3 Supersymmetry

3.1 SUSY algebra and superspace

- Recall Poincare algebra
$$[P_\mu, P_\nu] = 0$$
$$[M_{\mu\nu}, P_\rho] = i\eta_{\mu\rho} P_\nu - i\eta_{\nu\rho} P_\mu$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = i\eta_{\mu\rho} M_{\nu\sigma} + \cdots + \cdots + \cdots$$

as a symmetry algebra of $\mathbb{R}^{1,3}$, parameterized by $x^0, \cdots, x^3$.

- This algebra can be represented by diff. operators acting on fcts. on $\mathbb{R}^{1,3}$, e.g. $P_\mu = i\partial_\mu \quad \left(\partial_\mu \equiv \frac{\partial}{\partial x^\mu}\right)$.

This indeed generates translations:

$$\exp[\varepsilon^\mu i P_\mu] f(x) = f(x) - \varepsilon^\mu \partial_\mu f(x) + \cdots = f(x-\varepsilon).$$

(Analogously for $M_{\mu\nu}$)

- Any larger symmetry (i.e. Lie algebra) of a relativistic QFT is the direct sum of the Poinc. alg. and an "internal" symm., such as U_1, SU_2, etc. (Coleman-Mandula theorem).

- This theorem can be avoided if one allows for "Super-Lie-algebras". The resulting non-trivial enlargement of the symm. of space-time is <u>unique</u> (more or less, given appropriate assumptions) and is called the supersymm. algebra. (Haag-Lopuszanski-Sohnius theorem)

- The new generators <u>have</u> to be (Weyl) spinors Q_α. The crucial new <u>anti</u>-commutator is
$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2(\sigma^\mu)_{\alpha\dot\alpha} P_\mu \;.$$
(inv. tensor!)

Furthermore, the Q's anticommute with eachother,

$$\{Q_\alpha, Q_\beta\} = 0 \quad ; \quad \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0 \quad ,$$

and transform like (x-independent) spinors,

$$[P_\mu, Q_\alpha] = 0 \quad ; \quad [M_{\mu\nu}, Q_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta \quad , \text{ where}$$

$$\sigma_{\mu\nu} \equiv -\frac{1}{4}\left(\sigma_\mu \bar\sigma_\nu - \sigma_\nu \bar\sigma_\mu\right).$$

- To construct a QFT invariant under this symm., let us represent the algebra on an appropriate larger space ("superspace"). This space is "parametrized" by

$$x^\mu \;(\mu = 0 \cdots 3) \quad \& \quad \underbrace{\underset{\substack{\uparrow \\ \text{spinor index}}}{\theta^\alpha} \;(\alpha = 1,2)}_{\text{"fermionic coordinates"}}$$

$$\Big[(\theta^\alpha)^* = \bar\theta^{\dot\alpha} \; ; \; \{\theta^\alpha, \theta^\beta\} = 0 \;\; \& \text{ h.c.} \; ; \; \{\theta^\alpha, \bar\theta^{\dot\alpha}\} = 0 \; ;$$

$$\text{explicitly: } (\theta^1)^2 = (\theta^2)^2 = 0 \; ; \; \theta^1\theta^2 = -\theta^2\theta^1 \Big]$$

- Derivatives: $\partial_\alpha \equiv \frac{\partial}{\partial\theta^\alpha}$; $\partial_\alpha \theta^\beta = \frac{\partial}{\partial\theta^\alpha}\theta^\beta = \delta_\alpha{}^\beta$

$$\bar\partial_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} \quad ; \quad \bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \;\cdots\; = \delta_{\dot\alpha}{}^{\dot\beta}$$

$$\partial_\alpha \bar\theta^{\dot\beta} = 0 \quad ; \quad \bar\partial_{\dot\alpha}\theta^\beta = 0$$

- θ's anticommute $\Rightarrow$ ∂'s anticommute:

$$\{\partial_1, \partial_2\}\theta^1\theta^2 = \partial_1\partial_2\theta^1\theta^2 + \partial_2\partial_1\theta^1\theta^2 =$$
$$= -\partial_1\partial_2\theta^2\theta^1 + \partial_2\partial_1\theta^1\theta^2 = -1 + 1 = 0$$

$\Rightarrow$ Natural to try something like $Q_\alpha \sim \partial_\alpha + \cdots$.

