. resonant modes of pendulum oscillator
2. Internal material oscillator modes



- **Seismic noise** $\lesssim$ 60 Hz

1. Vibrations of the ground
2. Gravity gradient noise (caused by seismic surface waves)



<u>Options</u>:

1. Monitor and subtraction method
2. Far from sources

⇨ quiet location

⇨ underground

⇨ space missions

Main noice sources

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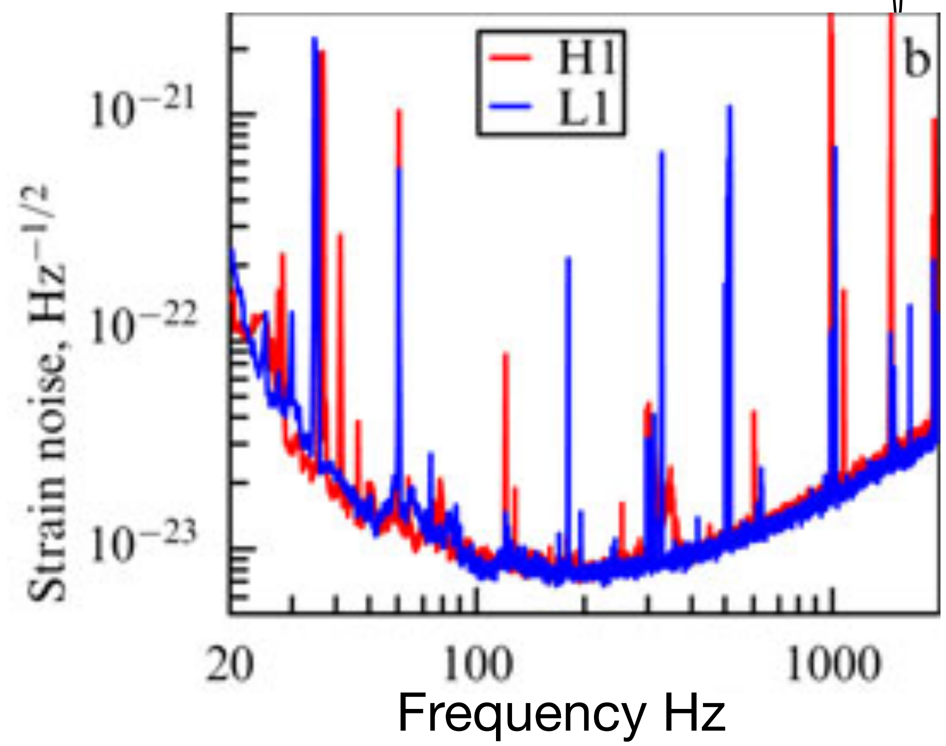

- **Seismic noise** ≲ 60 Hz

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<u>Options</u>:

