3.3 Dynkin diagram techniques; larger GUT groups

(mostly without proofs; cf. Slansky's Phys. Rep., for more detail see Georgi's book + many Math-texts; original papers: Dynkin ~1950..1960)

- let $\mathfrak{g}$ be the Lie-Alg. of G $(G \ni g = e^{iX};\ X \in \mathrm{Lie}(G))$
- $\mathfrak{g}$ acts on itself (adjoint repr.); $X: Y \longrightarrow [X,Y]$

  $[X,Y]$ ↑ Lie-alg.-operation

- complexify $\mathrm{Lie}(G) = \mathfrak{g}$ so that it becomes a vector space over $\mathbb{C}$.
- in the adj. repr., the generators are now matrices, where the original (before complexification) generators are hermitian.
- as in quantum mechanics, we now diagonalize a maximal set of generators:

$$\mathfrak{h} = \{H_i\}_{i=1}^{r}$$

This is the "max. abelian" subalg. or "Cartan subalg."
$e^{i\mathfrak{h}} \subset G$ is the "max. torus" (it is a product of U_1's). $\underline{r \equiv \text{"rank } G\text{"}}$.

Note: up to "inner automorphisms" $(X \to U X U^+)$ the Cartan subalg. is unique (without proof).

- the adj. repr. provides a natural metric on g:
  $(X,Y) = tr_{adj.}(X \cdot Y)$ ("Killing metric").
- this allows us to choose an orthogonal basis of g (which we continue to call H_i):

$$tr(H_i H_j) = \lambda \delta_{ij} \quad (\lambda \text{ arbitrary but fixed}).$$

- by definition, a basis of the remaining generators can be chosen in such a way that

$$[H_i, E_\alpha] = \alpha_i E_\alpha .$$

α is a real vector (in $\mathbb{R}^r$) with elements α_i. α determines E_α uniquely (justifying this notation).

Since $H_i^+ = H_i$ we have

$$-[H_i, E_\alpha^+] = \alpha_i E_\alpha^+ \quad \Rightarrow \quad E_\alpha^+ = E_{-\alpha}$$

- α are the "root vectors", E_α are the "roots"
- $[E_\alpha, E_\beta] \sim E_{\alpha+\beta}$

Demonstration: $[H_i, [E_\alpha, E_\beta]] = [[H_i, E_\alpha], E_\beta] + [E_\alpha, [H_i, E_\beta]]$
$= (\alpha_i + \beta_i) \cdot [E_\alpha, E_\beta]$.

Introducing a normalization constant we have

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}.$$

- From the above it is clear that $[E_\alpha, E_{-\alpha}]$ commutes with all H_i. Thus, it is in $\mathfrak{g}$ and we can write

$$[E_\alpha, E_{-\alpha}] = \beta_i H_i .$$

- go to the adj. repr., multiply with H_j and take the trace:

$$tr(H_j [E_\alpha, E_{-\alpha}]) = \beta_i \lambda \delta_{ij}$$

$$tr(H_j E_\alpha E_{-\alpha} - H_j E_{-\alpha} E_\alpha) = \lambda \beta_j$$

$$tr(E_{-\alpha} \underbrace{(H_j E_\alpha - E_\alpha H_j)}_{\alpha_j \cdot E_\alpha}) = \lambda \beta_j$$

$$\Rightarrow \beta_i = \alpha_i \frac{tr(E_{-\alpha} E_\alpha)}{\lambda}$$

- Choose normalization of E_αs such that $tr(E_\alpha^+ E_\alpha) = \lambda \delta_{\alpha\beta}$ (as for H_is), implying

$$\beta_i = \alpha_i \quad \text{or} \quad [E_\alpha, E_{-\alpha}] = \alpha_i H_i$$

(Note that $N_{\alpha,\beta}$ are now unambiguously defined.)

Define order in root space:

$$\alpha - \beta > 0 \iff \{\text{first} \neq 0 \text{ component of } \alpha - \beta\} > 0$$

also: a root is "positive" if its first non-zero component is positive.

The smallest r positive roots are called "simple roots." We denote them by $\{\alpha_{(i)}\}_{i=1}^{r}$.

They are lin. indep. (without proof) so that

$$\alpha = \sum_{i=1}^{r} k^i \alpha_{(i)} \quad \text{for any } \alpha.$$

Conventional basis:

$$e_{(i)} = \frac{2}{|\alpha_{(i)}|^2} \cdot \alpha_{(i)}$$

In this basis, the familiar euclidean metric of the root space $\mathbb{R}^r$ is characterized by $g_{ij} = e_{(i)} \cdot e_{(j)}$

$$(v = v^i e_{(i)}\,; \; w = w^i e_{(i)}\,; \; v \cdot w = v^i w^j g_{ij})$$

It is common to consider the dual space and its dual basis $\mu^{(i)}$. Since a metric exists, the dual space can be identified with the space itself,

$$\mu^{(i)} \cdot e_{(j)} = \delta^i{}_j$$

"Dynkin basis" "dual basis"

