9 Grand Unification

9.1 SU_5 - GUT : Quantum numbers

(→ e.g. G.G. Ross: Grand Unified Theories, Benjamin/Cummings '84)

- $G_{SM} = SU_3 \times SU_2 \times U_1 \subset SU_5$ as follows:

- $Lie(SU_5) = \{5\times 5$ herm., traceless matrices$\}$
- $Lie(SU_2) \oplus Lie(SU_3) \subset Lie(SU_5)$ in obvious way:

$$\begin{array}{c} 3\,\{ \\ 2\,\{ \end{array} \underbrace{\left(\begin{array}{c|c} Lie(SU_3) & 0 \\ \hline 0 & 0 \end{array}\right)}_{3 \quad\ 2} \oplus \left(\begin{array}{c|c} 0 & \\ \hline & Lie(SU_2) \end{array}\right) \subset Lie(SU_5)$$

- There exists exactly one element (up to rescaling) in $Lie(SU_5)$ commuting with $Lie(SU_3) \oplus Lie(SU_2)$:

$$T_1 = \frac{1}{\sqrt{60}} \begin{pmatrix} -2 &&&& \\ & -2 &&& \\ && -2 && \\ &&& 3 & \\ &&&& 3 \end{pmatrix} \qquad \left(\text{we imposed } \operatorname{tr} T_1^2 = \tfrac{1}{2}\right)$$

$\downarrow$

generates $U_1 = U_{1,Y}$

- Exponentiation of $Lie(SU_3) \oplus Lie(SU_2) \oplus Lie(U_1)$ gives the desired "max. subgroup" $G_{SM} \subset SU_5$.

- The "extra" bosons are called "X, Y - gauge - bosons":

$$A_\mu(SU_5) = \left(\begin{array}{c|c} G_\mu^a T_{SU_3}^a - \frac{2}{\sqrt{60}} B_\mu & \frac{1}{\sqrt{2}} \begin{pmatrix} X^1 & Y^1 \\ X^2 & Y^2 \\ X^3 & Y^3 \end{pmatrix} \\ \hline \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{X}^1 & \bar{X}^2 & \bar{X}^3 \\ \bar{Y}^1 & \bar{Y}^2 & \bar{Y}^3 \end{pmatrix} & A_\mu^i T_{SU_2}^i + \frac{3}{\sqrt{60}} B_\mu \end{array}\right)$$

($m_{X,Y} \gtrsim 10^{15}$ GeV because of proton decay - see later)

- Crucial concept: <u>"Branching rule"</u>

$$SU_5 \supset SU_3 \times SU_2 \times U_1$$

$$5 = (3,1)_{-2} + (1,2)_3$$

This is a common and (hopefully) self-explanatory notation.

For these U_1-charges, one often takes only the <u>relative</u> normalization seriously. In our case, these are $\sqrt{60} \times \{U_1\text{-charge with standard } T_1\text{-normalization}\}$

Analogously, we have:

$$\bar{5} = (\bar{3},1)_2 + (1,\bar{2})_{-3}$$ (as obvious as for 5)

$$10 = (3,2)_1 + (\bar{3},1)_{-4} + (1,1)_6$$ (requires some work → problems)

antisymm. tensor of SU_5, i.e.

$$x_{ij} \to U_{ik} U_{jl} x_{kl} \quad , \quad U \in SU_5 \quad , \quad x_{ij} = -x_{ji}$$

<u>Comments / Ideas:</u> • To understand the above, it is crucial to use that:

$$(3 \otimes 3)_A \sim \bar{3} \quad \text{for } SU_3 \quad (\text{using } \varepsilon_{ijk})$$

antisymm. equivalence of repr.

$$\bar{2} \sim 2 \quad \text{for } SU_2$$

$$(2 \otimes 2)_A \sim 1 \quad \text{for } SU_2$$

- Equivalence representations R_1, R_2 means commutativity of

$$\begin{array}{ccc} u & \xrightarrow{U} & R_1(U)\cdot u \\ V\downarrow & & \downarrow V \\ v & \xrightarrow{U} & R_2(U)\cdot v \end{array}$$

, where V is an appropriate isomorphism of vector spaces.

