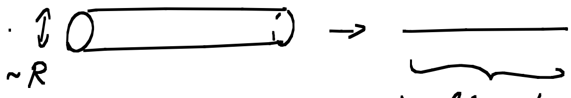
11 Extra Dimensions (and some further issues)

11.1 The idea of extra dimensions

Motivations:

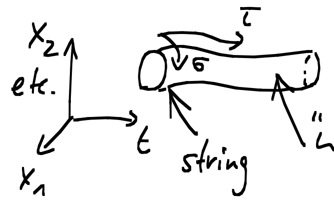
a) It is a consistent logical possibility for physics beyond the SM, not excluded by experiment if the size of the extra dims. is sufficiently small. (vis.



visible at energies $\ll R^{-1}$)

b) string theory (as the best-understood candidate for a theory of quantum gravity) requires extra dims.

[In detail:



"world-sheet" swept out in time

$\Rightarrow$ need to consider "world-sheet"- (2d) field theory of variables $x^1(\tau,\sigma)$, $x^2(\tau,\sigma)$, ...
This can (at present?!) only be treated if this theory is a 2d-CFT. Conformal invariance is anomalous unless $d = 26$.

In fact, the 26d-Minkowski-vacuum of string theory turns out to be unstable (i.e. there exists a scalar with $m^2 < 0$, a "tachyon"). This can be cured by introducing "world-sheet supersymmetry" (= fermionic partners for $x^1, x^2 \cdots$). The resulting 2d SCFT is non-anomalous for

$$\underline{\underline{d = 10}}.$$

(in fact, one gets a 10d supergravity theory at "low" energies).

c) Kaluza-Klein theories (predicting string theory!)

Consider 5d gravity:

$$S = \int d^5x \sqrt{-g_5}\ \frac{1}{2} M_5^3 R$$

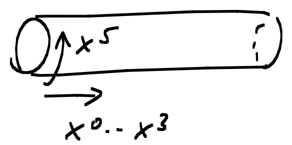
$$R = R_{\mu\nu} g^{\mu\nu} \ ; \quad R_{\mu\nu} = R_{\mu\rho\nu}{}^{\rho} \ ; \quad R_{\mu\nu\rho}{}^{\sigma} = \partial_\mu \Gamma_{\nu\rho}{}^{\sigma} + \cdots ,$$

$$\Gamma_{\mu\nu}{}^{\rho} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \cdots \right). \qquad \mu, \nu, \ldots \in \{0,1,2,3,5\}$$

(Many authors use $M, N, \ldots$ for d-dim. indices and reserve $\mu, \nu, \ldots$ for 4d-indices.)

- Consider an "S^1-compactification" of this theory:

$$\int d^5x \longrightarrow \int d^4x \int_0^{2\pi R} dx^5 \ :$$



- At low energies $(E \ll 1/R)$, one finds a 4d theory of $\underbrace{\text{gravity}}_{g_{\mu\nu}} + \underbrace{\text{electrodynamics}}_{g_{\mu 5}} + \underbrace{\text{real scalar}}_{g_{55}}$ (with $\mu, \nu \in (0, \ldots, 3)$)

- More generally, compactifying on a space with isometry group G ($G(S^1) = U_1$; $G(S^2) = SO_3$ etc.), one finds a gauge theory with group G in 4d.

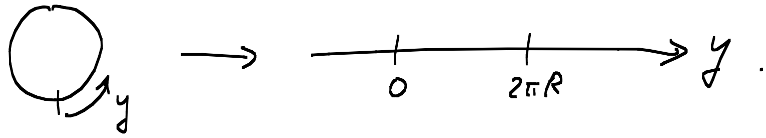
[in our S^1-case, the scalar g_{55} parameterizes a "flat direction",

$$R_{phys.} = \int_0^{2\pi R} dx^5 \sqrt{-g_{55}} \, .$$

For curved compactifications (e.g. S^2), the volume is <u>not</u> a flat direction and extra fields are needed to keep it stable. (Also in the S^1-case loop effects destabilize the extra-dim. volume or radius.)]