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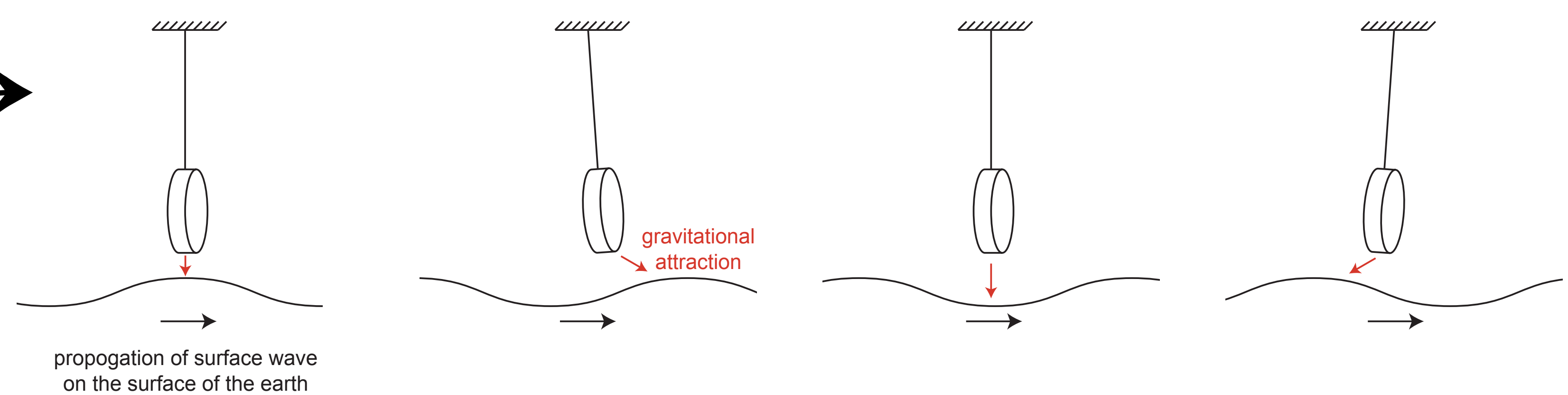
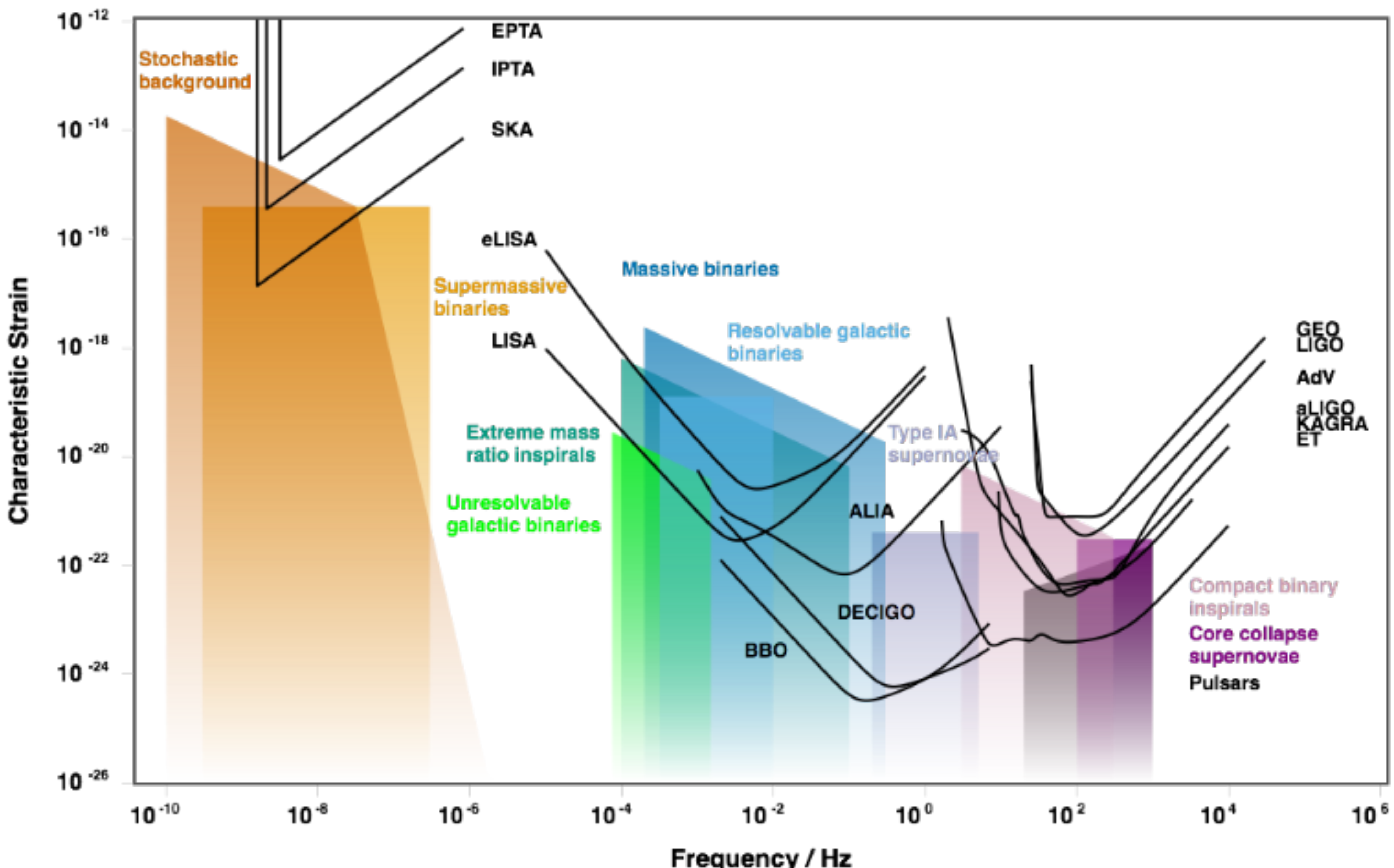


Reach of gravitational wave detectors



http://rhcole.com/apps/GWplotter/

Technology (and going underground)

⇩

Objects further away

More massive objects

⇦

From earth to space

Conclusions

- Entered a new area of gravitational wave astronomy
- First direct black hole evidence with aLIGO (5 BH events by now)
- Space-baced missions will increase sensitivity to new sources like rotating neutron stars, etc…
- A bright future lies ahead of us