- Crucial fact: $10 + \bar{5}$ is <u>precisely</u> one generation of SM fermions with correct trf. properties under $SU_3 \times SU_2 \times U_1 \subset SU_5$.

- More precisely: $\bar{5} = \left(\frac{d}{\ell}\right) \qquad 10 = \left(\begin{array}{c|c} u & \text{-}q \\ \hline q & e \end{array}\right)$

  - If the U_1 generator is SU_5-normalized, d has U_1-charge $2/\sqrt{60}$ (see above).

  - With our previous $U_{1,Y}$-normalization, d has $U_{1,Y}$-charge $1/3$

$$\Rightarrow \quad q_d = \frac{2}{\sqrt{60}} = \frac{6}{\sqrt{60}} \cdot \frac{1}{3} = \sqrt{\frac{3}{5}}\, q_{Y,d}$$

This works for <u>all</u> fermions:

$$\underbrace{q}_{\substack{SU_5\text{-normalized} \\ U_1\text{-charge}}} = \sqrt{\frac{3}{5}}\, \underbrace{q_Y}_{\text{SM-hypercharge}}$$

- Unfortunately, this does not work for the Higgs:

$$5 = (3,1)_{-2} + \underbrace{(1,2)_3}_{\substack{\text{This is ok for SM Higgs, but the triplet} \\ \text{has no room in SM spectrum}}}$$

Moreover, the scalar triplet coming from this scalar 5 of SU_5 can be nowhere near the el. weak scale since it would induce fast proton decay.
[We need it to be heavy (just like the X, Y gauge bosons, for the same reason): $m_{X,Y} \sim m_3 \gtrsim 10^{15}\,\text{GeV}$]

Comment: In SUSY, H_u & H_d can come from chiral SFs in 5 & $\bar{5}$ of SU_5. Again, the triplets must be heavy.

9.2 Higgs Breaking of SU_5

Need: $SU_5 \xrightarrow[\substack{\text{some high scale,} \\ \text{e.g. } 10^{16}\,\text{GeV}}]{H} SU_3 \times SU_2 \times U_1 \xrightarrow[\sim 10^2\,\text{GeV}]{\phi} SU_3 \times U_{1,EM}$

Take $H \in 24$ of SU_5 (adjoint, i.e. traceless herm. 5×5 matrices)

$(H \longrightarrow UHU^+ \; ; \; U \in SU_5)$

Take $\phi \in 5$ of SU_5 (as explained above)

Assume renormalizable scalar potential $V(H, \phi)$. For simplicity require extra symmetry $\phi \to -\phi$ & $H \to -H$.

$\Rightarrow$ General form of V:
$$V = -m_H^2 \, tr(H^2) - m_\phi^2 |\phi|^2 + \lambda_1 (tr H^2)^2 + \lambda_2 \, tr(H^4) + \lambda_3 (|\phi|^2)^2 + \lambda_4 (tr H^2)|\phi|^2 + \lambda_5 (\bar\phi H^2 \phi)$$

Without proof: The parameters can be chosen such that, setting $\phi = 0$ and minimizing in H one finds vacuum
$$H_0 = v_H \begin{pmatrix} 2_{2_2} & 0 \\ 0 & -3_{-3} \end{pmatrix} \quad \text{with } v_H^2 \sim m_H^2$$

Note: Such a value of H_0 is not as hard to get as it might seem: While a generic H-VEV has the form $\text{diag}(a_1, \cdots, a_5)$ with $\sum_i a_i = 0$, points where some of the a_i are degenerate are special (such vacua have a higher symm.) and can be realized using just certain inequalities between masses & couplings (no fine-tuning is required).

- Generic H-VEV: $SU_5 \to (U_1)^4$
- "Our" H_0: $SU_5 \to SU_3 \times SU_2 \times U_1$ (obvious).

- In analogy to SM-Higgs mechanism, $|DH_0|^2 \subset \mathcal{L}$ provides masses $m_{X,Y} \sim g_5 v_H$ for X,Y-bosons.