Decompose a given vector v in the $\mu^{(i)}$-basis:

$$v = \mu^{(i)} \cdot v_i$$

"Dynkin labels"

$$v \cdot e_{(j)} = v_i \, \mu^{(i)} \cdot e_{(j)} = v_j$$

Applying this to a simple root $\alpha_{(i)}$ we find:

$$\alpha_{(i)\,j} = \alpha_{(i)} \cdot e_{(j)} = 2 \frac{\alpha_{(i)} \cdot \alpha_{(j)}}{|\alpha_{(j)}|^2} = g_{ij} \frac{|\alpha_{(i)}|^2}{2} \equiv A_{ij}$$

Cartan matrix

(encodes all data of a given Lie alg.)

Choice of λ: choose λ such that $|\alpha_{(i)}|^2 = 2$ for the longest of the simple roots.

$\Rightarrow$ A_{ij} all integer (without proof)

$\Rightarrow$ even more: Dynkin labels of all weight vectors are integer (without proof)

Weights: • Basis vectors of a repres. of g (are represented in an obvious way by "weight vectors" in root space)

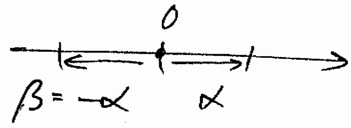
$$R(H_i) \cdot W = w_i \, W \quad ; \quad W \in \text{repres. space}$$

$\{w_i\}$ – weight vector (analogue of $\{\alpha_i\}$)

Some examples (very brief; to be worked out in more detail in the "problems".)

SU_2: rank 1 ; 1 Cartan generator, two roots

Root space:



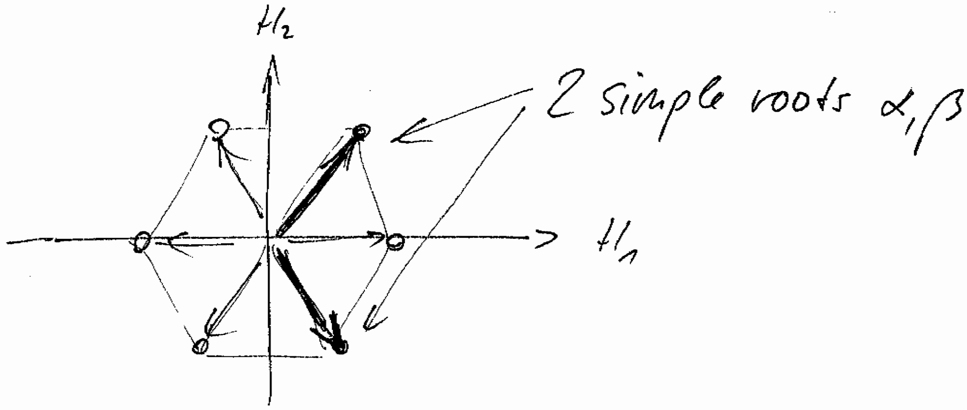
$$[H_1, E_\alpha] = \alpha E_\alpha$$

explicitly: e.g. $H_1 \sim \sigma_3$

$$E_{\pm\alpha} \sim \sigma_1 \mp i\sigma_2$$

(Normalization to be fixed in problems.)

SU_3: rank 2 ; 2 Cartan generators, 6 roots



All other roots can be obtained as lin. combinations of α, β and $-\alpha, -\beta$. This geometric operation is reflected in the Lie alg. structure, e.g.

$$[E_\alpha, E_\beta] \sim E_{\alpha+\beta} \quad \text{etc.}$$

($[E_\gamma, E_{\gamma'}] = 0$ means that $\gamma+\gamma'$ is not a root.)

<u>More generally:</u>

The simple roots (and their hermitian conjugates) generate <u>all roots</u>. (without proof)

$\Downarrow$

The (purely geometric) information about the location of the r simple roots in rootspace $\mathbb{R}^r$ specifies a (simple, complex) Lie alg. completely.

(A Lie-alg. $\mathfrak{g}$ is called <u>simple</u> if it has no non-trivial invariant subalgebra ("ideal"), i.e., $\eta \subset \mathfrak{g}$ with $[X, Y] \in \eta$ for any $X \in \mathfrak{g}$ and $Y \in \eta$.)

A simple Lie alg. is most easily characterized by its <u>Dynkin-diagram</u>:

$\downarrow$

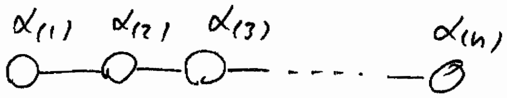
A diagram built from r symbols "$\circ$" or "$\bullet$" (standing for the long or short simple roots — — there are never more than two different lengths in one diagram) pairwise connected by "$-$", "$=$" or "$\equiv$".

"$-$"	"$=$"	"$\equiv$"	
$\uparrow$	$\uparrow$	$\uparrow$	angle
$120°$	$135°$	$150°$	between the two roots in question.

