Where the Lorentz-Abraham-Dirac equation for the radiation reaction force fails, and why the "proofs" break down

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Abstract:
We calculate the energy radiated coherently by a system of N charged non relativistic particles. It disagrees with the energy loss which is obtained if one employs the Lorentz Abraham Dirac (LAD) equation for each particle, and sums up the contributions. This fact was already clearly stated in the classical literature long ago. The reason for the discrepancy is the omission of the mixing terms in the Poynting vector. For some simple systems we present a generalized equation for the radiation reaction force which cures this defect. The counter examples show that the LAD equation cannot be generally valid and that all "proofs" must fail somewhere. We demonstrate this failure for some popular examples in the literature.

1 Introduction

It is known for more than hundred years now, that an accelerated charge radiates, and that the energy for this radiation should be provided by a radiation reaction force f^μ which acts like a friction to the motion of the particle. The earliest and most popular form of this reaction force is the Lorentz-Abraham-Dirac (LAD) force [1][2][3],

$$f^\mu = \frac{2}{3}e^2(\ddot{u}^\mu + \dot{u}^2 u^\mu), \tag{1}$$

with u^μ the four velocity, and the dot denoting differentiation with respect to the proper time τ. Due to strange properties like the possible appearance of runaway solutions one often modifies this equation by introducing the zero order solution of the equation of motion into the rhs, arriving at the equation, usually called the Landau-Lifschitz (LL) equation [4]:

$$f^\mu = \frac{2}{3}\frac{e^2}{m}(\frac{d}{d\tau}F^\mu + (F\dot{u})u^\mu), \tag{2}$$

with F^μ the external force. See also [5][6][7]. Some of these authors consider this equation as the fundamental one and even claim it as being "exact" or providing the "correct" equation.

The non relativistic limit of equations (1), (2) is

$$\mathbf{f} = \frac{2}{3}e^2\ddot{\mathbf{v}} \approx \frac{2}{3}\frac{e^2}{m}\frac{d}{dt}\mathbf{F}. \tag{3}$$

There is still a lively discussion about these equations. In particular there are various papers which claim to have provided a general proof. This is astonishing, because already Abraham [2], (& 15, p. 119) clearly states that the equation cannot be always correct! As an example he mentions a number of equally distributed electrons which move along a circle with equal velocities. Each electron radiates less than predicted by (1). In the limit of many electrons the motion of the electrons corresponds to a stationary current; there is no radiation at all.

The effect has nothing to do with the subtle differences between equations (1) and (2). The problem arises already for weak fields where these equations are equivalent, and where the non relativistic limit (3) is applicable. No complicated mathematics or subtle physics is involved in the arguments, just old fashioned textbook electrodynamics.

The relevant point is simple. The energy density as well as the Poynting vector are quadratic in the fields. Therefore it does not make sense to add up the energy of the radiation fields created by various particles. One has first to superimpose the fields, and subsequently calculate energy densities and momentum flows. This introduces non diagonal mixing terms which can change the standard arguments considerably. This simple fact, already emphasized by Abraham, is well known and discussed in books on accelerator physics [8]. But it is usually not even mentioned at all in reviews on radiation reaction [9][10][11][12]. One can only find a brief remark in a different context in [10], p. 382. To our knowledge the problematics appears to be completely unknown, forgotten, or ignored by the authors who try to proof the LAD- or the LL equation, apparently being unaware that they try to proof an equation which fails already for simple counter examples. All general “proofs” must be incorrect.

In the present paper we point out all this in some detail. In sect. 2 we calculate the radiation of N non relativistic particles and show that the application of the LAD equation for each of the particles would violate energy conservation. In sect. 3, restricting to simple situations, we present a modified equation for the reaction force $\mathbf{f}_i$ which acts on particle i. It gives the correct energy balance. Our equation necessarily depends not only on the motion of the particle i under consideration, but also on the motion of all the other particles. In sect. 4 we present some simple examples. Sect. 5 analyzes some of the “proofs” in the literature and shows where they break down. In sect. 6 we investigate under which circumstances the standard LAD equation could be correct. Sect. 7 gives a summary.

