10 Flavor, Neutrinos, Electroweak Precision Analysis

10.1 Yukawa couplings

For simplicity, we start with just one family. Convince yourself that the only gauge-inv. fermion-fermion-scalar couplings are:

$$\lambda_e \underset{\overset{\uparrow}{2}}{l_i}\, e\, \underset{\overset{\nearrow}{\bar{2} \text{ of } SU_2}}{\bar{\phi}_i} \;; \quad \lambda_d \underset{\overset{\uparrow}{3}}{q_{ia}}\, \underset{\overset{\uparrow}{\bar{3} \text{ of } SU_3}}{d_a}\, \bar{\phi}_i \;; \quad \lambda_u \underset{\overset{\uparrow}{2}}{q_{ia}}\, u_a\, \underset{\overset{\nearrow}{2 \text{ of } SU_2}}{\phi_j}\, \varepsilon^{ij}$$

(and their hermitian conjugates)

Note: Here and below we suppress Weyl indices using the standard convention $le = l^\alpha e_\alpha$ etc.

Using this convention, we have $\psi\chi = \chi\psi$ in spite of anticommutation:

$$\psi\chi = \psi^\alpha \chi_\alpha = -\chi_\alpha \psi^\alpha = -\varepsilon_{\alpha\beta}\chi^\beta \varepsilon^{\alpha\gamma}\psi_\gamma$$

$$\underset{\uparrow}{=} \chi^\beta \psi_\beta = \chi\psi$$

Here we used $\varepsilon_{\alpha\beta}\varepsilon^{\alpha\gamma} = -\delta_\beta{}^\gamma$ $\left(\varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ; \; \varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right)$

$\uparrow \qquad \uparrow$

This is the convention of W/B. Some authors (e.g. Sohnius, Phys. Rep.) use other conventions).

Using $\phi = \begin{pmatrix} 0 \\ v \end{pmatrix}$, we now immediately find the mass lagrangian

$$\mathcal{L} \supset m_e e_L e_R + m_d d_L d_R + m_u u_L u_R + h.c.$$

with $m_e = \lambda_e v$ etc.

(Check that this is equivalent to Dirac mass terms of the type $m \bar{e}_D e_D$ with $e_D = \begin{pmatrix} e_L \\ \bar{e}_R \end{pmatrix}$.)

Note: The phase (in particular the sign) of a fermionic mass term is irrelevant since it can be changed by a phase redefinition of the Weyl fermions.

The generalization to 3 generations is straightforward; one simply replaces the λ's by matrices λ_{ab} with $a, b \in \{1,2,3\}$. Using matrix notation, we have

$$\mathcal{L} \supset \bar{e}_L^T M_e \bar{e}_R + \bar{d}_L^T M_d \bar{d}_R + \bar{u}_L^T M_u \bar{u}_R + h.c.$$

where $e_L = \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix}$ etc.

[Note: I used the dotted spinors to be consistent with some of literature writing

$$\mathcal{L} \supset \bar{e}_{D,L} M_e e_{D,R} + \cdots \quad \text{with } e_{D,L} = \begin{pmatrix} e_L \\ 0 \end{pmatrix};\ e_{D,R} = \begin{pmatrix} 0 \\ \bar{e}_R \end{pmatrix}.$$]

10.2 The CKM matrix

Important fact: Any complex matrix M can be diagonilized by a biunitary trf.:

$$L^\dagger M R = M^{diag.}\ ;$$

The unitary matrices L & R can be chosen such that all entries of $M^{diag.}$ are real & non-negative.

Problem: Prove this!

- Writing, e.g., $M_e = L_e M_e^{diag.} R_e^+$ and introducing new fields $\bar{e}_R' \equiv R_e^+ \bar{e}_R$; $\bar{e}_L'^T = \bar{e}_L^T L_e$ etc.

we obtain $\mathcal{L} \supset \bar{e}_L'^T M_e^{diag.} \bar{e}_R' + \bar{d}_L'^T M_d^{diag.} \bar{d}_R' + \bar{u}_L'^T M_u^{diag.} \bar{u}_R'$.

- In this generation-vector-notation, the kinetic term reads

$$\mathcal{L} \supset \bar{e}_L^T \underset{\overset{\uparrow}{\equiv \bar{\sigma}^\mu \partial_\mu}}{\not{\partial}} e_L = \bar{e}_L^T \cdot \mathbb{1} \cdot \not{\partial} e_L .$$

Hence, its form is not changed by the above field redefinition.

- The same is true for the Z & A-couplings since they originate from the A^3 & B-couplings, which are governed by the diagonal matrices T^3 and $Y \cdot \mathbb{1}$.

- It is not true for the $W^\pm$-couplings, which have their origin in the non-diagonal generators $T^{1,2}$ mixing, e.g., up- & down-type quarks.

