3 Geodesics, Curvature, Einstein-Hilbert action

3.1 Geodesics

- A geodesic is the generalization of the concept of a straight line of euclidean geometry to metric manifolds.

- An obvious way to define this mathematically is to demand that the tangent vector does not change its direction along the curve:

$$\text{curve } x(\tau) \; ; \quad \underbrace{\left(\frac{dx^\mu}{d\tau} \cdot D_\mu\right)}_{\substack{\text{covar. directional}\\ \text{derivative}\\ \text{along curve}}} \underbrace{\frac{dx^\rho}{d\tau}}_{\substack{\text{tangent}\\ \text{vector}}} = \underbrace{\alpha \cdot \frac{dx^\rho}{d\tau}}_{\substack{\text{some vector}\\ \text{proportional}\\ \text{to tangent}\\ \text{vector}}} .$$

- Since the length of the tangent vector of a curve at any point depends on the parametrization, we can always choose a parametrization where

$$\boxed{\left(\frac{dx^\mu}{d\tau} D_\mu\right) \frac{dx^\rho}{d\tau} = 0}$$

We will work with this simpler definition of a geodesic from now on.

- More explicitly, the above reads

$$\frac{dx^\mu}{d\tau}\left(\frac{\partial}{\partial x^\mu} \cdot \left(\frac{dx^\rho}{d\tau}\right) + \Gamma_{\mu\nu}{}^{\rho} \frac{dx^\nu}{d\tau}\right) = \underline{\underline{\frac{d^2 x^\rho}{d\tau^2} + \Gamma_{\mu\nu}{}^{\rho} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0}}$$

or

$$\frac{d^2 x^\rho}{d\tau^2} + g^{\rho\sigma}(\partial_\mu g_{\nu\sigma}) \dot{x}^\mu \dot{x}^\nu - \frac{1}{2} g^{\rho\sigma}(\partial_\sigma g_{\mu\nu}) \dot{x}^\mu \dot{x}^\nu = 0 .$$

3.2 Extremizing the invariant length

- Consider first manifolds with a positive definite metric (i.e., all eigenvalues of $g_{\mu\nu}$ are positive; such manifolds are also known as "Riemannian manifolds").

- On such manifolds, there will in general be curves with minimal invar. length connecting two points. We will show that such curves are geodesics:

- Let $x(t)$ be a curve connecting p_1 and p_2 ($p_1, p_2 \in M$). (We may assume, e.g., $t \in [0,1]$.)

- $\ell = \int_0^1 d\tau \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$ is the inv. length.

- $\delta\ell = \int_0^1 d\tau \, \frac{1}{2\sqrt{\dot{x}^2}} \left(2 g_{\mu\nu} \dot{x}^\mu \left(\frac{d\,\delta x^\nu}{d\tau}\right) + \frac{\partial g_{\mu\nu}}{\partial x^\rho} \delta x^\rho \dot{x}^\mu \dot{x}^\nu \right)$

[Without loss of generality, we choose a parametrization in which $\dot{x}^2 = \text{const.}$]

$$= \int_0^1 \frac{d\tau}{\sqrt{\dot{x}^2}} \left(-\frac{d}{d\tau}\left(g_{\mu\nu}\dot{x}^\mu\right) + \frac{1}{2}\left(\partial_\nu g_{\mu\rho}\right)\dot{x}^\mu \dot{x}^\rho \right) \delta x^\nu$$

$$= -\int_0^1 \frac{d\tau}{\sqrt{\dot{x}^2}} \underbrace{\left(g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} + \left(\partial_\sigma g_{\mu\nu}\right)\dot{x}^\sigma \dot{x}^\mu - \frac{1}{2}\left(\partial_\nu g_{\mu\rho}\right)\dot{x}^\mu \dot{x}^\rho \right)} \delta x^\nu$$

raising the index μ, this becomes precisely the expression that was shown to vanish for a geodesic at the end of Sect. 3.1.

- Thus, minimal-length curves on Riemannian manifolds are geodesics.

3.3 Motion of point-particles in an external gravit. field

- As we will see below, the metric characterizes the physical gravitational field. Thus, the motion of point-particles is simply the motion on a Lorentzian manifold.

- The action for a massive point-particle is the obvious generalization of the Minkowski-space action:

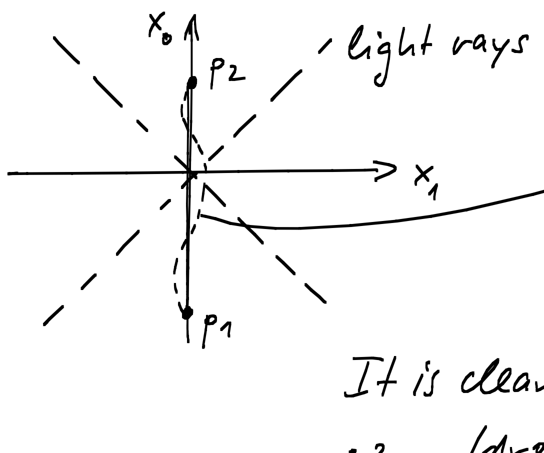
$$S = -m\int d\tau\,\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = -m\int d\tau\,\sqrt{-\dot{x}^2}\,.$$

- The curve (trajectory) is time-like at every point.