2 Coherent radiation of N particles

Consider N particles with masses m_i and charges e_i, moving with non relativistic velocities $v_i \ll c$. We assume that for early times $t < t_a$ and late times $t > t_b$ the particles are far apart, such that any interaction and acceleration is negligible. During the intermediate time between t_a and t_b the charges can approach each other, are accelerated, and can radiate. For large negative times $t < t_a$ there shall be no fields except the fields of the freely moving charges. This implies that one has to use retarded fields in the following. The energy which is radiated during the interval $t_a < t < t_b$ can be calculated by slightly generalizing the usual procedure. Choose a sphere with a large radius r, with the particles roughly in the center of the sphere. The non relativistic limit of the retarded radiation fields created by particle i is given by the well known expressions

$$\mathbf{E}_i(\mathbf{r},t) = e_i \frac{\mathbf{R}_i \times [\mathbf{R}_i \times \mathbf{a}_i(t'_i)]}{{R_i}^3}, \quad \mathbf{B}_i(\mathbf{r},t) = \frac{\mathbf{R}_i}{R_i} \times \mathbf{E}_i(\mathbf{r},t). \qquad (4)$$

Here $\mathbf{R}_i = \mathbf{r} - \mathbf{x}_i(t'_i)$ is the difference between the point $\mathbf{r}$ on the sphere where the fields are considered, and the position $\mathbf{x}_i(t'_i)$ of particle i at the retarded time t'_i, defined implicitly by the relation $R_i = t - t'_i$. The acceleration $\mathbf{a}_i(t'_i) = d\mathbf{v}_i(t'_i)/dt'_i$ of particle i enters at the retarded time t'_i. If $r > t_b - t_a$, the radiation arrives at the surface of the sphere at a late time $> t_b$ where the charges are essentially free and do no longer emit radiation. Furthermore choose r large compared to the maximal distance Δ_{max} of $|\mathbf{x}_i(t'_i) - \mathbf{x}_j(t'_j)|$ in the interval $t_a < t'_i, t'_j < t_b$. Then one has $\mathbf{R}_i = \mathbf{r} + O(\Delta_{max})$ for all $i = 1, \cdots, N$. One may replace $\mathbf{R}_i$ by $\mathbf{r}$, and the relative error can be made as small as one likes, if r is taken large enough. The total radiation fields $\mathbf{E}(\mathbf{r}, t) = \sum_{i=1}^N \mathbf{E}_i(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t) = \sum_{i=1}^N \mathbf{B}_i(\mathbf{r}, t)$ become

$$\mathbf{E}(\mathbf{r}, t) = \frac{\mathbf{r} \times [\mathbf{r} \times \sum_{i=1}^N e_i \mathbf{a}_i(t'_i)]}{r^3}, \quad \mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{r}}{r} \times \mathbf{E}(\mathbf{r}, t). \tag{5}$$

The retarded times t'_i in (5) are all different, but for non relativistic motions the differences of the accelerations at different times are of order v. Therefore one can replace all retarded times t'_i by some common average retarded time t'. Indeed this argument needs a more careful investigation which we postpone to sect. 4. But for the cases where the argumentation is valid one may replace $t'_i \to t'$. Eq. (5) then has the same form as the radiation field of a single particle, with the replacement $e\mathbf{a}(t') \to \sum_{i=1}^N e_i \mathbf{a}_i(t')$. Therefore one can simply repeat the standard derivation of the Larmor formula and ends up with the following expression for the energy which was radiated off during the time interval $t_a < t < t_b$:

$$E_{rad} = \int_{t_a}^{t_b} \frac{2}{3} (\sum_{i=1}^N e_i \mathbf{a}_i(t))^2 dt. \tag{6}$$

One could have guessed this formula immediately, but we took care to derive it explicitly. Indeed such a straightforward generalization is only possible for simple systems. For more complicated N-particle systems as well as for relativistic motions the differences between the accelerations at different times t'_i become relevant. In this case the derivation of a closed expression like (6) appears hopeless.