- These "charged bosons" couple to fermions via

$$\mathcal{L} \supset - \frac{g_2}{\sqrt{2}} \left(J_\mu^+ W^{+\mu} + h.c. \right) \equiv \mathcal{L}_{cc} ,$$

where $J_\mu^+ = \bar{\nu}_L^T \bar{\sigma}_\mu e_L + \bar{u}_L^T \bar{\sigma}_\mu d_L$ (This is called the "charged current", as oppossed to the neutral current coupling to Z.)

- In terms of the mass eigenstates

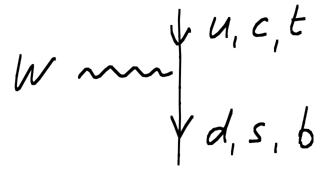
$$\bar{u}_L'^T = \bar{u}_L^T L_u \quad ; \quad d_L' = L_d^+ d_L \quad ; \quad e_L' = L_e^+ e_L$$

we have $$J_\mu^+ = \bar{\nu}_L^{\,T} \bar{\sigma}_\mu L_e e_L' + \bar{u}_L'^{\,T} \bar{\sigma}_\mu \underbrace{L_u^+ L_d}_{\equiv V_{CKM}} d_L' .$$

This matrix can simply be absorbed in a redefinition of the vector ν_L.

(→ Cabbibo, Kobayashi, Maskawa, '73)

This matrix governs flavor changes in charged current transitions, e.g.



The actual numbers:

Masses :
(in GeV)

family	u	d	e
1	0.003	0.005	0.0005
2	1.2	0.1	0.1
3	175	4.2	1.7

(very rough) pattern :

[slightly better after running to $M_{GUT} \sim 10^{16}$ GeV, where it can be "explained" in certain SU_5 models]

$$\alpha^4 m_t \qquad \alpha^2 m_{b,\tau}$$
$$\alpha^2 m_t \qquad \alpha\, m_{b,\tau}$$
$$m_t \qquad m_{b,\tau}$$

with $\alpha \sim 1/200$

CKM-matrix :

$$V_{CKM} \simeq \begin{pmatrix} 0.975 & \cdots & \cdots \\ 0.22 & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix} \simeq \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix}$$

to be multiplied by $O(1)$-numbers ("Wolfenstein parameterization)

10.3 CP-violation

Let us count the parameters of V_{CKM} for the general case of N flavors:

- a $U(N)$-matrix has N^2 real parameters
- demanding reality, we have the smaller group of $O(N)$-matrices with $N(N-1)/2$ real parameters (angles).

$\Rightarrow$ $U(N)$-matrices have $N^2 - N(N-1)/2 = N(N+1)/2$ phases.

- The fermions can absorb $2N-1$ phases.

  ↑

  (an overall phase does not affect V_{CKM})

$\Rightarrow$ V_{CKM} has $N(N+1)/2 - (2N-1) = (N-1)(N-2)/2$ physical phases. (zero for $N=2$, $\underline{\underline{\text{one}}}$ for $N=3$)

Phases, being truely complex, physical lagrangian parameters, induce CP-violation.

- To understand this statement, recall first that, for a charged Dirac fermion $\begin{pmatrix}\psi\\ \bar{\chi}\end{pmatrix} \xrightarrow{C} \begin{pmatrix}\chi\\ \bar{\psi}\end{pmatrix}$,

  or, in the language of Weyl fermions, $\psi, \chi \xrightarrow{C} \chi, \psi$

- P (parity) by definition exchanges l.h. & r.h. fields, i.e.

$$\begin{pmatrix}\psi\\ 0\end{pmatrix}, \begin{pmatrix}0\\ \bar{\chi}\end{pmatrix} \xrightarrow{P} \begin{pmatrix}0\\ \bar{\chi}\end{pmatrix}, \begin{pmatrix}\psi\\ 0\end{pmatrix}.$$

(This can not be formulated as a trf. of a set of Weyl fermions.)

- Specifically for a (Dirac) mass term:

$$m\,\bar{\Psi}_{D,L}\,\Psi_{D,R} \xrightarrow{P} m\,\bar{\Psi}_{D,R}\,\Psi_{D,L}. \quad \left(\text{where } \Psi_{D,L} = \begin{pmatrix}\psi\\0\end{pmatrix}\right.$$

$$\left. \& \ \Psi_{D,R} = \begin{pmatrix}0\\ \bar{\chi}\end{pmatrix}\right)$$

- This can be translated in "Weyl language" as

$$m\,\bar{\psi}\,\bar{\chi} \xrightarrow{P} m\,\chi\,\psi.$$

- Promoting ψ & χ to vectors of Weyl fermions and introducing corresponding mass matrices we thus have

$$\mathcal{L}_M = \psi^T M \chi + \bar{\psi}^T \bar{M} \bar{\chi}$$

$$\downarrow P$$

$$\mathcal{L}_M = \bar{\chi}^T M \bar{\psi} + \chi^T \bar{M} \psi$$

$$\downarrow C$$

$$\mathcal{L}_M = \bar{\psi}^T M \bar{\chi} + \psi^T \bar{M} \chi$$

$$= \psi^T \bar{M} \chi + \bar{\psi}^T M \bar{\chi}$$

Thus, $\mathcal{L}_M$ is "CP-invariant" if M is a real matrix. (Of course, this is only meaningful if the remaining part of $\mathcal{L}$ prevents us from simply absorbing any phases in ψ & χ.)

