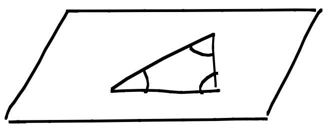
General Relativity

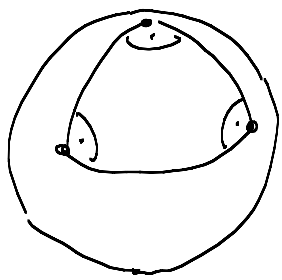
1 Introduction

1.1 General Idea

- In GR, the flat $\mathbb{R}^4$ of special relativity is replaced by a (in general curved) metric manifold

- Example: $\mathbb{R}^2$ vs. S^2 (sphere)





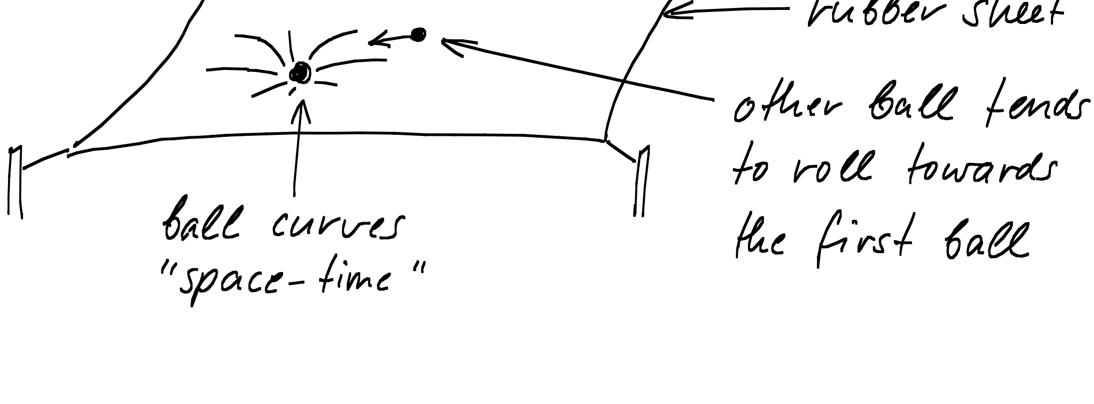
- "Metric" means that we can (locally) measure distances & angles

- S^2 is curved $\Leftrightarrow$ angles of large triangles (see above) sum to more than $180°$

- However, at small distances ($l \ll$ radius), S^2 looks locally like a piece of $\mathbb{R}^2$. For small triangles, the angles sum to $\sim 180°$.

- Gravity: Curvature affects the motion of bodies; Bodies (masses) curve space-time.

- Simple (not very precise) analogy:



- To understand all this in detail, we will need a lot of <u>differential geometry</u>.

1.2 Topics

- Manifolds, connection, curvature
- Motion in external gravitational field
- Einstein-Hilbert action and Einstein equations
- Newtonian limit
- Expansion of the universe
- Schwarzschild solution (bending of light, black holes, etc.)
- Post-Newtonian approximations
- Gravity waves
- Unruh effect & Hawking radiation
- Vielbein formalism and differential-form-formulation of GR
- Extra dimensions, Kaluza-Klein theory

...

(the order of presentation may change)

1.3 Dual vector spaces and tensors

- Let V be a vector space (e.g. $\mathbb{R}^3$ of class. mechanics).
- The dual vector space V^* is the space of all linear functionals on V (i.e. linear maps $V \to \mathbb{R}$).
- Let $\{\bar{e}_i\}$ be a basis of V (e.g. $\bar{e}_1, \bar{e}_2, \bar{e}_3 \in \mathbb{R}^3$)
- The dual basis $\{\underline{e}^i\}$ of V^* is defined by

$$\underline{e}^i(\bar{e}_j) = \delta^i{}_j \, .$$

- The evaluation of a general element
$$\underline{y} = y_i \underline{e}^i$$
of V^* on a general element
$$\bar{x} = x^i \bar{e}_i$$
of V is specified by
$$\underline{y}(\bar{x}) = y_i \underline{e}^i (x^j \bar{e}_j) = y_i x^j \underline{e}^i(\bar{e}_j) = y_i x^i$$