3 Radiation reaction force

Recall the procedure in the case of a single particle. The integral over the work performed by the radiation reaction force $\mathbf{f}$ on the particle is identified with the radiated energy, subsequently one performs a partial integration:

$$\int_{t_a}^{t_b} \mathbf{v}(t)\mathbf{f}(t)dt =$$
$$-\int_{t_a}^{t_b} \frac{2}{3}e^2\mathbf{a}^2(t)dt = \int_{t_a}^{t_b} \frac{2}{3}e^2\mathbf{v}(t)\dot{\mathbf{a}}(t)dt - \frac{2}{3}e^2\mathbf{v}(t)\mathbf{a}(t)|_{t_a}^{t_b}. \tag{7}$$

The last term vanishes if the acceleration vanishes at the endpoints. By identifying the integrands one obtains the well known non relativistic version (3) of the LAD equation, $\mathbf{f}(t) = \frac{2}{3}e^2\dot{\mathbf{a}}(t)$. Clearly this formula fails in our case. If one simply would put $\mathbf{f}_i(t) = \frac{2}{3}e_i^2\dot{\mathbf{a}}(t)$ for the radiation reaction force acting on particle i, one would only reproduce the diagonal terms $\sim e_i^2\mathbf{a}_i^2$ in the energy expression (6), but miss all the non diagonal terms $\sim e_ie_j\mathbf{a}_i\mathbf{a}_j$ for $i \neq j$. This shows already clearly that the LAD equation cannot be always valid.

To obtain a consistent expression for $\mathbf{f}_i$, we can proceed analogous to the one particle case. There is, however, more freedom now, because the mixed terms $\sim e_ie_j$ in (6) can be compensated by a reaction force acting either on particle i or j, or some combination.

Due to the symmetry in i and j one can write (6) in the form

$$E_{rad} = \int_{t_a}^{t_b} \frac{2}{3}\sum_{i,j=1}^{N} e_ie_j\lambda_{ij}\mathbf{a}_i(t)\mathbf{a}_j(t)dt, \tag{8}$$

with $\lambda_{ij} = \mathit{const}$ for simplicity, satisfying $\lambda_{ij} + \lambda_{ji} = 2$. In particular this implies, of course, λ_{ii} (*no sum*) $= 1$. The energy expression (8) is obviously independent of the choice of the λ_{ij}, but the manifest symmetry in i, j will be broken, after we perform the partial integration.

Consider the energy equation analogous to (7), and perform the partial integration by integrating $\mathbf{a}_i(t)$ and differentiating $\mathbf{a}_j(t)$:

$$\int_{t_a}^{t_b} \sum_{i=1}^{N} \mathbf{v}_i(t)\mathbf{f}_i(t)dt = -\int_{t_a}^{t_b} \frac{2}{3}\sum_{i,j=1}^{N} e_ie_j\lambda_{ij}\mathbf{a}_i(t)\mathbf{a}_j(t)dt$$
$$= \int_{t_a}^{t_b} \frac{2}{3}\sum_{i,j=1}^{N} e_ie_j\lambda_{ij}\mathbf{v}_i(t)\dot{\mathbf{a}}_j(t)dt - \frac{2}{3}\sum_{i,j=1}^{N} e_ie_j\lambda_{ij}\mathbf{v}_i(t)\mathbf{a}_j(t)|_{t_a}^{t_b}. \tag{9}$$

Again the last term can be made arbitrarily small if the times t_a, t_b are chosen appropriately, and a possible solution for the reaction forces which fulfills (9) is

$$\mathbf{f}_i(t) = \frac{2}{3} e_i \sum_{j=1}^{N} e_j \lambda_{ij} \dot{\mathbf{a}}_j(t), \tag{10}$$
$$\text{with} \quad \lambda_{ij} + \lambda_{ji}(t) = 2.$$

The term in the sum with $j = i$ is identical with the usual non relativistic form of the LAD force. But there are necessarily also the additional terms which involve the accelerations of the other particles.

We mention that Eliezer [9], (30) - (32), gives an equation of motion for a system of many electrons. It is not identical with that obtained from our eq. (10), because it only contains the Lorentz forces arising from the other charges, but not the $\dot{\mathbf{a}}_j(t)$ for $j \neq i$, thus violating energy conservation.