Correct definition: $Q_\alpha = \partial_\alpha - i(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu$

$$\bar Q_{\dot\alpha} = -\bar\partial_{\dot\alpha} + i\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu .$$

Problem: Derive from this that $\{Q_\alpha, Q_\beta\} = 0$, $\{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0$

$$\{Q_\alpha, \bar Q_{\dot\beta}\} = 2i(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu$$

3.2 Superfields

A (complex) general SF is a fct.

$$F(x,\theta,\bar\theta) = f(x) + \theta\phi(x) + \bar\theta\bar\chi(x) + \theta^2 m(x) + \bar\theta^2 n(x)$$

$$+ \theta\sigma^\mu\bar\theta v_\mu(x) + \theta^2\bar\theta\bar\lambda(x) + \bar\theta^2\theta\psi(x) + \theta^2\bar\theta^2 d(x)$$

Taylor expansion

all higher terms vanish

[Notation: $\theta\phi(x) = \theta^\alpha\phi_\alpha(x)^{(*)}$; $\theta^2 = \theta^\alpha\theta_\alpha = \varepsilon^{\alpha\beta}\theta_\beta\theta_\alpha = 2\theta_2\theta_1$;

$({}^\alpha{}_\searrow{}_\alpha\ ;\ {}_{\dot\alpha}{}^\nearrow{}^{\dot\alpha})$ $\quad \theta\sigma^\mu\bar\theta = \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}$; $\varepsilon^{\alpha\beta\,(\dot\alpha\dot\beta)} = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$; $\varepsilon_{\alpha\beta\,(\dot\alpha\dot\beta)} = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$]

- To derive what is intuitively a "symm. trf." from our rather abstract objects Q, recall

(*) also: $\psi\chi = \psi^\alpha\chi_\alpha = -\chi_\alpha\psi^\alpha = -\varepsilon_{\alpha\beta}\chi^\beta\varepsilon^{\alpha\gamma}\psi_\gamma = \chi^\beta\psi_\beta = \chi\psi$

– translation: $\delta_\varepsilon\varphi = i\varepsilon^\mu P_\mu\varphi = -\varepsilon^\mu\partial_\mu\varphi$

– SUSY trf.: $\delta_\xi F \equiv (\xi Q + \bar\xi\bar Q)F = \left[\left(\xi\partial - i\xi\sigma^\mu\bar\theta\partial_\mu\right) + \text{h.c.}\right]F$

Note: By "h.c." we mean here the application of a "formal star operation" on the algebra of fcts. & diff. operators. In essence, this means compl. conjugation, except for $(\partial_\alpha)^* = -\bar\partial_{\dot\alpha}$, which is required by consistency.

Explanation of $(\partial_\alpha)^* = -\bar\partial_{\dot\alpha}$: $(\partial_\alpha \theta^\beta)^* = \delta_\alpha{}^\beta = \bar\partial_{\dot\alpha} \bar\theta^{\dot\beta}$

(with $\alpha = \dot\alpha$, $\beta = \dot\beta$)

$$(\partial_\alpha \theta^\beta)^* = (\theta^\beta)^* (\partial_\alpha)^* = \bar\theta^{\dot\beta}(-\bar\partial_{\dot\alpha})$$

$$= \bar\partial_{\dot\alpha} \bar\theta^{\dot\beta}$$

extra "−" sign from exchange of two fermionic objects

- $\delta_\xi F$ continued:
  - calculate $\delta_\xi F$ by simple differentiation
  - expand the result in $\theta, \bar\theta$ analogously to the expansion of F
  - "call" the coefficient of 1 : $\delta_\xi f$
    — " — θ : $\delta_\xi \psi$
    etc.
  - This defines the SUSY-tfs. of the component fields

3.3 Chiral SFs

- It will prove useful to introduce "SUSY covariant" derivatives (in analogy to the Q's): $D_\alpha = \partial_\alpha + i(\sigma^\mu)_{\alpha\dot\alpha} \bar\theta^{\dot\alpha} \partial_\mu$

$$\bar D_{\dot\alpha} = -\bar\partial_{\dot\alpha} - i\theta^\alpha (\sigma^\mu)_{\alpha\dot\alpha} \partial_\mu .$$

- They fulfil $\{D_\alpha, D_\beta\} = 0$ (& h.c.), $\{D_\alpha, \bar D_{\dot\alpha}\} = -2i(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu$
- In addition, they anticommute with the Q's: $\{D_\alpha, \bar Q_{\dot\beta}\} = 0$ etc.
- From this last feature, we can conclude that

$$\bar D_{\dot\alpha} F = 0 \quad \Rightarrow \quad \bar D_{\dot\alpha} \delta_\xi F = 0 .$$

- Thus, superfields fulfilling $\bar D_{\dot\alpha} F = 0$ form a subrepresentation of the representation of the SUSY-alg. provided by general SFs.