- This can now be immediately applied to the SM case: Go to the formulation with generic mass matrices & no "V_{CKM}" in the gauge part (gauge eigenstates). The physical phase of V_{CKM} then finds its way into the mass Lagrangian and induces CP-violation according to the above argument.

Note: CP is also referred to as "particle-antiparticle symmetry". [C does this job for scalar fields, but in the fermionic case it also changes the handedness. CP is, by contrast, the true "particle-antiparticle" symm. of generic lagrangians, e.g.

$$\begin{pmatrix} e_L \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \bar{e}_R \end{pmatrix} \xrightarrow{P} \begin{pmatrix} 0 \\ \bar{e}_R \end{pmatrix}, \begin{pmatrix} e_L \\ 0 \end{pmatrix} \xrightarrow{C} \begin{pmatrix} 0 \\ \bar{e}_L \end{pmatrix}, \begin{pmatrix} e_R \\ 0 \end{pmatrix} \Bigg]$$

10.4 GIM mechanism

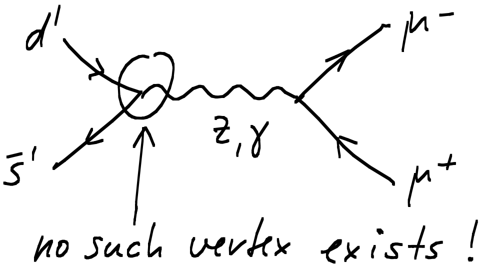
Recall: At tree level in the SM there are no FCNC's since

$W^\pm_\mu$ couple to $\bar{u}_L^T \bar{\sigma}_\mu d_L$ (→ FC after going to mass eigenstates)

γ, Z_μ couple to $\bar{u}_L^T \bar{\sigma}_\mu u_L$, $\bar{d}_L^T \bar{\sigma}_\mu d_L$ (→ no FC after going to mass eigenstates since rotation matrix drops out)

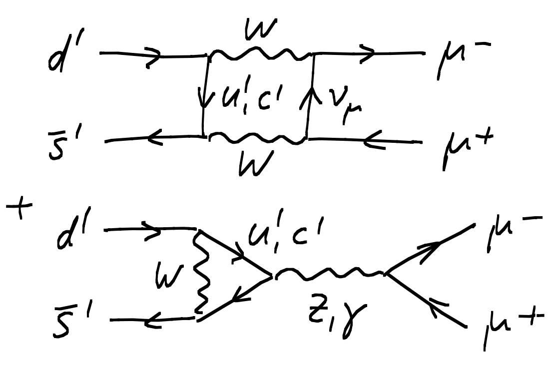
Example:

$K \longrightarrow \mu^+\mu^-$ does not arise at tree level since
$(d'\bar{s}')$



no such vertex exists!

However, at one-loop-level this process can occur:



\+ ...

[We restrict ourselves to a 2-generation-SM, which is a good approx. since the t-quark is very heavy and mixes very weakly.]

- This amplitude can be written as

$$A \sim V_{du} \bar{V}_{su} f(\Lambda, m_u, m_W) + V_{dc} \bar{V}_{sc} f(\Lambda, m_c, m_W).$$

- Since it calculates the coeff. of a higher-dim. operator (of the type $(\bar{\psi}\psi)^2$), which is not part of the tree-level lagrangian, renormalizability requires A to be finite.

- This is indeed the case since $m_u \neq m_c$ is irrelevant at high energies and

$$V_{du} \bar{V}_{su} + V_{dc} \bar{V}_{sc} = \sum_{i=u,c} V_{di} \bar{V}_{si} = (VV^+)_{ds} = 0$$

(since V is unitary).

- Once we know that A is finite, we can ask about its behaviour in the limit $m_u \to m_c$. By the same cancellation as above, A must vanish in this limit.

- The precise behaviour is $A \sim m_c^2 - m_u^2$ for $m_u \to m_c$ (see book by G. Ross on unified theories for details), implying

$$A \sim \frac{m_c^2 - m_u^2}{m_W^4}$$ for dimensional reasons.