4 Examples

Consider first a system of two particles. Here one can rigorously justify the replacements $t'_i \rightarrow t'$ in (5) without any further assumptions. The difference of the retarded times is at most as large as the distance x between the charges, $|t'_1 - t'_2| \leq x$. It is irrelevant whether one uses the distance at equal times or at the respective retarded times, they differ at most by a factor $(1+v)$. In leading order the force is the Coulomb force. Shorthand, suppressing irrelevant vector notation or factors of order 1, one has for any of the accelerations $ma \sim e^2/x^2, \quad m\dot{a} \sim (e^2/x^3)v$ i.e. $\dot{a}/a \sim v/x$. From the mean value theorem one then gets $\ln(a(t'_i)/a(t')) = \ln(a(t'_i)/a_0) - \ln(a(t')/a_0) = (t'_i - t')\dot{a}(\bar{t})/a(\bar{t}) \sim v$, with $\bar{t}$ some time between t'_i and t'. Therefore $a(t'_i)/a(t') \sim (1 + v)$, and it is legitimate to replace $\mathbf{a}_i(t'_i)$ by $\mathbf{a}_i(t')$.

If we specialize to particles with identical masses, $m_1 = m_2$, and with charges either identical or opposite, $e_1 = \pm e_2$, symmetry considerations can fix the λ_{ij}. Symmetry between the particles in the first case, and charge conjugation symmetry in the second case, now implies $\lambda_{ij} = 1$. The center of mass moves with constant velocity, the sum of the accelerations vanishes, $\sum_i \mathbf{a_i} = \mathbf{0}$.

For the case of identical charges, eq. (6) then implies that there is no radiation at all. Consequently there is also no radiation reaction force, in spite of the fact that the particles are accelerated!

Opposite charges provide an example of the other extreme. Each particle experiences a reaction force which is twice the LAD force (3).

The two particle case is the only one where one can immediately justify the performed approximations without further assumptions. The reason is that the time difference, the distance of the particles, and the acceleration are directly connected. If the acceleration and its derivative is large, the time difference is small. Vice versa, if the time difference is large, the acceleration is small.

For a multi particle system this is not necessarily the case. Consider, e.g. a four particle system where (1,2) as well as (3,4) are close together, while (1,2) are far apart from (3,4). All accelerations and their derivatives, but also the time differences $t'_{1,2} - t'_{3,4}$ are large. The relative error from the replacements $\mathbf{a}_i(t'_i) \to \mathbf{a}(t')$ is still of order v, but the factor in front may become inadmissible large. It needs a special investigation whether one can do the replacement. Essentially the approximation should be valid if all particles have similar distances.

The expressions (5) for the fields stay valid for all (non relativistic) multi particle systems, but already the calculation of the radiated energy would become clumsy if one keeps the different times t'_i. It appears unlikely that a closed formula for the reaction forces can be derived.

In all cases one has to be aware that there are always mixing terms, even between charges which are rather far apart. One should understand how the mixing terms decrease if the separation increases. Consider two particles i, j which both essentially radiate only between t_a and t_b. The distance x_{ij} may be very large, even larger than $t_b - t_a$. On most points of the sphere with the large radius r there is no interference, because the radiation from the two charges cannot arrive at the same time. Only in directions which are essentially orthogonal to $\mathbf{x}_{ij}$ one has $t'_i \sim t'_j$, there will be always interference. If x_{ij} increases these regions become smaller, so finally the fields from the two charges decouple.

A remark concerning causality appears appropriate. The mutual influence of the various charges does not violate causality. It even would not violate it in a relativistic treatment. The use of retarded potentials poses initial conditions at $t = -\infty$, therefore the whole time evolution is fixed, in particular also all the accelerations $\mathbf{a}_j(t)$ are completely determined from the initial conditions. No chance to send signals from j to i.

5 Where "proofs" fail

The simple counter example already mentioned by Abraham, together with the examples just presented more explicitly in sect. 4, clearly and irrefutably demonstrate that the LAD form of the radiation reaction force cannot be universally valid. A strong but unavoidable consequence is that all general "proofs" of this formula must be incorrect. In view of this it appears appropriate to point out where proofs fail. It is, of course, impossible to cover the vast literature here, and it is as well impossible to analyze proofs which are unintelligible, at least for the present author. We concentrate on some classical literature and on some more recent papers which claim a proof for the LAD force. We emphasize that it is not our intention to criticize the authors of these often rather ingenious considerations. But given the undeniable fact that there are situations where the LAD equation definitely fails, it appears necessary to show why these considerations cannot be generally correct. We will sometimes modify the original notation to our conventions.