- Such SFs are called <u>chiral SFs</u> (and are often denoted by ϕ)

- Every chiral SF can be written as $\phi = \phi(y, \theta)$ with $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$ and decomposed as

$$\phi = A(y) + \sqrt{2}\,\theta\psi(y) + \theta^2 F(y).$$

- Expanding $\delta_\xi\phi$ in this way, we find:

$$\delta_\xi A = \sqrt{2}\,\psi\xi$$
$$\delta_\xi \psi = i\sqrt{2}\,\sigma^\mu\bar\xi\,\partial_\mu A + \sqrt{2}\,\xi F$$
$$\delta_\xi F = i\sqrt{2}\,\bar\xi\bar\sigma^\mu\partial_\mu\psi\,.$$

<u>Note:</u>
$\bar D_{\dot\alpha}\phi = 0 \not\Rightarrow \bar D_{\dot\alpha}\bar\phi = 0$
$\Rightarrow D_\alpha\bar\phi = 0$
(antichiral SF)

3.4 SUSY-inv. lagrangians

$$\mathcal{L} = K(\phi,\bar\phi)\Big|_{\theta^2\bar\theta^2} + \Big(W(\phi)\big|_{\theta^2} + \bar W(\bar\phi)\big|_{\bar\theta^2}\Big)$$

i.e. + h.c.

This is the "D-term" $d_K(x)$ of the general SF K.

This is the "F-term" F_W of the chiral SF W (Note: ϕ chiral $\Rightarrow W(\phi)$ chiral).

K = "Kähler potential" (real fct. of ϕ & $\bar\phi$)
W = "superpotential" (holomorphic fct. of ϕ)

<u>Reason for SUSY-invariance of action:</u> The highest component of any SF transforms into a total derivative. (for dimensional reasons).

- Another way to write this lagrangian: $(\cdots)\big|_{\theta^2} = \int d^2\theta (\cdots)$

$$(\cdots)\big|_{\theta^2\bar\theta^2} = \int d^2\theta\, d^2\bar\theta (\cdots)$$

where $\int d\theta^1 \theta^1 = 1$ & $\int d\theta^1 \cdot 1 = 0$.

(This abstract integral satisfies the fundamental relations $\int d\theta^1 \frac{\partial}{\partial\theta^1}(\cdots) = 0$ etc., which implies the invariance under SUSY trfs.)

- <u>Wess-Zumino model</u> : $K = \bar\phi\phi$; $W = \frac{m}{2}\phi^2 + \frac{\lambda}{3}\phi^3$

$$\Rightarrow \mathcal{L} = -|\partial A|^2 - i\bar\psi\bar\sigma^\mu\partial_\mu\psi + \left(-\frac{m}{2}\psi^2 + \lambda\psi^2 A\right) + \text{h.c.} + (mA + \lambda A^2)F + \text{h.c.} + |F|^2$$

$\Rightarrow$ F has no propagator ("auxiliary field")

EOMs for F $\Rightarrow$ $F = -m\bar A - \lambda\bar A^2$

$$\Rightarrow \mathcal{L} = \text{kinetic} + \text{fermionic mass} + \text{Yukawa int.} - \underbrace{V(A,\bar A)}_{\text{"scalar potential"}}$$

with $V(A,\bar A) = |F|^2 = |mA + \lambda\bar A^2|^2$

<u>3.5 Real SFs ($\equiv$ vector SFs)</u>

$$V = \bar V \Rightarrow V(x,\theta,\bar\theta) = C + \theta\chi + \bar\theta\bar\chi + \theta^2 M + \bar\theta^2\bar M - \theta\sigma^\mu\bar\theta A_\mu + i\theta^2\bar\theta\bar\lambda - i\bar\theta^2\theta\lambda + \frac{1}{2}\theta^2\bar\theta^2 D$$

($\{C, \ldots, D\}$ - fcts. of x)

- Let $\Lambda = \Lambda(y,\theta) = A + \sqrt{2}\theta\psi + \theta^2 F$ ($\{A,\psi,F\}$ - fcts. of y) be chiral SF.

