2. The Standard Model

2.1 Gauge interactions

recall: $\mathcal{L}_{gauge} = - \sum_{i=1}^{3} \frac{1}{2g_i^2} \, tr \, F^{(i)}_{\mu\nu} F^{(i)\mu\nu}$

(the three terms correspond to SU_3, SU_2, U_1)

$F_{\mu\nu} = \frac{1}{i} [D_\mu, D_\nu]$; $D_\mu = \partial_\mu + i A_\mu$; $A_\mu \in Lie(G)$

Focus on the case $G = SU_N$. Any element $g \in SU_N$ can be written as $g = \exp(iT)$ with $T \in H$ ($H \equiv$ hermitian & traceless $N \times N$ matrices)

For some neighbourhood of $\mathbb{1} \in G$ and a corresponding neighbourhood of $0 \in H$ the map "exp" is 1-to-1 & differentiable (together with its inverse). (i.e. a "diffeomorphism")

[This "definition" of the Lie-algebra $iH = Lie(G)$ generalizes to all Lie-groups.]

<u>Note</u>: $F_{\mu\nu}$ is <u>not</u> a diff. operator ; $F_{\mu\nu} \in H$

It is useful to choose a basis $\{T^a\}$ of H such that $tr \, T^a T^b = \frac{1}{2} \delta^{ab}$.

(widely used convention)

[Since we have basically defined SU_N by its natural action on complex N-vectors, our generators T^a are "in the fund. representation.]

We write $F_{\mu\nu} = F_{\mu\nu}^a T^a$; $A_\mu = A_\mu^a T^a$ and hence

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f^{abc} A_\mu^b A_\nu^c \, , \text{ where}$$

the "structure constants" f^{abc} are defined by

$$[T^a, T^b] = i f^{abc} T^c. \qquad \text{(Check this!)}$$

Note: rescaling $A_\mu^{(i)} \to g_i A_\mu^{(i)}$, we get a canonical kinetic term for a vector field for each $A_\mu^{(i)}$ $(i=2,3)$.

The U_1 case is special since $U_1 \ni g = e^{iT}$, wher T is any real number and no "canonical" to normalize it exists. To comply with our SU_N-conventions, we could write

$$A_\mu = A_\mu^1 T^1 \quad \text{with} \quad T^1 = \frac{1}{\sqrt{2}} \quad \text{such that}$$

$$\underbrace{\mathcal{L}_{U_1} = -\frac{1}{2g_1^2}\, \text{"tr"}\, \underset{\text{"Lie group elements"}}{F_{\mu\nu} F^{\mu\nu}}}_{\text{in practice, one always skips this step}} = -\frac{1}{4g_1^2}\, \underset{\text{"real numbers"}}{F_{\mu\nu} F^{\mu\nu}}$$

2.2 Fermions

Befor writing down the fermionic part of $\mathcal{L}_{SM}$, we need to fix our conventions concerning fermions in general:

$\mathcal{L} = \bar{\psi}_D i \not{\partial} \psi_D$ for any Dirac spinor ψ_D.

The crucial Lorentz trf. property is

$$\psi_D \xrightarrow{\Lambda} \exp\left(t^{\mu\nu} \tfrac{1}{4} [\gamma_\mu, \gamma_\nu]\right) \psi_D$$

with $\Lambda = \exp(i t^{\mu\nu} \underset{\text{canonically normalized } SO_{1,3} \text{ generator}}{M_{\mu\nu}}) \in SO_{1,3}$

(Give $M_{\mu\nu}$ explicitly and check that the generators $\frac{1}{4}[\gamma_\mu, \gamma_\nu]$ fulfil the same commutation relations.)

For the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = -2\eta_{\mu\nu}$, we will use the Weyl basis:

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix} \quad ; \quad \sigma_\mu = (\sigma_0, \sigma_i) = \left(\mathbb{1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

$$\bar{\sigma}_\mu = (\sigma_0, -\sigma_i)$$

$$\gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad ; \quad P_{L,R} = \frac{\mathbb{1} \pm \gamma_5}{2}$$

$$\left(P_L = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \; ; \; P_R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \right)$$

$$\psi_D = \begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix}$$

The 2-component or Weyl spinors transform independently under $SO_{1,3}$.

($\frac{1}{4}[\gamma_\mu, \gamma_\nu]$ is block-diagonal !)

Instead of using Weyl spinors, one can also use left- & right-handed Dirac spinors:

$$\psi_{D,L} = \begin{pmatrix} \psi \\ 0 \end{pmatrix} = P_L \psi_D \quad ; \quad \psi_{D,R} = \begin{pmatrix} 0 \\ \bar{\chi} \end{pmatrix} = P_R \psi_D \; .$$

We will also use the notation

$$\psi_D = \begin{pmatrix} \psi_L \\ \bar{\psi}_R \end{pmatrix} \; ; \; \psi_{D,L} = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \; ; \; \psi_{D,R} = \begin{pmatrix} 0 \\ \bar{\psi}_R \end{pmatrix} .$$

In this notation, ψ_L & ψ_R are both Weyl spinors (with the same trf. properties). "L" & "R" are just indices.

