2 The Standard Model

2.1 Some further details concerning the (non-abelian) gauge interactions

- Recall: $\mathcal{L}_{gauge} = -\sum_{i=1}^{3} \frac{1}{2g_i^2} \operatorname{tr} F_{\mu\nu}^{(i)} F^{(i),\mu\nu}$; $G_{SM} = U_1 \times SU_2 \times SU_3$
- Some important facts (cf. e.g. books by Pokorski ; Peskin/Schröder etc.)
  - Let $G = SU_N$. For any $g \in G$ $\exists$ $T \in H$ such that

$$g = \exp(iT)$$

(Here H = {vector space of hermitian, traceless matrices} = Lie (G).)

  - For some neighbourhood of $\mathbb{1} \in G$ the map "exp" is a diffeomorphism.

(This generalizes to all Lie groups, including those which can not be defined as matrix groups.)

- $A_\mu \in \mathrm{Lie}(G)$; $D_\mu = \partial_\mu + iA_\mu$; $F_{\mu\nu} = \underbrace{\frac{1}{i}[D_\mu, D_\nu]}_{\text{not a diff. operator!}} \in \mathrm{Lie}(G)$

  - Choose basis $\{T^a\}$ of Lie (G) such that $\operatorname{tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$

"generators" (widely used convention).

  - We can write $A_\mu = A_\mu^a T^a$; $F_{\mu\nu} = F_{\mu\nu}^a T^a$

(Check that $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f^{abc} A_\mu^b A_\nu^c$ with "structure constants" f^{abc} defined by $[T^a, T^b] = i f^{abc} T^c$.)

Note:
- $A_\mu^a \to g A_\mu^a$ makes kinetic term canonical
- $G = G_1 \times G_2 \times \cdots \iff H = H_1 \oplus H_2 \oplus \cdots$
- For U_1 there is no "tr" and canonical normalization of generator (need charged particles for normalization).

2.2 Scalar lagrangian and electroweak symm. breaking

- Recall: $V(\phi) = -\nu|\phi|^2 + \lambda|\phi|^4 \qquad (|\phi|^2 \equiv \phi^+\phi)$

  $\langle\phi\rangle = \binom{0}{v} \qquad (v \text{ real})$

- Crucial assumption: ϕ is an SU_2-doublet with hyperchange $Y = 1/2$

  (The convention $Y \to 2Y$ is also widely used.)

- The masses of vector fields come from

$$|D_\mu\phi|^2 \supset \left|\left(g_2 A_\mu^a T^a + g_1 B_\mu Y\right)\cdot\binom{0}{v}\right|^2 \quad \text{with } T^a = \tfrac{1}{2}\sigma^a$$

$$= \frac{v^2}{4}\left\{ g_2^2\left(A_\mu^1 A^{1\mu} + A_\mu^2 A^{2\mu}\right) + \left(g_2 A_\mu^3 - g_1 B_\mu\right)\left(g_2 A^{3\mu} - g_1 B^\mu\right)\right\}$$

$\downarrow$ mass for W-bosons

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(A_\mu^1 \mp i A_\mu^2\right)$$

$$(\sim m_W^2 W^+W^-)$$

$\downarrow$ mass for the Z-boson

$$Z_\mu = A_\mu^3 \cos\theta - B_\mu \sin\theta$$

(We must use orthog. trf. to keep kinetic term canonical.)

- The above requires $\sin\theta/\cos\theta = g_1/g_2$

  or $\underline{\tan\theta = g_1/g_2}$ $\qquad$ [$\theta \equiv \theta_W$ is the "Weinberg angle"]

- $A_\mu = A_\mu^3 \sin\theta + B_\mu \cos\theta$ remains massless.

- The A^3, B-part of the covariant derivative (acting on any field, not just ϕ) can now be worked out:

$$g_2 A^3 T^3 + g_1 B Y = g_2 Z \cos\theta\, T^3 + g_2 A \sin\theta\, T^3$$
$$- g_1 Z \sin\theta\, Y + g_1 A \cos\theta\, Y$$

$$= Z \frac{g_2}{\cos\theta}\left(T^3 - Y\sin^2\theta\right) + A \underbrace{g_2 \sin\theta}_{\equiv e} \underbrace{(T^3 + Y)}_{\equiv Q}$$

[In the problems, you were asked to check all this in detail.]