- Generically, the d.o.f. of H not eaten by X,Y will also be heavy ($m^2 \sim m_H^2$).
- At lower energies, we get eff. pot.

$$V(H_0, \phi) = -m_\phi^2 |\phi|^2 + \lambda_3 (|\phi|^2)^2 + \lambda_4 \cdot 30 v_H^2 |\phi|^2$$
$$+ \lambda_5 v_H^2 (4|\phi_T|^2 + 9|\phi_D|^2)$$

where $\phi = \begin{pmatrix} \phi_T \\ \phi_D \end{pmatrix}$.

$$\Rightarrow \quad m_T^2 = -m_\phi^2 + (30\lambda_4 + 4\lambda_5) v_H^2$$
$$m_D^2 = -m_\phi^2 + (30\lambda_4 + 9\lambda_5) v_H^2$$

- Since $m_T^2 \sim (10^{16}\,\text{GeV})^2$, $m_D^2 \sim -(10^2\,\text{GeV})^2$ can only be achieve at the cost of an extreme tuning of parameters in $\mathcal{L}$.

("Doublet-triplet-splitting problem").

Note: In SUSY, everything works very similarly: One choses a generic superpotential $W(H, \phi_u, \phi_d)$ for chiral SFs H, ϕ_u, ϕ_d in $24, 5, \bar{5}$. After minimization of the scalar potential and GUT-breaking, v_H contributes differently to μ_T and μ_D of $\mu_T \phi_{u,T} \phi_{d,T}|_{\theta^3}$ and $\mu_D \phi_{u,D}\phi_{d,D}|_{\theta^2} = \mu H_u H_d|_{\theta^2}$. The fine-tuning is needed to achieve $|\mu| \equiv |\mu_D| \ll |\mu_T|$.

- String-theoretic & extra-dimensional GUTs as well as GUTs based on $SO_{10} \supset SU_5$ offer nice solutions to the 2-3-splitting problem.

9.3 Gauge coupling unification

- It is convenient to think in terms of $\left(\frac{2\pi}{\alpha_i}\right)$:

$$\left(\frac{2\pi}{\alpha_i}\right)(\mu) = \left(\frac{2\pi}{\alpha_i}\right)(M) + b_i \ln(M/\mu)$$

- Our data, corresponding to $\mu = m_Z$, is

$$\underbrace{\frac{2\pi}{\alpha_1} = 370.7 \; ; \; \frac{2\pi}{\alpha_2} = 185.8}_{\substack{\text{from } e \text{ and } m_W/m_Z \\ \text{(better: from } e,\, m_Z \text{ and } G_F)}} \; , \; \frac{2\pi}{\alpha_3} = 53.2$$

(i.e. $\alpha_3 = 0.118$)

- <u>Note</u>: from now on we think in terms of $\alpha_1 = \frac{5}{3}\alpha_{1,Y}$, <u>not</u> in terms of $\alpha_{1,Y}$.

$$\text{SM}: \quad b_i = \left(\frac{41}{10}, -\frac{19}{6}, -7\right)$$

$$\text{MSSM}: \quad b_i = \left(\frac{33}{5}, 1, -3\right)$$

(We will calculate this below.)

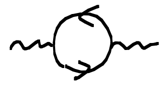
- Obviously, the α_i do not agree at m_Z. We <u>must</u> rely on the large logarithm $\ln(M/\mu) \to \ln(M_{GUT}/m_Z)$ if we want to describe the data using $\alpha_i = \alpha_{GUT}$ at $M = M_{GUT}$.

- We refer to QFT textbooks for a proper treatment of the running coupling. A simple shortcut is as follows:

  - Focus on a U_1 gauge theory and ignore $O(1)$ constants. Then

$$\mathcal{L} \supset \alpha^{-1} F^2 + \bar{\psi}(i\not{D} - m)\psi .$$

  - In this normalization, the propagator is $\sim \alpha$ and the vertex  carries no factor $\sqrt{\alpha}$.

$$\Rightarrow \quad \alpha^{-1}_{1\text{-loop}} \cdot p^2 = \alpha^{-1}_{\text{tree}} \cdot p^2 + $$  $$\Big|_{p^2=0}$$

- The fermion loop is UV divergent and hence gives a contribution
$$\# \cdot \ln(M/m).$$
- $\alpha_{\text{tree}} = \alpha(M)$ is the coupling one would observe at collider energies $\sim M$.

$$\Rightarrow \quad \alpha^{-1}(\mu) = \alpha^{-1}(M) + \frac{b}{2\pi} \ln(M/\mu)$$

- b is known as the "β-fct. coefficient" (The β-fct. is defined by $dg/d\ln\mu = \beta(g) = bg^3/16\pi^2 + \cdots$).