Kaluza-Klein reduction

Consider again the 5d case with $x^5 \equiv y$ and compact space $= S^1$. Fcts. on S^1 $\equiv$ periodic fcts. on $\mathbb{R}$:



generic field: $\varphi = \sum_{n=0}^{\infty} \varphi^c_{(n)}(x) \cos(ny/R) + \sum_{n=1}^{\infty} \varphi^s_{(n)}(x) \sin(ny/R)$

$$\int d^5x \; \varphi\left(\partial_\mu \partial^\mu + \partial_5 \partial^5\right)\varphi$$

$\mu = 0 \cdots 3$; metric: diag $(-1,1,1,1,1)$

$m_n = n/R$

$$2\pi R \int d^4x \left\{ \underbrace{\varphi^c_{(0)} \partial_\mu \partial^\mu \varphi^c_{(0)}}_{\text{"zero-mode"}} + \frac{1}{2} \sum_{n=1}^{\infty} \underbrace{\varphi^c_{(n)} \left(\partial_\mu \partial^\mu + m_n^2\right) \varphi^c_{(n)}}_{\text{"KK-modes"}} + (c \to s) \right\}$$

- In all realistic scenarios, "our" known particles are described by the zero-modes of the extra-dimensional model.

11.2 Fermions & Orbifolds

Fundamental Problem: An S^1-compactification can not produce chiral (or Weyl) fermions in 4d. [This generalizes to all "simple" compactifications of $d > 5$-theories on smooth compact spaces.]

<u>Reason</u>: representations of Clifford algebras

- Spinors in general dimensions are constructed as representations of the corresponding Clifford algebras γ^μ etc. ($\mu = 0, 1, 2, 3, 5, 6, \cdots, d-1$). For $d = 5$, we can simply use the standard set of γ-matrices, including γ^5 "on equal footing": $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$ remains true including $\mu = 5$.

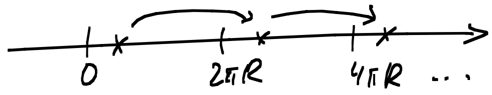
- However, the projection operator $P_L = \frac{\mathbb{1} - \gamma^5}{2}$ breaks 5d-Lorentz invariance; hence we can not define a chiral 5d fermion. ("γ^6" $= \gamma^0\gamma^1\gamma^2\gamma^3\gamma^5 \sim \mathbb{1}$ and can hence not be used to reduce the representation)

- Thus, our fermionic zero-mode will always be a Dirac fermion (= 2 Weyl fermions). The SM-situation with, e.g. ψ_L an SU_2-doublet & ψ_R - two singlets can not be realized in an S^1- compactification.

- One generic way out: use compact spaces with singularities (Another way out is to use compact spaces with non-zero "background fields".)

- The simplest singular spaces are <u>orbifolds</u>:

- As a warm-up, construct S^1 as $S^1 = \mathbb{R}/\mathbb{Z}$, where the action of $\mathbb{Z}$ is defined by $n: x \to x + 2\pi R$



- More generally: M – manifold
  K – discrete symm. group of M
  M/K – manifold of equivalence classes
  (where $x \sim x'$ iff $\exists\, k \in K$ with $x' = k \cdot x$)

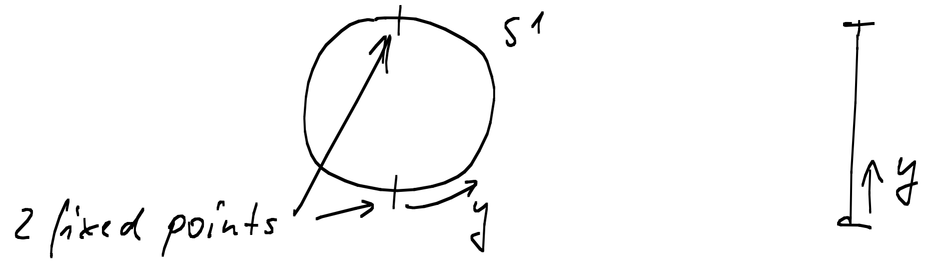
Fact: M/K is a (smooth) manifold if K acts freely, i.e.
$kx = x$ for some $x \Rightarrow k = \mathbb{1}$.

- M/K is an <u>orbifold</u> if K acts non-freely (i.e. the action of K has fixed points). Each fixed point gives rise to a singularity.

- Example 1: $\mathbb{R}/\mathbb{Z}_2$ whith $\mathbb{Z}_2 \ni \mathbb{1} : x \to x$
  $\mathbb{Z}_2 \ni -\mathbb{1} : x \to -x$.

  (Obviously: $\mathbb{R}/\mathbb{Z}_2 = \mathbb{R}^+$; This "manifold" has a singularity (in this case a "boundary" or "brane" at $x = 0$.)

- Example 2: $S^1/\mathbb{Z}_2$ = "Interval"
  $\mathbb{Z}_2$ acting on $y \in \mathbb{R}/\mathbb{Z}$ by $-\mathbb{1}: y \to -y$



- A field theory on M/K (both for free & non-free actions) is defined by restricting the field-space of M to those fields invariant under K.