The first type of approaches identifies the work done by the radiation reaction force with the energy radiated off from the particle by the Larmor formula. From this one extracts the reaction force. There are two problems in this approach. The counter examples presented above show the first of these problems. Neither the radiated energy nor the reaction force can be treated separately for each particle. They are collective phenomena and necessarily involve other particles as well.

The second problem is that energy can be stored and released in the electromagnetic field. Although we know, of course, the expressions for energy and momentum density of an electromagnetic field, this does not help to calculate the total energy, because there are the non integrable singularities at the position of the point particle. Unless one is willing to introduce extended particles, an alternative which we discuss below, apparently the only way to avoid this problem is to restrict the discussion of energy to times where the particle is far away from the "external" field, or from the fields of other particles. In these regions the particle moves with constant velocity, and the divergent field of the particle poses no problem. The whole four momentum of the system consisting of particle plus its own field is simply given by $P^\mu = mu^\mu$. Thus one can only derive necessary conditions which make statements on the energy balance between times t_a and t_b where the particle under consideration is outside of all other fields. Therefore, notwithstanding the previous difficulty, one only obtains an integrated relation. One cannot derive relations for a definite time t where the particle moves within the field. We therefore also cannot maintain all the statements in our previous paper [15]. But we definitely reemphasize the point made there, that the reaction force depends not only on the motion of the particle of interest, but also on the way how the acceleration is achieved.

A more direct approach tries to derive the equation of motion for an electron by taking into account both an external as well as its own radiation field. This procedure is again plagued by the singularity of the electromagnetic field at the position of the particle. There are two strategies in order to attack this problematics.

The first method, first employed by Lorentz [1] and Abraham [2] and still advocated for by some modern authors [13][14], is to treat the electron as an extended particle. The original idea of Lorentz, that the mass of the electron should be of purely electromagnetic origin, was already criticized by Abraham. One needs in addition some other forces in order to hold the whole structure together. Since a rigid charge distribution is incompatible with special relativity, the determination of this distribution would become part of the dynamical problem and depend on all the forces present in the system. We don't know of any detailed model of this kind. The problems connected with an extended particle appear to be at least as serious as those arising from the singular field around a point particle.

In a more recent paper Gralla, Harte, and Wald [16] claim "A rigorous derivation ...". They argue that a point like limit of a particle with fixed mass and charge is physically impossible. Therefore they consider an extended particle where in the point like limit mass and charge go to zero, hardly a candidate for a physical electron. We are not able to comment on their arguments.

If one wants to avoid a model with an extended particle one needs some other limiting procedure. Let us take a look on the classical derivation of Dirac [3]. He uses the singularity free and time symmetric combination $f^{\mu\nu} = (F^{\mu\nu}_{in} + F^{\mu\nu}_{out})/2$, but this is not the relevant point here. Dirac derives the equation $\frac{1}{2}e^2\epsilon^{-1}\dot{u}^\mu - eu_\nu f^{\mu\nu} = \dot{B}^\mu$, where ϵ is the small radius of a tube surrounding the electron. Obviously one must have $u\dot{B} = 0$.

This B^μ cannot be calculated, therefore Dirac makes use of the most simple ansatz $B^\mu = ku^\mu$, discarding more complicated expressions like e.g. $B^\mu = k'[\dot{u}^4 u^\mu + 4(\dot{u}\ddot{u})\dot{u}^\mu]$. In fact, there are more general possibilities as given by Eliezer [9]. The result for the force is ambiguous.

An interesting procedure was suggested by Barut [17]. To avoid the divergences of the electromagnetic field at the position $x^\mu(t)$ of the particle he starts with the Lorentz force at a slightly displaced position, with the field taken at $y^\mu = x^\mu(t+\epsilon) = x^\mu(t) + \epsilon\dot{x}^\mu(t) + \epsilon^2\ddot{x}^\mu(t)/2 + \epsilon^3 \dddot{x}^\mu(t)/6 + \cdots$. A possible singular term $\sim 1/\epsilon^2$ in the force vanishes, the next singular term $\sim 1/\epsilon$ is absorbed into a mass renormalization, while the term $\sim \epsilon^0$ gives the LAD reaction force. All the higher terms vanish in the limit $\epsilon \to 0$.