- This extra suppression of FCNC's at 1-loop in the SM is known as <u>GIM mechanism</u> (Glashow, Iliopoulos, Maiani, '70). It occurs in many rare processes and is crucial for the correct description of flavor physics. BSM-models usually have difficulties to maintain this suppression.

<u>Comment:</u> The GIM paper <u>postulated</u> the c-quark to realize this suppression.

10.5 Neutrino masses

If the SM is only a low-energy eff. field theory (valid below some scale M), we expect higher-dim. operators even at tree level. The only such operator at mass-dim. 5 is

$$\mathcal{L} \supset \frac{1}{M}(l\cdot\phi)^2 = \frac{1}{M}\, l_i^\alpha l_{\alpha j}\, \varepsilon^{ik}\varepsilon^{jl}\, \underset{\uparrow}{\phi_k}\, \underset{\uparrow}{\phi_l}$$

(and its h.c.). SU_2 - indices

- $\phi_{vac} = \binom{0}{v} \Rightarrow \mathcal{L} \supset \frac{v^2}{M}\nu^\alpha \nu_\alpha + h.c.$

- This "Majorana mass term" can also be written using a "Majorana fermion" $\nu_M = \binom{\nu}{\bar{\nu}}$: $\mathcal{L} \supset \frac{v^2}{M}\bar{\nu}_M \nu_M$.

- Interesting fact: $m_\nu \approx 5\cdot 10^{-2}\,eV$ (expected value)

  implies $M \approx 1.2\cdot 10^{15}\,GeV$ (hint at GUT scale?)

- In any case: The SM must break down at that scale (or earlier).

10.6 See-saw mechanism

(Minkowski, '77; Yanagida; Gell-Mann/Ramond/Slansky, ~79)

- The above operator naturally arises if a total singlet ν_R (the "r.h. neutrino") is integrated out:

$$\mathcal{L} \supset \lambda\, l\phi\nu_R - \frac{1}{2}M\nu_R\nu_R + h.c. \Rightarrow \mathcal{L} \supset -m_D\nu_R\nu_L - \frac{1}{2}M\nu_R\nu_R + h.c.$$

$$\xRightarrow[\text{varying } \nu_R]{} \delta\mathcal{L} \supset -\delta\nu_R\,(m_D\nu_L + M\nu_R)$$

$$\Rightarrow \nu_R = -\frac{m_D}{M}\nu_L \qquad \Rightarrow \qquad \mathcal{L} \supset \frac{1}{2}\underbrace{\frac{m_D^2}{M}}_{\equiv -m_\nu}\cdot \nu_L \nu_L$$

$\Rightarrow$ tiny ν-mass induced by "normal" m_D and large M through "see-saw".

- Redoing this calculation for 3 generations and N r.h. neutrinos, we find:

$$m_\nu = -\underset{\substack{\nearrow \\ 3\times N \text{ matrix}}}{m_D}\;\underset{\substack{\uparrow \\ \text{symm.} \\ N\times N \text{ matrix}}}{M^{-1}}\;\underset{\substack{\nwarrow \\ N\times 3 \text{ matrix}}}{m_D^T}$$

10.7 MNS-matrix & ν-oscillations

(analogue of "CKM"; Maki, Nakagawa, Sakata, '62)

- The complex symm. matrix m_ν can be diagonalized by a unitary trf.: $m_\nu = L\, m_\nu^{diag} L^T$.

- Thus, we can write the lagrangian as

$$\mathcal{L} \supset -\bar{e}_L^T M_e \bar{e}_R - \frac{1}{2}\bar{\nu}^T m_\nu \bar{\nu} - \frac{g_2}{\sqrt{2}} W^{+\mu}\bar{\nu}^T \bar{\sigma}_\mu e_L + h.c.$$

$$= -\bar{e}_L^T L_e M_e^{diag} R_e^+ \bar{e}_R - \frac{1}{2}\bar{\nu}^T L_\nu m_\nu^{diag.} L_\nu^T \nu$$

$$- \frac{g_2}{\sqrt{2}} W^{+\mu}\bar{\nu}^T L_\nu \bar{\sigma}_\mu L_\nu^+ L_e L_e^+ e_L + h.c.$$

$$= -\bar{e}_L'^T M_e^{diag} \bar{e}_R' - \frac{1}{2}\bar{\nu}'^T m_\nu^{diag} \bar{\nu}'$$

$$- \frac{g_2}{\sqrt{2}} W^{+\mu}\bar{\nu}'^T \bar{\sigma}_\mu \underbrace{V_{MNS}}_{\text{origin of } \nu\text{-mixing; } \equiv L_\nu^+ L_e} e_L' + h.c.$$

The actual numbers:

(unfortunately, only mass-square-differences are presently accessible)

solar $\longrightarrow$ $\Delta m_{12}^2 = m_2^2 - m_1^2 = (8.0 \pm 0.3) \cdot 10^{-5}\ eV^2$

atmosph. $\longrightarrow$ $|\Delta m_{23}^2| = \quad - \quad = (2.5 \pm 0.2) \cdot 10^{-3}\ eV^2$

$$V_{MNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} e^{i\varphi} \\ 0 & 1 & 0 \\ -s_{13} e^{-i\varphi} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\uparrow$

Note: There are in addition two physical phases in $m_\nu^{\text{diag.}}$

Here: $s_{12} \equiv \sin\theta_{12}$ etc. Data: $\tan^2\theta_{12} = 0.45 \pm 0.05$

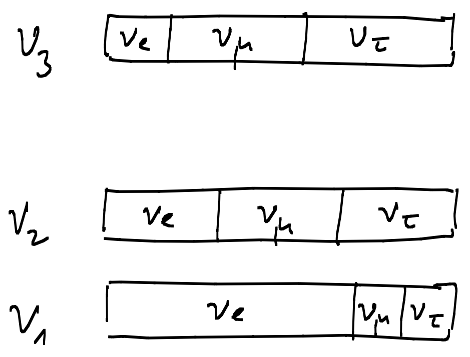
$\sin^2 2\theta_{23} = 1.02 \pm 0.04$

$\sin^2 2\theta_{13} = 0 \pm 0.05$

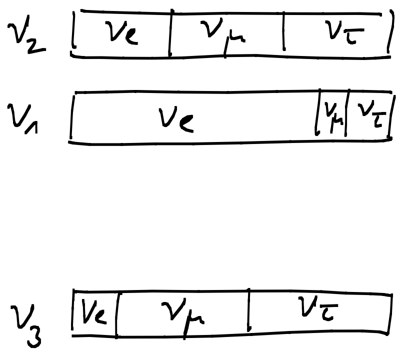
Theoretical speculations: • Is $\theta_{23} = 45°$ for "deep reason"

• Is $\theta_{13} = 0$ (or parametrically small)?

- Frequently used illustration:



("normal spectrum")



("inverted spectrum")

10.8 Electroweak Precision Analysis

(Closely following lecture notes of J.D. Wells, hep-ph/0512342)

I - Tree level observables (observable → "^")

$\hat{\alpha} = 1/137.03599$ (in IR)

$\hat{G}_F = 1.16639 \cdot 10^{-5}\ \mathrm{GeV}^{-2}$ (μ-decay)

$\hat{m}_Z = 91.187\ \mathrm{GeV}$

$\hat{m}_W = 80.43\ \mathrm{GeV}$

$\hat{s}^2_{eff} = 0.2315$ (LR-asymmetry, see below)

$\hat{\Gamma}_{e^+e^-} = 84.0\ \mathrm{MeV}$ ($Z \to e^+e^-$)

- Most relevant lagr. parameters: g (g_{SU_2}) ; g' ($g_{U_{1,Y}}$) ; v

  or e ; $s = \sin\theta$; v

  ($g = e/s$; $g' = e/c$)

- Tree level relations:

$$\hat{\alpha} = e^2/4\pi$$

$$\hat{G}_F = 1/\sqrt{2}v^2$$

[$v \to \sqrt{2}v$ relative to our earlier conventions]

$$\hat{m}_Z^2 = e^2v^2/4s^2c^2$$

$$\hat{m}_W^2 = e^2v^2/4s^2$$

$$\hat{s}^2_{eff} = s^2$$

$$\hat{\Gamma}_{e^+e^-} = \frac{v}{96\pi} \cdot \frac{e^3}{s^3c^3}\left[\left(-\frac{1}{2}+2s^2\right)^2 + \frac{1}{4}\right]$$

- Can determine lagr. parameters from χ^2-fit:

$$\chi^2(e,s,v) = \underset{\substack{\uparrow \\ \hat{m}_W^2,\ \hat{s}^2_{eff},\ \text{etc.}}}{\sum_i} \frac{(\hat{O}_i - O_i(e,s,v))^2}{(\Delta\hat{O}_i)^2}$$

- Alternatively: Calculate e, s, v in terms of $\hat{\alpha}, \hat{G}_F, \hat{m}_Z^2$
  (best-measured obs.)

  and express the other obs. through them:

  e.g. $\hat{m}_W^2 = \pi \sqrt{2}\, \hat{G}_F^{-1} \hat{\alpha} \left(1 - \sqrt{1 - 4\pi\hat{\alpha}/(\sqrt{2}\,\hat{G}_F \hat{m}_Z^2)}\right)^{-1}$

  $\hat{s}_{eff}^2 = ..$

  $\hat{\Gamma}_{e^+e^-} = ..$

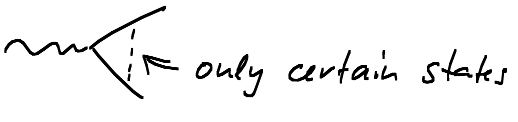
- One finds: These three obs. are off by 15σ, 120σ, 10σ resp.!