Using the isomorphism $SL_{2,\mathbb{C}} \simeq SO_{1,3}$ (near $\mathbb{1}$, not globally!), we can also write Weyl spinors with their explicit $SL_{2,\mathbb{C}}$-index:

$$\psi = \{\psi_\alpha\} \quad , \quad \alpha = 1,2 \; .$$

Then $\psi_D = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot\alpha} \end{pmatrix}$, where the dotted index transforms under the compl. conj. $SL_{2,\mathbb{C}}$ - repres.

Weyl indices are raised & lowered with the invariant tensor $\varepsilon_{\alpha\beta}$ or $\varepsilon_{\dot\alpha\dot\beta}$ of $SL_{2,\mathbb{C}}$. χ_α is a Weyl spinor just like ψ_α. $\bar\chi_{\dot\alpha}$ is its complex conjugate and

$\bar\chi^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\chi_{\dot\beta}$.

(→ Tutorials & Appendix of "Wess & Bagger" for more details)

<u>Summary:</u>

$$\begin{aligned} &1 \text{ Dirac fermion}\\ &\quad\parallel\\ &1 \text{ l.h. fermion} + 1 \text{ r.h. fermion}\\ &\quad\parallel\\ &2 \text{ Weyl fermions}\end{aligned}$$

<u>Note:</u> The Dirac fermion kinetic term corresponds to two Weyl fermion kinetic terms:

$$\begin{aligned}
\bar\psi_D i\not\partial\psi_D &= i\psi_D^\dagger\gamma_0\gamma^\mu\partial_\mu\psi_D\\
&= i(\bar\psi^T, \chi^T)\begin{pmatrix}0 & 1\!\!1\\ 1\!\!1 & 0\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu\\ \bar\sigma^\mu & 0\end{pmatrix}\partial_\mu\begin{pmatrix}\psi\\ \bar\chi\end{pmatrix}\\
&= i\bar\psi^T\bar\sigma^\mu\partial_\mu\psi + i\chi^T\sigma^\mu\partial_\mu\bar\chi\\
&= i\bar\psi^T\bar\sigma^\mu\partial_\mu\psi + i\bar\chi^T\bar\sigma^\mu\partial_\mu\chi + \text{total derivative}
\end{aligned}$$

These two terms (and hence ψ & χ) come completely on the same footing.

- <u>Charge conjugation</u>

Given a complex scalar φ which transforms as $\varphi \to e^{i\alpha}\varphi$ under a U_1 gauge symm., φ^* obviously transforms as

$\varphi^* \to e^{-i\alpha}\varphi^*$ and hence describes a particle of opposite charge. Thus, renaming the fields according to $\varphi^c = \varphi^*$ & $\varphi^c = \varphi^*$ obviously corresponds to "charge conjugation.

For a Dirac spinor, the situation is more involved since ψ_D^* is not a Dirac spinor. Anyway, the conventionally used compl. conj. object is not ψ_D^* but $\bar{\psi}_D = \psi_D^\dagger \gamma^0$. One might try to use $\bar{\psi}_D^T$, but this is also not a Dirac spinor.

However, there exists a matrix C satisfying

$$C\gamma^\mu C^{-1} = (\gamma^\mu)^T.$$

It can be shown ($\to$ problems) that $\psi_D^c \equiv C\bar{\psi}_D^T$ $\underline{\underline{\text{is}}}$ a Dirac spinor. Thus, for Dirac spinors charge conjugation corresponds to using ψ_D^c instead of ψ_D. (C is the "charge conjugation matrix".

Using $\psi_D = \begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix}$, it is easy to guess that $\psi_D^c = \begin{pmatrix} \chi \\ \bar{\psi} \end{pmatrix}$ (which is obviously a Dirac spinor of opposite charge).

[Given an explicit form of the γ_μ's, an explicit C can easily be given, but we will not need it.]

It is obvious that if ψ_D is l.h., then ψ_D^c is r.h.

Thus, we have the following options for writing down fermionic lagrangians:

Dirac: ψ_D – not useful for SM, since l.h. & r.h. parts have different gauge trf. properties ("The SM is chiral"!)

$\updownarrow$

l.h./r.h. Dirac: $\psi_{D,L}$; $\psi_{D,R}$

$\updownarrow$

l.h. Dirac / charge conj. of r.h. Dirac: $\psi_{D,L}$; $(\psi_{D,R})^c$ – This has the advantage that we always use l.h. fields.