- Define SUSY gauge trf.: $2V \to 2V + \Lambda + \bar{\Lambda}$

  (in components: $2C \to 2C + A + \bar{A}$

  $\vdots$

  $2A_\mu \to 2A_\mu - i\partial_\mu (A - \bar{A})$

  $\vdots$

- Choose Λ such that $C = \chi = M = 0$ ("Wess-Zumino gauge")

  $\Rightarrow V = -\theta\sigma^\mu\bar{\theta}A_\mu + i\theta^2\bar{\theta}\bar{\lambda} - i\bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D$

  (technically very convenient since $V^2 = -\frac{1}{2}\theta^2\bar{\theta}^2 A_\mu A^\mu$ & $V^3 = 0$ in this gauge)

- Field strength superfield: $W_\alpha \equiv -\frac{1}{4}\bar{D}^2 D_\alpha V$ (chiral & gauge inv.)

  component form:

  $$W = i\lambda(y) + [D(y) + i\sigma^{\mu\nu}F_{\mu\nu}(y)]\cdot\theta + \theta^2\sigma^\mu\partial_\mu\bar{\lambda}(y)$$

  where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

  $\sigma^{\mu\nu} = -\frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$.

- $$\mathcal{L} = \frac{1}{4g^2}\left(W^\alpha W_\alpha\big|_{\theta^2} + \bar{W}_{\dot{\alpha}}\bar{W}^{\dot{\alpha}}\big|_{\bar{\theta}^2}\right)$$

  $$= \frac{1}{g^2}\Big\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\bar{\lambda}\bar{\sigma}^\mu\partial_\mu\lambda + \frac{1}{2}D^2\Big\}$$

  gauginos ↑ auxiliary field ↑

- Charged matter: chiral SF ϕ with trf. $\phi \to e^{-\Lambda}\phi$.

- For invariant lagrangian replace

  $$\mathcal{L} = \bar{\Phi}\phi\big|_{\theta^2\bar{\theta}^2} \longrightarrow \mathcal{L} = \bar{\Phi}e^{2V}\phi\big|_{\theta^2\bar{\theta}^2}$$

- Simplest model: $\mathcal{L} = \bar{\Phi}e^{2V}\phi\big|_{\theta^2\bar{\theta}^2} + \frac{1}{2g^2}W^2\big|_{\theta^2}$

Non-abelian generalization:

- V takes values in Lie(G): $V(x,\theta,\bar\theta)_{ij} = (T^a)_{ij}\,\underbrace{V^a(x,\theta,\bar\theta)}_{\text{set of real SFs}}$

- gauge trfs.: $e^{2V} \to e^{\Lambda^+} e^{2V} e^{\Lambda}$

$$\phi \to e^{-\Lambda}\phi$$

matrix $\Lambda_{ij} = T^a_{ij}\Lambda^a$ vector ϕ_i

- gauge inv. Kähler potential: $\phi^+ e^{2V}\phi$

- field strength SF: $W_\alpha = -\frac{1}{8}\bar D^2 e^{-2V} D_\alpha e^{2V}$

(Note: W is not gauge-inv. any more. It transforms as
$$W_\alpha \to e^{-\Lambda} W_\alpha e^{\Lambda}.)$$

- $\mathcal{L} = \frac{1}{2g^2}\,\mathrm{tr}\left(W^2|_{\theta^2} + h.c.\right) + \phi^+ e^{2V}\phi\big|_{\theta^2\bar\theta^2} + W(\phi)\big|_{\theta^2} + h.c.$

could, in principle, transform in a complicated, reducible repres. of G

must be made from gauge-singlets, which can be built on the basis of the repres. of ϕ.

Component form:

(given that $\phi = \{\phi, \psi, F\}$ & $V = \{A_\mu, \lambda, D\}$)

$$\mathcal{L} = \frac{1}{g^2}\,\mathrm{tr}\left\{-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 2i\bar\lambda\bar\sigma^\mu D_\mu\lambda + D^2\right\}$$

$$- |D_\mu\phi|^2 - i\bar\psi\bar\sigma^\mu D_\mu\psi + |F|^2 \qquad \left[D_\mu = \partial_\mu + iA_\mu^a T^a\right]$$

$$+ i\sqrt{2}\left(\phi^+\lambda\psi - \bar\psi\bar\lambda\phi\right) + \phi^+ D\phi$$

"D-term"

Note the involved index structure of the Yukawa-type interactions: $\phi^+ \lambda \psi = \bar{\phi}_i (\lambda^a)^\alpha (T^a)_{ij} (\psi_j)_\alpha$.

- Integrating out D gives $D^a = -g^2 \phi^+ T^a \phi$ and

$$\mathcal{L} \supset -\frac{g^2}{2} (\phi^+ T^a \phi)(\phi^+ T^a \phi).$$

(In the SUSY-SM this "D-term potential" is crucial since, at tree level, it is the only source of a quartic Higgs potential term. This implies that the Higgs mass is "predicted" in SUSY.)