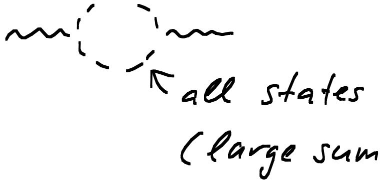
  ⇒ Tree-level "theory" ruled out!

Jumping ahead: Incl. loop effects, this is Ok if $114\,\text{GeV} < m_H < 219\,\text{GeV}$
↑
85% C.L.

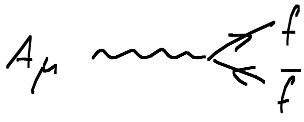
II One-loop

- Focus only on self-energy corrections of γ, $W^\pm$, Z — "oblique corrs."
  (most relevant for new-physics effects since:

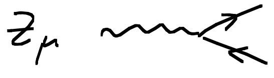




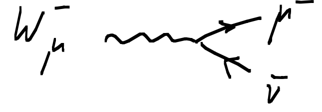
- Feynman rules:



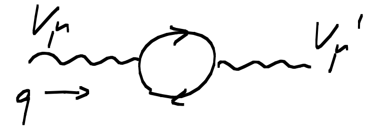
$ieQ_f\gamma_\mu$



$\frac{ie}{sc}\gamma_\mu\left[(T_f^3 - Q_f s)P_L - Q_f s^2 P_R\right]$



$\frac{ie}{s\sqrt{2}}\gamma_\mu P_L$

⇒ 

$$\Rightarrow\; i\left[\Pi_{VV'}(q^2)g^{\mu\nu} - \underbrace{\Delta_{VV'}(q^2)q^\mu q^\nu}_{\substack{\text{irrelevant for us}\\ \text{since we couple to on-shell}\\ \text{fermions only } \&\ m_f = 0}}\right]$$

Note also: γ massless $\Rightarrow$ $\Pi_{\gamma\gamma}(0) = \Pi_{\gamma Z}(0) = 0$

from gauge inv.

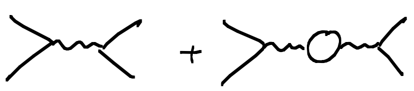
(Not quite true: $W^\pm$ - loop contributes to $\Pi_{\gamma Z}(0)$ because of non-abelian SU_2 structure. But this won't happen again with (most) new physics contributions...)

III One-loop predictions for observables:

$$(\hat{m}_Z^2)^{th} = \frac{e^2 v^2}{4 s^2 c^2} + \Pi_{ZZ}(m_Z^2)$$

$$(\hat{m}_W^2)^{th} = \frac{e^2 v^2}{4 s^2} + \Pi_{WW}(m_W^2)$$

$$(\hat{\alpha})^{th} = \frac{e^2}{4\pi}\left(1 + \Pi'_{\gamma\gamma}(0)\right)$$



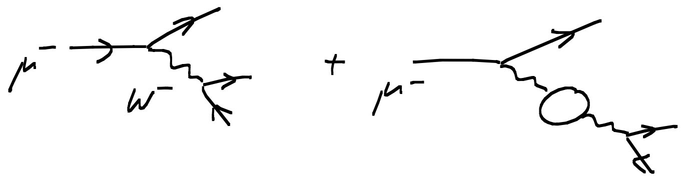
here $\Pi'_{\gamma\gamma}$ arises from $\lim \frac{\Pi_{\gamma\gamma}(q^2)}{q^2}$.

$\hat{G}_F$ is defined by

$$\tau_\mu^{-1} = \frac{\hat{G}_F^2 m_\mu^5}{192\pi^3} \underbrace{K(\alpha, m_e, m_\mu, m_W)}$$

$\to 1$ for $\alpha \to 0$, $\frac{m_e}{m_\mu} \to 0$, $\frac{m_\mu}{m_W} \to 0$

$(\to$ PDG$)$



$$\Rightarrow \quad (\hat{G}_F)^{th} = \frac{1}{\sqrt{2} v^2}\left(1 - \frac{\Pi_{WW}(0)}{m_W^2}\right)$$

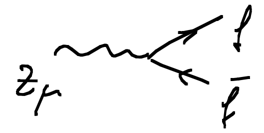
$\hat{s}_{eff}^2$ is defined by

$$\hat{A}_{LR} = \frac{(1/2 - \hat{s}_{eff}^2)^2 - (\hat{s}_{eff}^2)^2}{(1/2 - \hat{s}_{eff}^2)^2 + (\hat{s}_{eff}^2)^2}$$

(this is just the tree-level relation)

- $\hat{A}_{LR}$ is the cross-sect. asymm. for lepton-production from left/right polarized e^+e^- collisions (at Z-pole):

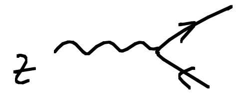
$$\hat{A}_{LR} = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = \frac{c_L^2 - c_R^2}{c_L^2 + c_R^2}$$

where 

$\longleftrightarrow \; i\gamma_\mu \left(c_L P_L - c_R P_R\right)$

(c.f. Feynman-rules above)

- The relevant loop correction is



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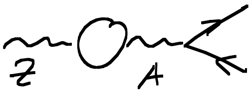


(Π_{ZZ} drops out !)