$\updownarrow$

Weyl: ψ , χ $\left(\psi_D = \begin{pmatrix}\psi \\ \bar{\chi}\end{pmatrix}\right)$

Note: We will mostly use Weyl spinors since they are conceptually simplest and are particularly well-suited for the superfield description of $N=1$ SUSY.

2.3 Gauge trfs. of fermions

Experimental fact:

(Weyl) fermions	SU_3	SU_2	$U_1 = U_{1,Y}$
$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	3	2	$1/6$
u_R	$\bar{3}$	1	$-2/3$
d_R	$\bar{3}$	1	$1/3$
$\ell_L = \begin{pmatrix} \nu \\ e_L \end{pmatrix}$	1	2	$-1/2$
e_R	1	1	1

$\uparrow$ $\uparrow$ here "1" means singlet

$\uparrow$ here "1" means a charge, just like any other number

(singlet would correspond to "0")

Note: • In the notation $q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$, u_R, d_R, the transformation in the 3 or $\bar{3}$ of SU_3 (and the corresponding color index) are implicit.

• The up-type Dirac quark $u_D = \begin{pmatrix} u_L \\ \bar{u}_R \end{pmatrix}$ transforms in the 3 of SU_3. This implies that the Weyl fermion u_R transforms in the $\bar{3}$ of SU_3, as stated in the table above

• We normalize the hyperchange, $U_{1,Y}$, such that $D_\mu \psi = (\partial_\mu + iB_y \cdot Y + \dots)\psi$, with a kinetic term $-\frac{1}{4g_1^2} F^{(1)}_{\mu\nu} F^{(1)\mu\nu}$; $F^{(1)}_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$.

Y is the real number given in the table.

2.4 Scalar lagrangian and electroweak symm. breaking

• $V(\phi) = -\mu^2 |\phi|^2 + \lambda(|\phi|^2)^2$; ϕ - SU_2-doublet with hypercharge $Y = 1/2$

• minimum of potential is at $|\phi|^2 = \frac{\mu^2}{2\lambda} \equiv v^2$

• by a gauge trf. (SU_2-rotation), we can always achieve $\phi_{vac} = \begin{pmatrix} 0 \\ v \end{pmatrix}$, i.e. $\mathrm{Re}\,\phi_2 = v$.

The masses of the vector fields come from

$$|D_\mu \phi_{vac}|^2 = \left|(A_\mu^a T^a + B_\mu Y) \cdot \begin{pmatrix} 0 \\ v \end{pmatrix}\right|^2 \quad \text{with } T^a = \frac{1}{2}\sigma^a.$$

To understand this quantitatively, it is convenient to replace A_μ^a by $g_2 A_\mu^a$ and B_μ by $g_1 B_\mu$, such that the kinetic terms become canonical.

$$\Rightarrow \quad \mathcal{L} \supset -\frac{v^2}{4}\left\{ \underbrace{g_2^2 \left(A^1_\mu A^{1\mu} + A^2_\mu A^{2\mu}\right)}_{\text{mass for W-bosons}} + \underbrace{\left(g_2 A^3_\mu - g_1 B_\mu\right)\left(g_2 A^{3\mu} - g_1 B^\mu\right)}_{\text{mass for Z-boson}} \right\}$$

$$W^\pm_\mu = \frac{1}{\sqrt{2}}\left(A^1_\mu \mp A^2_\mu\right)$$

To understand the origin of the massive Z-boson and the massless photon, we define (suppressing Lorentz indices):

$$Z = A^3 \cos\theta - B \sin\theta$$
$$A = A^3 \sin\theta + B \cos\theta .$$

[$\theta = \theta_W$ is the "Weinberg angle"]

(It is crucial to use an orthogonal trf. to keep the kinetic term canonical.)

For $\tan\theta = g_1/g_2$, we see that the mass langrangian contains terms $\sim W^+W^-$ and $\sim Z^2$, while A remains massless.

The A^3, B-part of the covariant can now be worked out to be (for any field, not just ϕ):

$$g_2 A^3 T^3 + g_1 B Y = g_2 Z \cos\theta\, T^3 + g_2 A \sin\theta\, T^3 - g_1 Z \sin\theta\, Y + g_1 A \cos\theta\, Y$$

$$= Z \frac{g_2}{\cos\theta}\left(T^3 - \sin^2\theta\, Y\right) + A \underbrace{g_2 \sin\theta}_{\equiv e} \underbrace{\left(T^3 + Y\right)}_{\equiv Q}$$

<u>Problems:</u> Check all of this in detail; work out the W- & Z-boson masses in terms of g_1, g_2, v; Check that Q is indeed the well-known electric charge of the various particles

given in our table ; check which of the two (complex) vector fields $W^{\pm}$ is positively/negatively charged.