2.3 Fermions

Let's first fix our (hopefully rather standard) conventions for Dirac fermions:

$$\mathcal{L} = \bar{\psi}_D \, i \partial\!\!\!/ \, \psi_D \quad ; \quad \psi_D \xrightarrow{\Lambda} \exp\left(t^{\mu\nu} \tfrac{1}{4}[\gamma_\mu, \gamma_\nu]\right) \psi_D$$

$$\text{with } \Lambda = \exp\left(i t^{\mu\nu} M_{\mu\nu}\right) \in SO_{1,3}$$

appropriately normalized $SO_{1,3}$-generators

Clifford algebra: $\{\gamma_\mu, \gamma_\nu\} = -2\eta_{\mu\nu}$.

Explicit form (in Weyl basis): $\gamma_\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}_\mu & 0 \end{pmatrix}$

$$\sigma_\mu = (\sigma_0, \sigma_i) = \left(\mathbb{1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \quad ; \quad \bar{\sigma}_\mu = (\sigma_0, -\sigma_i)$$

$$\gamma_5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \quad ; \quad P_{L,R} = \frac{\mathbb{1} \pm \gamma_5}{2} \qquad \left(P_L = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} ; \; P_R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \right)$$

$\psi_D = \begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix}$ These 2-component or Weyl spinors transform independently under $SO_{1,3}$ (since $\frac{1}{4}[\gamma_\mu, \gamma_\nu]$ is block-diagonal).

Instead of using Weyl-spinors, one can also use left-handed and right-handed Dirac-spinors:

$$\psi_{D,L} = \begin{pmatrix} \psi \\ 0 \end{pmatrix} = P_L \psi_D \quad ; \quad \psi_{D,R} = \begin{pmatrix} 0 \\ \bar{\chi} \end{pmatrix} = P_R \psi_D .$$

We will also use notation

$$\psi_D = \begin{pmatrix} \psi_L \\ \bar{\psi}_R \end{pmatrix} \quad ; \quad \psi_{D,L} = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix} \quad ; \quad \psi_{D,R} = \begin{pmatrix} 0 \\ \bar{\psi}_R \end{pmatrix}.$$

In this notation, ψ_L & ψ_R are both Weyl spinors (with identical trf. properties). "L" & "R" are just indices.

Using the isomorphism $SL_{2,\mathbb{C}} \simeq SO_{1,3}$ (near $\mathbb{1}$, not globally!), we can also write Weyl spinors with their explicit $SL_{2,\mathbb{C}}$-index:

$$\psi = \{\psi_\alpha\} \quad , \quad \alpha = 1, 2 .$$

Then $\psi_D = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix}$, where the "dotted index" transforms under the compl. conj. repres. of $SL_{2,\mathbb{C}}$.

Weyl indices are raised and lowered with the invariant tensor $\varepsilon_{\alpha\beta}$ or $\varepsilon_{\dot\alpha\dot\beta}$ of $SL_{2,\mathbb{C}}$. χ_α is a Weyl spinor, just like ψ_α.

$\bar\chi_{\dot\alpha}$ is its complex conjugate and $\bar\chi^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\chi_{\dot\beta}$

($\rightarrow$ tutorials & Appendix of Wess/Bagger for more details.)

Intermediate summary:

$$1 \text{ Dirac fermion} = \begin{matrix} 1 \text{ l.h. fermion} \\ + \\ 1 \text{ r.h. fermion} \end{matrix} = 2 \text{ Weyl fermions.}$$

Kinetic term(s): $\bar\psi_D i\partial\!\!\!/ \psi_D = i\psi_D^+\gamma^0\gamma^\mu\partial_\mu\psi_D$

$$= i(\bar\psi^T, \chi^T)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}\partial_\mu\begin{pmatrix} \psi \\ \bar\chi \end{pmatrix} = i\bar\psi^T\bar\sigma^\mu\partial_\mu\psi + i\chi^T\sigma^\mu\partial_\mu\bar\chi$$

$$= i\bar\psi^T\bar\sigma^\mu\partial_\mu\psi + i\bar\chi^T\bar\sigma^\mu\partial_\mu\chi + \text{total derivative}$$

apply the identity $(\bar\sigma^\mu)^{\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}$

$$= \underbrace{i\bar\psi_{\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\alpha}\partial_\mu\psi_\alpha + i\bar\chi_{\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\alpha}\partial_\mu\chi_\alpha}_{\text{Two completely equivalent terms!}}$$

note the lower index, inherited from the original ψ_α

2.4 Charge conjugation

- For scalars: $\varphi \rightarrow e^{i\alpha}\varphi$ for some U_1-symm.