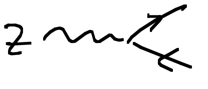
$$\Rightarrow \quad c_L = \frac{e}{sc}\left[T^3 - Q\left(s^2 - sc\,\frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2}\right)\right]$$

$$c_R = -\frac{eQ}{sc}\left[s^2 - sc\,\frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2}\right]$$

hence: $(\hat{s}^2_{eff})^{th} = s^2 - sc\,\frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2}$

And finally:

$$\left(\hat{\Gamma}_{e^+e^-}\right)^{th} = \frac{Z_Z}{48\pi} \cdot \frac{e^2}{s^2c^2}\,\hat{m}_Z\left[\left(-\frac{1}{2} + 2(\hat{s}^2_{eff})^{th}\right)^2 + \frac{1}{2}\right]$$



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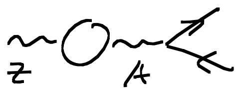

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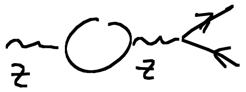


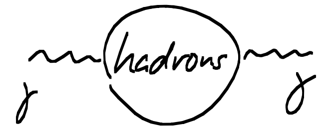
$$\Downarrow \qquad\qquad \Downarrow$$

$$s^2 \to (\hat{s}^2_{eff})^{th} \qquad 1 \to Z_Z$$

with $Z_Z = 1 + \delta_Z = 1 + \Pi'_{ZZ}(m_Z^2) \simeq 1 + \frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\cdots(0)}{m_Z^2}$

Important Comment:

We have naively achieved nothing since $e^2 = \frac{4\pi\hat\alpha}{1+\Pi'_{\gamma\gamma}(0)}$ and

$\Pi'_{\gamma\gamma}(0)$ requires , which we cant calculate.

Trick:
$$\Pi'_{\gamma\gamma}(0) = \underbrace{\mathrm{Re}\,\frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2}}_{\text{calculate}} - \underbrace{\left[\frac{\mathrm{Re}\,\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2} - \Pi'_{\gamma\gamma}(0)\right]}_{\equiv \Delta\alpha(m_Z)}$$

$$\Delta\alpha_{m_Z} = \underbrace{\Delta\alpha_\ell(m_Z)}_{\substack{0.03150 \\ \text{(tiny error!)}}} + \underbrace{\Delta\alpha_{top}(m_Z)}_{\substack{m_t\text{- dep.} \\ \text{but very small}}} + \underbrace{\Delta\alpha^{(5)}_{had}(m_Z)}_{\substack{\text{This is the} \\ \text{problem!}}}$$

- Optical theorem + analytic continuation

$$\Rightarrow \quad \Delta\alpha^{(5)}_{had} = -\frac{m_Z^2}{3\pi}\int_{4m_\pi^2}^{\infty} \frac{R_{had}(q^2)\,dq^2}{q^2(q^2-m_Z^2)} \quad ; \quad R_{had} = \frac{\sigma_{had}(q^2)}{\sigma_{e^+e^-}(q^2)}$$

Data $\Rightarrow$ $\Delta\alpha^{(5)}_{had} \approx 0.0276$

Can now replace $\hat\alpha$ by $\hat\alpha(m_Z^2)$: $\quad \hat\alpha(m_Z^2) = \frac{\hat\alpha}{1-\Delta\alpha(m_Z^2)}$

data

$$= \frac{e^2}{4\pi}\left(1 + \frac{\mathrm{Re}\,\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2}\right)$$

theory

$$\left(= 1/128.936 \pm 0.046\right)$$

Thus: $\quad e^2 = 4\pi\hat\alpha(m_Z^2)\left(1 - \frac{\mathrm{Re}\,\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2}\right)$ (see above)

$\quad v^2 = \frac{1}{\sqrt{2}\hat G_F}\left(1 - \frac{\Pi_{WW}(0)}{m_W^2}\right)$ (see $\hat G_F$ - formula)

$$s^2c^2 = \frac{\pi\hat\alpha(m_Z^2)}{2\sqrt{G_F}\,\hat m_Z^2}(1+\delta_S) \; ; \quad \delta_S \equiv \frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\Pi_{WW}(0)}{m_W^2} - \frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2}$$