- This clearly gives, as already anticipated, the periodic functions if one constructs S^1 as $\mathbb{R}/\mathbb{Z}$.

- For a scalar on $S^1/\mathbb{Z}$, it eliminates the sine-modes.

- Note however, that we are not obliged to define the action of $k \in K$ on a field as $(k\varphi)(x) = \varphi(kx)$. We can also use a non-trivial repres. of K on the space in which φ takes its values (in this case $\mathbb{R}$), e.g.

$$\mathbb{Z}_2 \ni -\mathbb{1} : \quad \varphi(y) \longrightarrow -\varphi(-y)$$

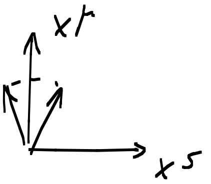
(in which case we lose all cosine-modes, including the zero-mode.)

- In the case of fermions on S^1, we have to be careful to respect 5d Lorentz-symm. in defining the action of $-\mathbb{1} \in \mathbb{Z}_2$:

$$-\mathbb{1} : \; x^0, \dots, x^3, x^5 \;\rightarrow\; x^0, \dots, x^3, -x^5 .$$

(K has to be a <u>symmetry</u> of the relevant theory on M.)

A rotation in the x^μ-x^3-plane (generated by $[\gamma^\mu, \gamma^5]$) must anticommute with the reflection:

[vis. 

.] $\quad [\gamma^\mu, \gamma^5]\, "{-\mathbb{1}}" = -("{-\mathbb{1}}")[\gamma^\mu, \gamma^5]$.

At the same time, it should commute with rotations in any x^μ-x^ν-plane $(\mu, \nu \in (0, \dots, 3))$.

$$\Rightarrow \; "{-\mathbb{1}}" = \pm\gamma^5 \quad (\text{assuming conventions where } (\gamma^5)^2 = \mathbb{1}).$$

free choice; for $+$: $\quad "{-\mathbb{1}}"\, \psi_L = -\psi_L$
$\quad "{-\mathbb{1}}"\, \psi_R = +\psi_R$

Thus, in the KK-decomposition of ψ on $\mathbb{R}^4 \times S^1$,

$$\psi = \sum_{\substack{n=0 \\ L,R}}^{\infty} \psi^{c}_{L,R\,(n)}(x)\cos(ny/R) + \sum_{\substack{n=1 \\ L,R}}^{\infty} \psi^{s}_{L,R\,(n)}(x)\sin(ny/R)$$

the "$\mathbb{Z}_2$-projection" (the "orbifolding") kills either all l.h. odd & r.h. even modes or vice versa.
In any case: a chiral zero-mode is left!

11.3 Gauge-Higgs unification

- Consider a 5d theory with $G = SU_3$
- Compactify on $S^1/\mathbb{Z}_2$:

$$\mathbb{Z}_2: \quad y \longrightarrow -y \qquad (y \in [0, 2\pi R) \text{ for the } S^1)$$

$$\phi(y) \longrightarrow P\phi(-y) \quad \text{with } P \in SU_3 \text{ for } \phi$$
a field transforming in any repr. of SU_3
($P^2 = \mathbb{1}$ required by consistency)

$$\left.\begin{array}{l} A_\mu(y) \longrightarrow P A_\mu(-y) P^{-1} \\ \qquad\qquad \text{for } \mu = 0 \cdots 3 \\ A_5(y) \longrightarrow -P A_\mu(-y) P^{-1} \end{array}\right\} \text{required by consistency}$$

- Choose $P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow$ for any T^a for which $P T^a P^{-1} = T^a$:

A_μ^a has zero mode ; A_5^a has no zero-mode for any T^a for which $PT^aP^{-1} = -T^a$:

A_μ^a has no zero mode ; A_5^a has zero mode

$\Rightarrow$ surviving gauge group: $SU_2 \times U_1$ (U_1 generated by $T^a \sim \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$)

$\downarrow$

(the corresponding A_μ's have zero mode)

- We also get a charged scalar doublet (from the A_5-components of the "broken" gauge bosons):

$$SU_3 \supset SU_2 \times U_1$$

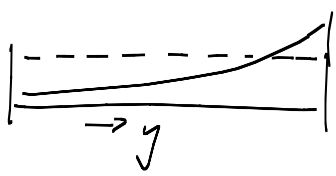
$$8 = 3 + 1 + 2 + \bar{2} \quad \Big\} \text{ for complex repr.}$$