$\Rightarrow \varphi^* \rightarrow e^{-i\alpha}\varphi^*$

$\Rightarrow$ The substitution $\varphi \rightarrow \varphi^c \equiv \varphi^*$

& $\varphi^* \equiv \varphi^c \rightarrow \varphi$

in a Lagrangian corresponds to "charge conjugation".

(Note: If $\mathcal{L}$ contains only real parameters, this corresponds to $S \to S^*$, which is a symm. because S is always real.)

- For Dirac spinors: The naive guess for C would be
$$\psi_D \to \psi_D^* \quad \text{or} \quad \psi_D \to \bar{\psi}_D^T.$$

This does not work since ψ_D^* & $\bar{\psi}_D^T$ do <u>not</u> transform as spinors. We need to use a matrix C (charge conj. matrix) satisfying
$$C^{-1}\gamma_\mu C = -(\gamma_\mu)^T. \qquad (\text{e.g. } C = i\gamma_2\gamma_0)$$
It can be shown (→ problems) that
$$\psi_D^c \equiv C\bar{\psi}_D^T$$
<u>is</u> a Dirac spinor. Thus:

Charge conjugation: $\psi_D \to \psi_D^c \equiv C\bar{\psi}_D^T$

- Writing $\psi_D = \begin{pmatrix}\psi \\ \bar{\chi}\end{pmatrix}$, it is easy to check that $\psi_D^c = \begin{pmatrix}\chi \\ \bar{\psi}\end{pmatrix}$.

[Crucial point: $i\sigma_2 \sim \varepsilon$] (There may be extra "−" signs, depending on the signs of $\varepsilon_{\alpha\beta}$, $\varepsilon^{\alpha\beta}$ – which is pure convention.)

- Obvious: ψ_D l.h./r.h. $\Leftrightarrow$ ψ_D^c r.h./l.h.

- <u>Summary</u> of options for writing fermionic lagrangians:

① Dirac: ψ_D – not useful for SM since l.h. & r.h. parts have different gauge trf. properties.

② l.h./r.h. Dirac: $\psi_{D,L}$, $\psi_{D,R}$

③ l.h. Dirac: $\psi_{D,L}$, $(\psi_{D,R})^c$

④ Weyl: ψ_L, ψ_R (where $\psi_{D,L} = \begin{pmatrix}\psi_L \\ 0\end{pmatrix}$, $\psi_{D,R} = \begin{pmatrix}0 \\ \bar{\psi}_R\end{pmatrix}$)

2.5 Gauge tf. of SM fermions

Experimental fact:

Weyl-fermions	SU_3	SU_2	U_1
$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	3	2	1/6
u_R	$\bar{3}$	1	-2/3
d_R	$\bar{3}$	1	1/3
$\ell_L = \begin{pmatrix} \nu \\ e_L \end{pmatrix}$	1	2	-1/2
e_R	1	1	1

for each generation

here "1" means "singlet"

here "1" means "charge = 1"
(singlet would mean "0")

Note: • The up-type Dirac quark $u_D = \begin{pmatrix} u_L \\ \bar{u}_R \end{pmatrix}$ transforms in the 3 of SU_3 (in agreement with u_R being in the $\bar{3}$).
(However, this Dirac notation is not useful for the SU_2 tf., as mentioned earlier.)

• Example:

$$\bar{q}_L \not{D} q_L = (\bar{q}_L)_{\underset{\text{color}}{A}\underset{\text{weak}}{\alpha}} \gamma^\mu (D_\mu q_L)_{A\alpha} \quad \text{with}$$

$$(D_\mu q_L)_{A\alpha} = \Big[\delta_{AB}\,\delta_{\alpha\beta}\,\partial_\mu + i\,G_\mu^{\overset{1\cdots 8}{a}} (T^a_{SU_3})_{AB}\,\delta_{\alpha\beta}$$
$$+ i A_\mu^{\overset{1\cdots 3}{a}} (T^a_{SU_2})_{\alpha\beta}\,\delta_{AB} + i B_\mu \cdot \frac{1}{6} \cdot \delta_{AB}\,\delta_{\alpha\beta} \Big] (q_L)_{A\alpha}$$

$Y = \frac{1}{6}$ is the number given in the U_1-column in the table.