  $b > 0 \Rightarrow$ Landau pole, as in QED.

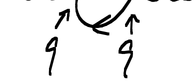
  $b < 0 \Rightarrow$ asymptotic freedom, as in QCD.

- To get b, one just needs the coeff. of the log.-divergence of the self-energy. We just state the results:

$U(1)$-gauge theory with charged matter, i.e. $D_\mu\varphi = (\partial_\mu + iqA_\mu)\varphi$ & $1/4g^2\, F_{\mu\nu}F^{\mu\nu}$:

$$b = \frac{q^2}{6} \cdot c \qquad \text{with} \qquad \begin{array}{ll} c = 2 & \text{for compl. scalar} \\ 4 & \text{for Weyl fermion} \\ -22 & \text{for vector} \\ & \text{(really only relevant} \\ & \text{for non-abel. case)} \end{array}$$

Non-abelian generalization:



$\longrightarrow$

$$\underbrace{T^a \qquad\qquad T^b}_{\sim\, tr(T^aT^b) \,=\, T_R\, \delta^{ab}} \qquad \text{(in some repr. } R\text{)}$$

"Dynkin index"

- hence: $g^2 \to T_R$
- Most relevant cases: fund repr. "F" of SU_N: $R = F$; $T_F = \frac{1}{2}$
  adjoint "A" of SU_N: $R = A$; $T_A = N$

  ($T_A = T(A) = C_2(A)$;
  "quadratic Casimir of adj. repres.)

- Example: b_3 of SM:

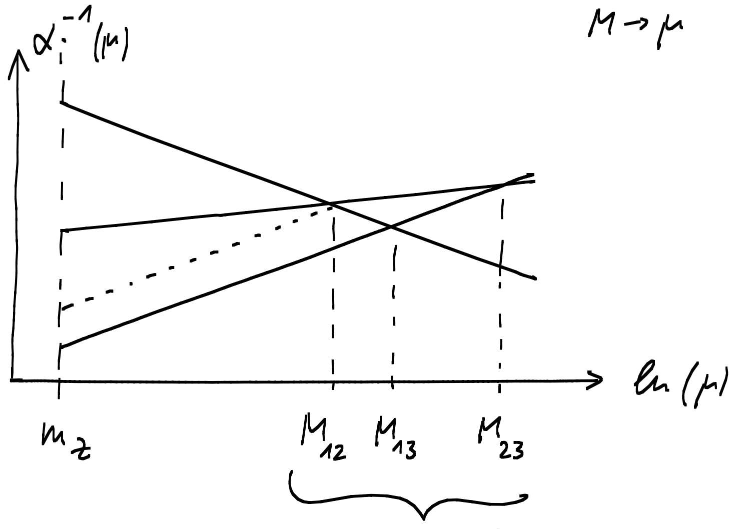
$$b_3 = \frac{1}{6}\left(4 \cdot 2 \cdot 2 \cdot N_f \cdot \frac{1}{2} - 22 \cdot 3\right) = \frac{4}{3} N_f - 11$$

Weyl ferm.   u,d   L,R   families T_F vector (gluons) T_A

analogously (see problems): $b_2 = \frac{4}{3} N_f - \frac{43}{6}$; $b_1 = \frac{4}{3} N_f + \frac{1}{10}$

($N_f = 3$ at high energies.)

$$\alpha_i^{-1}(\mu) = \alpha_i^{-1}(M) + \frac{b_i}{2\pi} \ln \frac{M}{\mu} \xrightarrow[\substack{\mu \to m_Z \\ M \to \mu}]{} \alpha_i^{-1}(\mu) = \alpha_i^{-1}(m_Z) - \frac{b_i}{2\pi} \ln \frac{\mu}{m_Z}$$



M_{GUT} ?

$10^{13} \dots 10^{17}$ GeV

Check this in detail!