$\uparrow \ \uparrow$ also charged under U_1

- In our case, the 8 is real. The 4 surviving real scalars combine in a doublet, which can play the role of the Higgs: The VEV of A_5^a, ($a \leftarrow$ broken generators) breaks $SU_2 \times U_1$ further to $U_{1,EM}$.
- Problem: wrong θ_W (ratio of g_1 & g_2) since U_1 & SU_2 gauge coupling unify at $\sim R^{-1} \sim$ TeV.
- Simplest cure: add terms $\sim F_{U_1}^2$ & $\sim F_{SU_2}^2$ at boundaries (not forbidden by bulk SU_3-symmetry).
- Better cure: larger bulk gauge group (where θ_W comes out right at tree level).

- A_5 massless by bulk gauge symm.
- small mass induced by loop correction (a collective effect, requiring the participation of $A_M(x^\mu, y)$ for <u>different</u> y's ; similar to little-Higgs idea.)

- Yukawa couplings can arise from gauge coupling of bulk fermions. (Hierarchy can be created since bulk fermion mass leads to non-trivial profile of zero-mode:



ψ_0 - profile

11.4 Randall-Sundrum model

Consider again S^1/Z_2 model, but introduce 5d-cosmol. constant & (brane) 4d cosmol. constants:

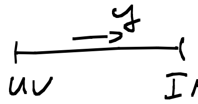
"induced metric" at boundary

$$S_{5d} \to S_{5d} + \int d^4x \sqrt{-g(y=0)}\, \Lambda_4$$
$$- \int d^4x \sqrt{-g(y=\pi R)}\, \Lambda_4 .$$

For appropriately tuned Λ_4 (given M_5 & Λ_5), one finds solution to Einstein-eqs:

"warp factor"

$$ds^2 = e^{-2ky}\, dx^\mu dx^\nu \eta_{\mu\nu} + dy^2 \qquad (AdS_5!)$$

- The 4d-metrics on UV & IR brane () differ by $e^{-2k\underbrace{\pi R}_{y_{max}}}$.

can be exponentially small for "reasonable" values of k & R.

- SM-fields can be brane- or bulk fields, but let the

Higgs be localized on IR-brane:

$$S \;\to\; S + \int d^4x \sqrt{-g_{4,IR}}\left[g_4^{\mu\nu}\partial_\mu\phi\partial_\nu\phi^\dagger + m^2\phi^2 + \dots\right]$$

- $M_5\,;\ \Lambda_5^{1/5}\,;\ \Lambda_4^{1/4}\,;\ y_{max}^{-1} \sim M_4$ ← stabilization of "somewhat" large y_{max} possible (→ Goldberger/Wise)
- $e^{-2ky_{max}}$ nevertheless very small.

$$\downarrow$$

$$S_{eff,4d} \supset \int d^4x \left[e^{-2ky_m}|\partial\phi|^2 + e^{-4ky_m}\phi^2 m^2\right] \quad (m \sim M_5 \sim M_4)$$

$$\boxed{\phi \to e^{ky_m}\phi}$$

$$\downarrow$$

$$\int d^4x \left[|\partial\phi|^2 + \underbrace{\underbrace{e^{-2ky_m}m^2}_{m^2_{Higgs,4d}}}_{\text{can thus be exponentially small!}}\phi^2\right]$$

<u>However:</u> little hierarchy problem not solved since strongly coupled gravity is at TeV (near IR brane) and Higgs-mass-loop-divergence cut of by these effects (at least generically).

11.5 Strong CP-problem & Peccei-Quinn axion

$$S_{QCD} \supset \frac{\theta}{16\pi^2}\int F_{\mu\nu}\tilde{F}^{\mu\nu} \quad ; \quad |\theta| < 10^{-9} \text{ experimentally}$$

(unnatural!)

Axion idea: $\rightarrow \quad \frac{1}{2}\partial_\mu a\,\partial^\mu a + \frac{a}{16\pi^2 f_a} F\tilde{F}$

QCD-instantons $\Longrightarrow \quad V_{inst.} \sim \Lambda_{QCD}^4 \left(1 - \cos\left(\frac{a}{f_\pi}\right)\right)$

$\Rightarrow \quad a = 0$ dynamically $\quad$ [$\exists$ anomalous symm. $a \to a + \varepsilon$]

$$m_a \sim \frac{\Lambda_{QCD}^2}{f_a}$$

Supernova cooling: $\quad f_a > 10^9\,\mathrm{GeV}$

Universe overclosure: $f_a < 10^{12}\,\mathrm{GeV}$