This approach appears elegant and convincing, but one has to realize that it uses quite a special choice for the limiting procedure. Even if one is willing to approach the point x^μ along the trajectory, one could use a more general limiting procedure of the form $y^\mu = x^\mu + \epsilon x^\mu_{(1)} + \epsilon^2 x^\mu_{(2)}/2 + \epsilon^3 x^\mu_{(3)}/6 + \cdots$. In order to remove the singularity $\sim 1/\epsilon^2$ one has to put $x^\mu_{(1)} = \dot{x}^\mu(t)$ as before. The next order $\sim 1/\epsilon$ is again absorbed by mass renormalization. But one is completely free to choose the function $x^\mu_{(3)}$ as one likes. Instead of (1) one then ends up with $f^\mu = \frac{2}{3}e^2(x^\mu_{(3)} - (ux_{(3)})u^\mu)$.

In general one may say that all "derivations" use some special ansatz and tacitly assume that the reaction force depends on the motion of the considered particle only, and on nothing else. We have seen that this is in general not true.

6 When can the LAD force be correct?

A hasty and naive comment on the counter examples given above could consist in postulating that the various derivations in the literature should (of course!) be only applicable in the case of an external field. Whatever the arguments for such a restriction could be, it would require a clear definition of an external field. This is more tricky than one might expect.

A first try for a definition could be: An external field is a field which is not changed by the presence of the test particle. This would be the case in the limit that the charge e of the test particle goes to zero, keeping everything else fixed. Our examples clearly show that this would not help.

To the contrary, the mixing terms $\sim ee_j$ would now become dominant compared to the diagonal term $\sim e^2$.

The examples also show that it would not help to define an external field as being a field which is created by many other particles.

A definition, though somewhat academic, which might work, is to create the field by one or more charged particles with a very large mass compared to the mass of the test particle. Of course, no further light particles should be around. To make this more explicit consider a two particle system with $m_1 \ll m_2$. Using $m_1\dot{\mathbf{a}}_1 + m_2\dot{\mathbf{a}}_2 = 0$ one can write the reaction forces (10) as

$$\begin{aligned} \mathbf{f}_1 &= \frac{2}{3}(e_1^2 - \lambda_{12}\frac{m_1}{m_2}e_1e_2)\dot{\mathbf{a}}_1, \\ \mathbf{f}_2 &= \frac{2}{3}(e_2^2 - \lambda_{21}\frac{m_2}{m_1}e_1e_2)\dot{\mathbf{a}}_2. \end{aligned} \tag{11}$$

If one had $\lambda_{ij} = 1$, the formula for the light particle would approach the LAD form, while that for the heavy particle would contain a large factor compared to LAD. If, on the other hand, one would have, e.g. $\lambda_{ij} = 2m_j^2/(m_i^2 + m_j^2)$, both forces would approach the LAD form in the limit $m_1/m_2 \to 0$.

Whenever one talks about an external field (which breaks, by the way, momentum conservation, and, if time dependent, also energy conservation) one should specify how this field is realized and analyze the whole system of particle plus “external” field.

A realistic chance of really measuring the radiation reaction force can probably only come from the use of intense laser beams. There has been some discussion on this recently [18]. At present we are unable to discuss the consequences of our considerations for this case. The motion of the electron is necessarily relativistic, besides the electron of interest there is a huge number of other electrons, part of which producing the coherent laser radiation, a process which apparently requires the consideration of quantum mechanics. The situation is complex, we don’t try to formulate a guess here.

7 Summary and conclusions

We called attention to some facts which are almost trivial, but have, to our knowledge, never been carefully exploited with respect to their consequences. Radiation reaction cannot be treated as a phenomenon which only concerns a particle and an external field. Energy density and Poynting vector are quadratic in the fields, therefore they are not simply obtained by summing up the contributions of all particles involved. There are mixing terms. This implies that the energy loss through the friction of the Lorentz Abraham Dirac force is usually not identical with the radiated energy. All general "proofs" of the LAD force are thus bound to fail. In (10) we suggested a form of the reaction force which respects energy conservation, but such a closed expression can be derived only for simple non relativistic systems. The question how to treat realistic situations like an electron in a strong laser field remains open.

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