Alternative perspective: Assume unification and predict $\alpha_3(m_Z)$ (→ dotted line) ⇒ α_3 comes out much too low!

- Crucial fact: In the MSSM, this works much better!
  1-loop prediction (with $M_{soft} \sim m_Z$): $\alpha_3(m_Z) = 0.117$
  (Check this!)

- Unfortunately: $\alpha_3(m_Z)_{2\text{-loop}} = 0.129$

  (This 10% discrepancy is well outside exp. errors and may be a serious issue. However, we know little about "GUT scale threshold effects" and the actual soft mass spectrum ...)

Note: If we assume that α_3 could be anything between $0 \cdots 1$, hitting the right value with 0.01 accuracy corresponds to a "3σ" effect (99% unlikely). This is significant!

9.4 SU_5-Yukawa couplings

Let us use the following names for the fermions & Higgs:

$$10 \rightarrow T \quad ; \quad \bar{5} \rightarrow \underline{F} \quad ; \quad 5\,(\text{Higgs}) \rightarrow \phi = \begin{pmatrix} \phi_T \\ \phi_D \end{pmatrix}$$

Poincare & gauge symm. allow couplings:

$$T\,T\,\phi \qquad\qquad \underline{F}\,T\,\bar{\phi}$$

$$(T_{ij}\,T_{kl}\,\phi_m\,\varepsilon^{ijklm}) \qquad\qquad ((\underline{F})_i\,T_{ij}\,\bar{\phi}_j)$$

$$\downarrow \qquad\qquad \swarrow \quad \searrow$$

$$q \cdot u \cdot \phi_D \qquad\qquad q \cdot d \cdot \bar{\phi}_D \qquad l \cdot e \cdot \bar{\phi}_D$$

(Check this translation into SM fields!)

Thus, λ_d & λ_e are related. The naive SU_5 prediction is

$$m_d = m_e$$

for all 3 generations. Amusingly, this works approximately for the 3rd generation:

$$\begin{aligned} m_b &\simeq 3\text{ GeV} & \xrightarrow[\text{to } M_{GUT}]{\text{running}} \quad & 1.0\text{ GeV} \\ m_\tau &\simeq 1.7\text{ GeV} & & 1.2\text{ GeV} \end{aligned}$$

- For the light generations this does not work. A possible reason is that the (tiny) renormalizable couplings are overwhelmed by higher-dim. operators:

$$\frac{1}{M_{Pl.}} \underbrace{\psi\psi}_{\text{SM matter}} \underbrace{H}_{\text{GUT Higgs}} \phi \xrightarrow{SU_5} \frac{v_H}{M_{Pl.}} \psi\psi\phi$$

9.5 SU_5 - proton decay

- Baryon number B is an "accidental global symm." of $\mathcal{L}_{SM}$. It is a U_1 symm. with charges:

$$\begin{array}{lcl} q & : & 1/3 \\ d, u & : & -1/3 \\ \text{all others} & : & 0 \end{array}$$

(Check that this is indeed a symm.!)

[By "accidental" we mean that, given the SM gauge symms. & field content, it is not possible to write down a renormalizable operator violating B.]

Note: The same is true for lepton number L (defined analogously). It turns out that $B-L$ is conserved also at the quantum level while $B+L$ is anomalous. However, it is only violated by negligibly small non-pert. effects.

- B conserved $\Rightarrow$ proton can not decay since it is the lightest particle with $B \neq 0$.
- However, X, Y gauge bosons violate B. This leads to proton decay as we now demonstrate by integrating out X, Y and analysing the resulting 4-fermion operators:

- Let us write $X \sim \begin{pmatrix} X^1 & Y^1 \\ X^2 & Y^2 \\ X^3 & Y^3 \end{pmatrix}$ such that $A_\mu = \left(\begin{array}{c|c} 0 & X \\ \hline X^+ & 0 \end{array}\right)_\mu \in Lie(SU_5)$.

- X acquires a mass from $\langle H \rangle \neq 0$:

$$\mathcal{L} \supset -(D_\mu H)^2 \implies \mathcal{L} \supset -m_X^2 |X|^2$$

Here the contraction of the SU_2 & SU_3 index of the matrix X is implicit.