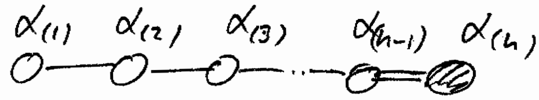
Two roots not connected by a line form a $90°$ angle.

Complete list:

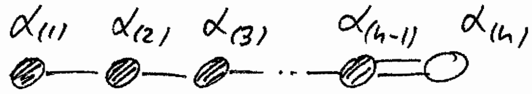
A_n $su(n+1)$



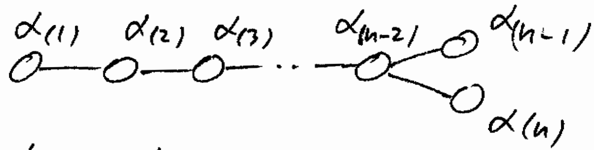
B_n $so(2n+1)$



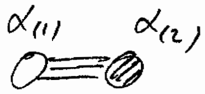
C_n $sp(2n)$



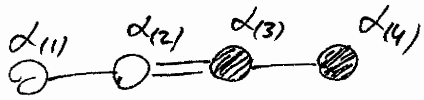
D_n $so(2n)$



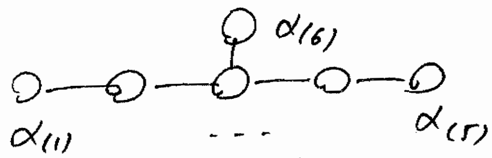
G_2



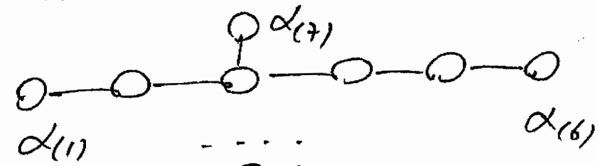
F_4



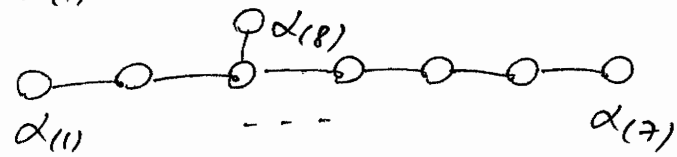
E_6



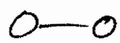
E_7



E_8



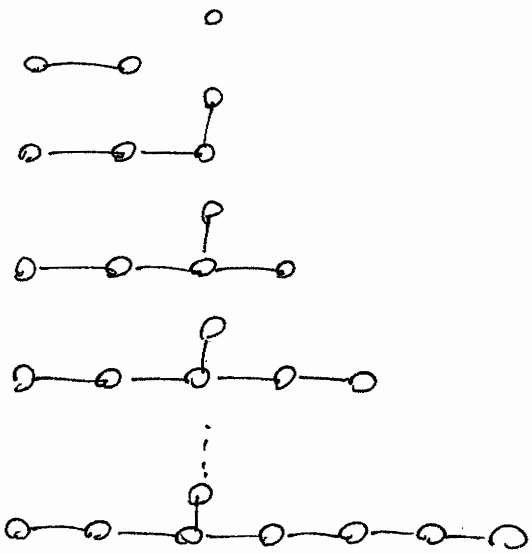
Already familiar: $su(2)$: 

$su(3)$: 

$so(4)$:  $(= su(2) \times su(2))$

($\rightarrow$ semi-simple groups (= direct products of simple groups) can also be represented in this way.)

Famous series of subgroups:



"SM"
$\cap$
SU_5
$\cap$
SO_{10}
$\cap$
E_6
$\cap$
E_7
$\cap$
E_8 (cf. the $E_8 \times E_8$ gauge group of the heterotic string.)

Other intriguing facts:

$$E_6 \supset SO_{10}$$

$$\underset{\text{(adj.)}}{78} = 45 + 16 + \overline{16} + 1$$

(just gauge fields contain the right quantum numbers for a SM generation; need SUSY for explicit models → "gauginos"!)

$$E_8 \supset SU_3 \times E_6$$

$$\underset{\text{(adj.)}}{248} = (8,1) + (1,78) + (3,27) + (\overline{3},\overline{27})$$

together with

$$E_6 \supset SO_{10}$$

$$27 = 1 + 10 + 16$$

⇒ "Everything could come from gauge fields (all <u>three</u> generations).

Comments:

$Sp(2n)$ – group of real matrices leaving invariant a bilinear form

$$f_{ij} = \begin{pmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & & & \\ & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{pmatrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\\a\\a\end{matrix}}\right\} n \text{ blocks}$$

$$(S \in Sp(2n) : \quad S^T f S = f)$$

(In complete analogy to $SO(n)$ respecting the bilinear form $g_{ij} = \mathrm{diag}(1, \cdots, 1)$;
$$O \in SO(n) : \quad O^T g O = g \;)$$

Note: $\quad \mathrm{Lie}(Sp(2)) = \mathrm{Lie}(SU(2))$