(see $\hat m_Z^2$-formula)

Now we can express the other observables through our basic (most precisely measured) observables at the one-loop-level:

$$(\hat m_W^2)^{th} = \frac{\pi\hat\alpha(m_Z^2)}{\sqrt{2}\,\hat G_F\,\hat s_0^2}\left[1 - \frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2} - \frac{c_0^2}{c_0^2-s_0^2}\cdot\delta_S - \frac{\Pi_{WW}(0)}{m_W^2} + \frac{\Pi_{WW}(m_W^2)}{m_W^2}\right]$$

where $\hat s_0^2\hat c_0^2 \equiv \dfrac{\pi\hat\alpha(\hat m_Z^2)}{\sqrt{2}\,\hat G_F\hat m_Z^2}$.

(Note: In the higher order terms, it's irrelevant which precise def. of c, s, m_Z etc. we use.)

$(\hat s_{\text{eff}}^2)^{th} = \dots$

$(\hat\Gamma_{e^+e^-})^{th} = \dots$

could add as many obs. as one wants.

<u>Note</u>: This is finite, i.e., the divergences cancel!

↓ to see this, work out the loops.

<u>Method</u>: <u>Passarino-Veltman fcts.</u>

$$16\pi^2\mu^{4-n}\int\frac{d^nq}{i(2\pi)^n}\cdot\frac{1}{q^2-m^2+i\varepsilon} = A_0(m^2)$$

$$16\pi^2\mu^{4-n}\int\frac{d^nq}{i(2\pi)^n}\cdot\frac{1}{[q^2-m_1^2+i\varepsilon][(q-p)^2-m_2^2+i\varepsilon]} = B_0(p^2,m_1^2,m_2^2)$$

$$\Big[\qquad -\text{''}- \qquad\Big]\cdot q_\mu = p_\mu B_1(-\text{''}-)$$

also: B_{21}, $B_{22} = \dots$

One finds: $A_0(m^2) = m^2\left(\Delta + 1 - \ln m^2/\mu^2\right)$, $\Delta \equiv \frac{1}{4-n} - \gamma_E + \ln 4\pi$

$B_0(\cdot\cdot) = \Delta - \ln(p^2/\mu^2) + \cdots$

etc.

This systematic approach makes the finiteness-check and the actual calculation (relatively) easy.

- The above expressions for (in principle all!) el. weak observables are valid both for SM-effects (up to the $\Pi_{\gamma Z}$ issue discussed earlier) and "new physics".
- Lets restrict ourselves to "new physics" ($\Pi_{ZZ} \to \Pi_{ZZ}^{new}$ etc.) and also assume $m_Z/m_{new} \ll 1$.
- It then turns out that the corrections to all oblique corrections to all observables can be expressed through

$$T = \frac{1}{\alpha}\left[\frac{\Pi_{WW}(0)}{m_W^2} - \frac{\Pi_{ZZ}(0)}{m_Z^2} - 2\frac{s}{c}\frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2}\right]$$

$$S = \frac{4s^2c^2}{\alpha}\left[\frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\Pi_{ZZ}(0)}{m_Z^2} + \cdots\right]$$

$(S+U) = \cdots$ similar to S, but with $Z \to W$.

(Extensions of this parameterization beyond $m_Z/m_{new} \ll 1$ are known.)

Finally:

$$\left.\begin{aligned}\Delta(\hat{s}^2_{eff})^{th} &= (3.59\cdot 10^{-3})\,S - (2.54\cdot 10^{-3})\cdot T\\ \Delta(\hat{m}_W^2/\hat{m}_Z^2)^{th} &= \cdots \quad S\ ,\ T\ ,\ U\\ \Delta(\hat{\Gamma}_{\ell^+\ell^-})^{th} &= \cdots \quad S\ ,\ T\end{aligned}\right\}\text{equiv. to our previous formulae!}$$

- Crucial point: It is relatively easy to get these S, T, U-values for a given new-physics scenario (e.g. a 4-th generation etc) and hence to check how well a given scenario does (i.e. whether it keeps $S^{new} = T^{new} = U^{new} = 0$ within suff. precision. (below $O(1)$!)

- One can also do an "oblique" χ^2-analysis including SM + new-physics parameters → Wells's review for details.

- Finally: Alternative (maybe more modern) perspective:

SM as low-energy eff. FT;

$$S \quad \leftrightarrow \quad O_S = H^\dagger \sigma^i H \, F(SU_2)^i_{\mu\nu} \, F(U_1)^{\mu\nu}$$

$$T \quad \leftrightarrow \quad |H^\dagger D_\mu H|^2$$

(→ W. Skiba, 1006.2142 for more details).