- X couples to SM fermions via usual (SU_5-) covariant derivative:

$$\mathcal{L} \supset (\bar{\underline{F}})\, i \not{D} \underline{F} \supset -g (\bar{\underline{F}}) \bar{\sigma}^\mu A_\mu \underline{F}$$

- remembering $\underline{F} = \begin{pmatrix} d \\ e \end{pmatrix}$, this gives

$$-g \bar{d} \bar{\sigma}^\mu X_\mu e + h.c.$$

$$\left[\text{in more detail: } \underbrace{\bar{d}_{\underset{\bar{3}}{A}} (X)_{\underset{3}{A}}}_{SU_3}\underbrace{{}_{\underset{2}{i}}\, e_{\underset{\bar{2}}{i}}}_{SU_2} \right]$$

- Analogously, but with slightly more work:

$$\bar{T} i \not{D} T \to -g \bar{T} \bar{\sigma}^\mu \underbrace{A_\mu T}$$

This is shorthand for the action of $A_\mu \in Lie(SU_5)$ on a 10 of SU_5. Taking A_μ to be an explicit matrix in the fund. of SU_5 and T an antisymm. 5×5 matrix, we have

$$(\text{"}A_\mu T\text{"})_{ik} = (A_\mu)_{ij} T_{jk} + (A_\mu)_{kj} T_{ij}$$
$$= (A_\mu \cdot T)_{ik} + (T \cdot A_\mu^T)_{ik} .$$

- Suppressing all group indices, one finds

$$\mathcal{L} \supset -g\, X_\mu \cdot \left(\bar{u}\bar{\sigma}^\mu q + \bar{q}\bar{\sigma}^\mu e \right) + \text{h.c.}$$

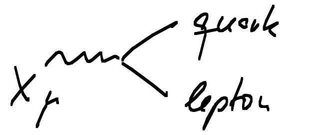
(One can easily guess the proper contractions, e.g. $X \in (3,2)$; $\bar{q} \in (\bar{3},\bar{2})$; $e \in (1,1)$ $\Rightarrow$ contraction obvious. Of course, to get numerical factors right, one has to do the work outlined above ...)

- The general structure we have found is

$$\mathcal{L} \supset -m_X^2 |X_\mu|^2 + X_\mu J^\mu + \text{h.c.}$$

"B-violating" current, combining both the F & T contributions discussed above.

[It is "B-violating" since it describes interactions of the type



.]

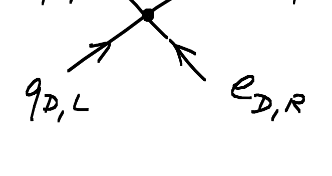
- "Integrating out" X_μ, we find

$$\mathcal{L} \supset \frac{1}{m_X^2} J_\mu \bar{J}^\mu \supset \sim \frac{g^2}{m_X^2} (\bar{e}\bar{u})(qq) \quad (+\text{ many more terms})$$

Here the σ's & $\bar{\sigma}$'s are removed using Fierz identities, in our case e.g. $\sigma^\mu_{\alpha\dot{\alpha}}\, \bar{\sigma}_\mu^{\dot{\beta}\beta} = -2\, \delta_\alpha{}^\beta\, \delta_{\dot{\alpha}}{}^{\dot{\beta}}$.

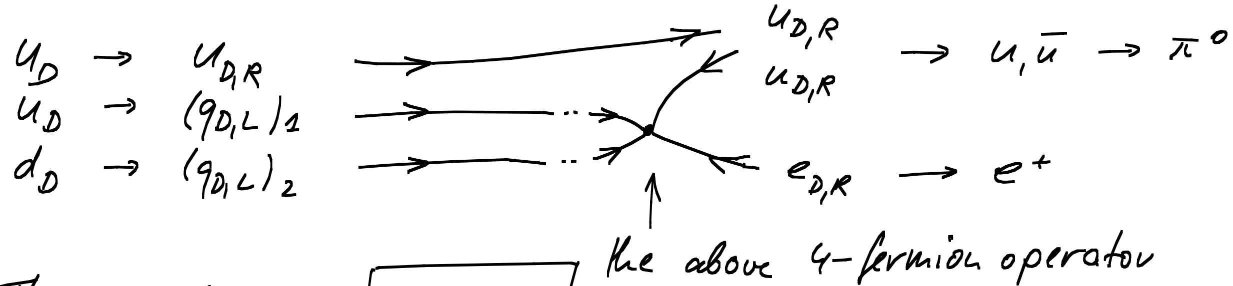
- Thinking of ψ as $\psi_{D,L} = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$ and of $\bar{\psi}$ as $\psi_{D,R} = \begin{pmatrix} 0 \\ \bar{\psi} \end{pmatrix}$, we can associate conventional (Dirac-type) lines with arrows with our Weyl-type fields:

$\bar{e}\bar{u}qq \longrightarrow$



- The proton is $\begin{pmatrix} u_D \longrightarrow \\ u_D \longrightarrow \\ d_D \longrightarrow \end{pmatrix}$ in Dirac notation,

where u_D contains both $u_{D,L} = (q_{D,L})_1$ and $u_{D,R}$ (and analogously for d_D). Hence:



- Thus, we have e.g. $\boxed{p \to \pi^0 e^+}$

- Amplitude: $\sim \dfrac{g^2}{m_X^2} \sim \dfrac{1}{m_X^2}$ (for us $g \sim O(1)$)

- Rate: $\sim \dfrac{1}{m_X^4}$; Lifetime: $\sim m_X^4$

- On dim. grounds: $\tau \sim \dfrac{m_X^4}{\Lambda_{QCD}^5}$ ← This scale enters via p & π^0 ($m_e = 0$ for our purposes)

(Translate that into years...)

9.5 SO_{10} GUTs (very brief)

- Identify $\mathbb{C}^5$ with $\mathbb{R}^{10}$ in obvious way. Any SU_5-trf. preserves the length of complex vectors in $\mathbb{C}^5$ and hence the length of the corresponding real vectors in $\mathbb{R}^{10}$. Hence $SU_5 \subset SO_{10}$ or

$$SO_{10} \supset SU_5 \supset SU_3 \times SU_2 \times U_1$$

- From the above, we see that
$$SO_{10} \supset SU_5$$
$$10_{real} = 5_{compl.}$$

- We also need the $10_{compl.}$ of SO_{10} defined by $z_i \to O_{ij} z_j$, $\{z_i\} \in \mathbb{C}^{10}$; $O \in SO_{10}$.

- The (complex) 10 of SO_{10} is said to be a "real" representation in the following sense:

$$\begin{tikzcd} z \arrow[r, "O"] \arrow[d, "*"'] \arrow[dr, phantom, "\text{commutes!}"] & Oz \arrow[d, "*"] \\ z^* \arrow[r] & Oz^* = (Oz)^* \end{tikzcd}$$

(≡ "group action commutes with complex conjugation")

- It is then natural (in fact unavoidable) that

$$SO_{10} \supset SU_5$$

$$10_{\text{compl.}} = 5 + \bar{5} \qquad \text{(Work this out in detail!)}$$

(Nice but minor implication: We get 2nd Higgs for free!)

- Note also:

$$SO_{10} \supset SU_5$$

$$45 = 24 + 10 + \overline{10} + 1$$

adjoint (antisymm. 10×10 matrices, - for the purposes of this branching rule they are <u>complex</u>)

- The absolutely crucial point:

$$SO_{10} \supset SU_5$$

$$16 = 10 + \bar{5} + 1$$

spinor   "r.h. neutrino"

(All of our matter may come just from 3 spinors of SO_{10}!)

[To demonstrate this branching rule, one needs to work a bit more on spinors. → later and/or problems]

- We can break SO_{10} using a GUT-Higgs H in 45:

- Fact: It is possible to write down a (generic, i.e. without tuning of parameters) scalar potential for H which is minimized by

$$H_0 = v_H \left(\begin{array}{ccc|cc} \begin{smallmatrix}0&-1\\1&0\end{smallmatrix} & & & & \\ & \begin{smallmatrix}0&-1\\1&0\end{smallmatrix} & & & \\ & & \begin{smallmatrix}0&-1\\1&0\end{smallmatrix} & & \\ \hline & & & \begin{smallmatrix}0&0\\0&0\end{smallmatrix} & \\ & & & & \begin{smallmatrix}0&0\\0&0\end{smallmatrix} \end{array}\right) \in Lie(SO_{10})$$

(rest all zeros)

- Claim: This breaks $SO_{10} \to G_{SM} \times U_1$.