- Basis change: $\bar{e}_i{}' = \bar{e}_j (M^{-1})^j{}_i$
(linear trf. of basis vectors)
$$\bar{x} = x^i \bar{e}_i = x'^i \bar{e}'_i \quad \Rightarrow \quad x'^i = M^i{}_j x^j$$

- Corresponding dual basis: $\underline{e}'^i = M^i{}_j \underline{e}^j$.
Coordinates of dual vector: $x'_i = x_j (M^{-1})^j{}_i$

- A tensor, contravariant of rank m and covariant of rank n, is $$t \in \underbrace{V \otimes \cdots \otimes V}_{m} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{n}$$

- It can be characterized by its components in a certain basis of V and the dual basis:
$$t = t^{i_1 \cdots i_m}{}_{j_1 \cdots j_n} \, \bar{e}_{i_1} \otimes \cdots \otimes \bar{e}_{i_m} \otimes \underline{e}^{j_1} \otimes \cdots \otimes \underline{e}^{j_n}.$$
(Vectors $\in V$ and dual vectors $\in V^*$ are special tensors.)

- A metric is specified by a rank-2 symmetric covariant tensor g_{ij}. It can be used to raise and lower indices:
$$x_i = g_{ij} x^j \; ; \; x^i = g^{ij} x_j \quad \text{where } g^{ij} \text{ is defined by } g^{ij} g_{jk} = \delta^i{}_k .$$

- Under a basis change, the components transform as

$$t'^{i_1 \cdots i_m}{}_{j_1 \cdots j_n} = (M^{i_1}{}_{k_1}) \cdots (M^{i_m}{}_{k_m})\, t^{k_1 \cdots k_m}{}_{l_1 \cdots l_n} (M^{-1})^{l_1}{}_{j_1} \cdots (M^{-1})^{l_n}{}_{j_n} .$$

1.4 Classical mechanics

- Space-time: $\mathbb{R} \times \mathbb{R}^3 \ni (t, \bar{x})$
- Symmetries:
  - translations in time & space.
  - euclidean boosts (going to a coordinate system moving with constant velocity $\bar{v}$).
  - rotations
- Rotations in more detail:
  - $\bar{x} = x^i \bar{e}_i$ with basis vectors $\bar{e}_1, \bar{e}_2, \bar{e}_3$
  - basis change: $\bar{e}'_i = \bar{e}_j (M^{-1})^j{}_i$; $x'^i = M^i{}_j x^j$
  - metric $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$; $(\bar{x}, \bar{y}) \longmapsto \underbrace{g_{ij}}_{\text{symm. in } ij\text{; invertible}} x^i y^j$
  - this allows for the measurement of distances:

$$d\bar{x}^2 = g_{ij}\, dx^i dx^j$$

  - in an orthonormal basis, the metric is $g_{ij} = \delta_{ij}$
  - Under a basis change,

$$g_{ij} \longrightarrow g'_{ij} = g_{kl} (M^{-1})^k{}_i (M^{-1})^l{}_j$$

  - In classical mechanics, one usually works with orthonormal bases. The lin. trfs. are then restricted by $\delta_{ij} = \delta_{kl} (M^{-1})^k{}_i (M^{-1})^l{}_j$

$\Rightarrow \mathbb{1} = M^T M$; $\Rightarrow M = R \in SO(3)$
(rotations)

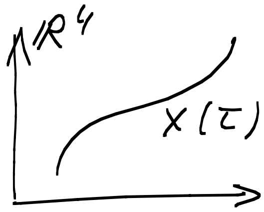
1.5 Special relativity

- Symmetries: – translations in time & space
  – Lorentz rotations $\Big\}$ Poincare group

- Lorentz rotations in more detail:

  - $x = x^\mu e_\mu$ with basis vectors $e_0, \dots, e_3$
  - basis change: $e'_\mu = e_\nu (M^{-1})^\nu{}_\mu$; $x'^\mu = M^\mu{}_\nu x^\nu$
  - time- & space distances between points in space-time ("events") are no longer well-defined. What is well-defined is the invariant space-time interval
  $$dx^2 = g_{\mu\nu} dx^\mu dx^\nu$$
  - For Lorentz-frames or inertial frames we have ($c = 1$)
  $$dx^2 = -dt^2 + d\vec{x}^2$$
  $$\text{or } g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}_{\mu\nu}$$
  (one often writes $ds^2 = -dt^2 + d\vec{x}^2$)
  - $\eta_{\mu\nu}$ is left unchanged by Lorentz-trfs.:
  $$\eta_{\mu\nu} = \eta_{\alpha\beta} (M^{-1})^\alpha{}_\mu (M^{-1})^\beta{}_\nu$$
  $\Rightarrow \eta = M^T \eta M$; $\Rightarrow M = \Lambda \in SO(1,3)$
  (special Lorentz rotations)

- Trajectories in the Minkowski-space $\mathbb{R}^4$ defined above: maps $\mathbb{R} \to \mathbb{R}^4$;



; parameterized by $\tau \in \mathbb{R}$

- The parameterization is irrelevant:
  We can replace τ by $\tau' = \tau'(\tau)$ as long as $d\tau'/d\tau > 0$ everywhere.

- A trajectory is time-like at every point:

$$\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = \eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu < 0$$

- Choose a parameterization $x(\tau)$ such that

$$\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \dot{x}^\mu \dot{x}_\mu = \dot{x}^2 = -1 .$$

The corresponding is the "eigentime". $u^\mu \equiv \dot{x}^\mu$ the 4-velocity ($u^2 = -1$).

[In the rest frame of the particle, $\vec{u} = 0$ and $u^2 = -1$ implies $dx^0/d\tau = 1$ or $dx^0 = d\tau$.]

- Action for a freely moving massive particle:

$$S = -m \int d\tau \quad (\tau\text{-eigentime}, \ m\text{-mass})$$

or $$S = -m \int d\tau \sqrt{-\dot{x}^\mu \dot{x}_\mu} \quad (\text{for general } \tau)$$

(check that the latter is invariant under $\tau \to \tau' = \tau'(\tau)$ and that the variation gives the EOM $\dot{u}^\mu = 0$.)

- Action for a field theory (electrodynamics):

$$S = -\int d^4x \, \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(Many other examples of actions will follow.)

<u>Crucial point:</u> These and most other relevant actions involve only <u>local</u> information about $\mathbb{R}^4$,

e.g. $\dot{x}^\mu = \lim_{\varepsilon \to 0} \frac{x^\mu(\tau+\varepsilon) - x^\mu(\tau)}{\varepsilon}$ requires only the difference of vectors characterizing nearby points.

We do not need quantities like $x^\mu - y^\mu$ (for significantly different x^μ, y^μ).

Similarly, $A_\mu = A_\mu(x)$ is defined at every point of $\mathbb{R}^4$.

$\partial_\mu A_\nu$ involves only information from nearby points.

- Thus we don't really need the linear structure of $\mathbb{R}^4$. We can allow for coordinate changes of the type

$$x'^\mu = \underset{\substack{\uparrow \\ \text{differentiable} \\ \text{function}}}{f^\mu(x)} \quad \left(\text{rather than} \quad x'^\mu = \Lambda^\mu{}_\nu x^\nu\right).$$

It is only important that our space <u>locally</u> looks like $\mathbb{R}^4$ (i.e. small patches can be identified with small patches of $\mathbb{R}^4$ in some well-defined way.

- $\Rightarrow$ Our space-time only needs to be a 4-dim. differentiable <u>manifold</u>. We will, however, still need a metric to measure distances locally.