(subgroup generated by $H_0 \in Lie(SO_{10})$)

- Demonstration: We can build any SU_5 generator from 5×5 matrices of the type
$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \text{ or } \begin{pmatrix} & -i \\ i & \end{pmatrix} \text{ or } \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & -1 & \\ & & & \ddots \end{pmatrix}.$$

- Considering SU_5 as a subgroup of SO_{10} in the standard way explained above (i.e. via $\mathbb{C}^5 \cong \mathbb{R}^{10}$), we can give the corresponding SO_{10}-generators (10×10 matrices):

$$\underbrace{\begin{pmatrix} & \mathbb{1}_2 \\ \mathbb{1}_2 & \end{pmatrix}}_{\text{obvious}} \text{ or } \underbrace{\begin{pmatrix} & \begin{smallmatrix}0&1\\-1&0\end{smallmatrix} \\ \begin{smallmatrix}0&-1\\1&0\end{smallmatrix} & \end{pmatrix}}_{z=x+iy \;\to\; iz = ix-y = -y+ix} \text{ or } \underbrace{\begin{pmatrix} \mathbb{1}_2 & & & \\ & \ddots & & \\ & & -\mathbb{1}_2 & \\ & & & \ddots \end{pmatrix}}_{\text{obvious}}$$

cf.

$$\begin{pmatrix}0&-1\\1&0\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-y\\x\end{pmatrix}.$$

- From this our claim follows immediately, observing in particular that the 3 (2×2-blocks) in H_0 are equal and hence commute with all generators of $SU_3 \subset SU_5$.

- Important implication: The coupling $(\phi^1_{10})^T H_{45} (\phi^2_{10}) \subset \mathcal{L}$ between to 10's of SO_{10} and a 45 gives a mass $\sim H_0^2$ to the triplets (from the $5, \bar{5} \subset 10$), keeping the $2, \bar{2}$ (SM Higgses) massless. This is known as the "Dimopoulos-Wilczek mechanism for 2-3-splitting", ~'81. (Works also in SUSY-GUT).

- Note however: Need $\underline{\underline{\text{two}}}$ 10's (i.e two 5's and two $\bar{5}$'s of SU_5 – i.e. non-minimal field content) since H_{45} is antisymm.

- <u>Further comments:</u>
  - SO_{10} allows only a single Yukawa: $16 \cdot 16 \cdot 10$ $\Rightarrow$ b, τ, t-unification (requires two light Higgs doublets, as in SUSY, and $\tan\beta \sim 50$).

- All SO_{10}-repr. are anomaly-free
- Other SO_{10} subgroups:

$$\begin{tikzcd}
 & SO_{10} \arrow[dl] \arrow[d] \arrow[dr] & \\
SU_5 \times U_{1,X} \arrow[dr] & \begin{array}{c} SU_5' \times U_1' \\ (\text{"flipped } SU_5\text{"}) \end{array} \arrow[d] & \begin{array}{c} SU_4 \times SU_2 \times SU_2 \\ (\text{"Pati-Salam", '73} \\ \text{-historically first GUT!)} \end{array} \arrow[dl] \\
 & \underbrace{SU_3 \times SU_2 \times U_{1,Y}}_{SM} \times U_{1,X} &
\end{tikzcd}$$

<u>Note:</u> $SU_4 \simeq SO_6$
$SU_2 \times SU_2 \simeq SO_4$

Pati-Salam $\Downarrow$ means simply $SO_{10} \supset SO_6 \times SO_4$

- $SU_5' \supset SU_3 \times SU_2$; $SU_5' \not\supset U_{1,Y}$
- Flipped SU_5 & Pati-Salam do not allow an α_3-prediction (unless supplemented by some larger symm. like SO_{10} or string UV-completion).
- $SO_{10} \supset SU_5 \times U_{1,Y}$ & $SO_{10} \supset SU_5' \times U_{1,X}$ are group-theoretically equivalent.
- $SO_{10} \supset PS \supset SM$ allows non-SUSY precision unification.