- $\delta S = 0$ implies, via a calculation identical to that of Sect. 3.2, that the trajectory is a time-like geodesic:

$$\frac{d^2x^\rho}{d\tau^2} + \Gamma_{\mu\nu}{}^\rho\,\frac{dx^\mu}{d\tau}\cdot\frac{dx^\nu}{d\tau} = 0\,.$$

- Note that the trajectory has maximal rather than minimal invariant length. To see this, consider 2d-Minkowski space:



parametrize both the straight and the bent line by x_0;

It is clear that for the bent line

$$-\dot{x}^2 = \left(\frac{dx^0}{dx^0}\right)^2 - \left(\frac{dx^1}{dx^0}\right)^2 = 1 - \left(\frac{dx^1}{dx^0}\right)^2$$

is always smaller than for the straight line.

- Geodesics on a Lorentz manifold are always time-like (resp. space-like, $\dot{x}^2 > 0$; resp. light-like; $\dot{x}^2 = 0$) at <u>every</u> point since $\dot{x}^\mu$ is covar. constant and hence $\dot{x}^2$ is constant.

- It can be shown* on the basis of Maxwell's equation in curved space that light propagates on light-like geodesics. (Although our above point particle action does not apply: The variation of a light-like geodesic will, in general, be a curve that changes between time-like and space-like. Such curves have no well-defined invar. length.)

  * see later in these notes

- Space-like geodesics connecting two given points are in general neither maxima nor minima of the invariant-distance-functional. They are just stationary points ("saddle points").

3.4 Action for massless point-particle

- As we have seen, the action

$$S = -m \int d\tau \sqrt{-\dot{x}^2}$$

is not applicable in the massless case. However, there exists a generalization of the above action in which the limit $m \to 0$ can be taken.

- To write down such an action, we introduce a metric on the trajectory (= world line = 1-dim. manifold) of the particle:

$$\text{metric: } \gamma_{\tau\tau}(\tau) < 0$$

in analogy to the "-1" of $\eta_{\mu\nu}$.

- invar. distance : $ds^2 = \gamma_{\tau\tau}\, d\tau^2$
- square root of the "determinant" : $\eta(\tau) \equiv \sqrt{-\gamma_{\tau\tau}(\tau)}$
- action: $$S = \frac{1}{2}\int d\tau \left(\eta^{-1}\dot{x}^2 - \eta m^2\right)$$
- problem: Check that this action is invariant under reparametrizations $\tau \to \tau' = \tau'(\tau)$.
  (For this you need to obtain the transformation of η under such reparametrizations.)
- Varying the above action with respect to η and demanding stationarity, we find
  $$-\eta^{-2}\dot{x}^2 - m^2 = 0 \quad \text{or} \quad \eta^2 = -\dot{x}^2/m^2 .$$
- Inserting this in the action, we find
  $$S = \frac{1}{2}\int d\tau \left(\frac{1}{\sqrt{-\dot{x}^2/m^2}} \cdot \dot{x}^2 - \sqrt{-\dot{x}^2/m^2}\, m^2\right)$$
  $$= -\frac{m}{2}\int d\tau \left(\sqrt{-\dot{x}^2} + \sqrt{-\dot{x}^2}\right) = -m\int d\tau \sqrt{-\dot{x}^2} ,$$
  i.e., the standard massive point-particle action.
- For $m=0$, the action reads $S = \frac{1}{2}\int d\tau\, \eta^{-1}\dot{x}^2$
  and variation w.r.t. η gives $\dot{x}^2 = 0$, i.e., the trajectory is light-like.
- Now we vary w.r.t. $x(\tau)$:
  $$\delta S = \frac{1}{2}\int d\tau\, \eta(\tau)^{-1}\left(2 g_{\mu\nu}\dot{x}^\mu \frac{d\,\delta x^\nu}{d\tau} + \partial_\rho(g_{\mu\nu})\,\delta x^\rho \dot{x}^\mu \dot{x}^\nu\right)$$
- If we now choose a parameterization such that $\eta(\tau) = \text{const.}$,

we can perform a calculation identical to that for time-like geodesics in Sect. 3.2 and demonstrate that the extremal trajectory is a <u>light-like geodesic</u>.

- <u>Comment:</u> The above action for point particles does not only have the advantage that the $m=0$ case is included. It is also convenient because of the absence of the $\sqrt{\cdots}$. Its generalization to the string, $\int d\tau \rightarrow \int d\sigma \cdot d\tau$

is called the "Polyakov action" and forms the basis for the quantization of the string in modern string theory.

3.5 Curvature tensor

- Now that we can measure distances and describe point-particle motion on a Lorentz-manifold, we need to characterize the dynamics of the metric ($\hat{=}$ the gravit. field).
- For this we need an action, which is normally given by integrating a scalar over all space-time (cf. $\int d^4x\, F_{\mu\nu} F^{\mu\nu}$ of electrodynamics).
- As in this example, this scalar (the Lagrange density or Lagrangian) is normally constructed by contracting indices of tensors.

- The only tensor we have so far is $g_{\mu\nu}$, but $g_{\mu\nu} g^{\mu\nu} = \mathrm{tr}\, \mathbb{1} = 4$, which is not suitable. Furthermore, we normally need derivatives in the lagrangian to get interesting dynamics. Thus, we should use D_μ. However, $D_\mu g_{\nu\rho} = 0$.

- Crucial idea: We can construct a tensor from D_μ which is independent of any specific tensor field to which D_μ may be applied.

- Consider $[D_\mu, D_\nu]$ as an operator mapping

  covector-field $\rightarrow$ covector field.

- Let D_μ be some connection (not necessarily the Riem. conn.) and calculate

$$[D_\mu, D_\nu] f = D_\mu \partial_\nu f - D_\nu \partial_\mu f = \Gamma_{\mu\nu}{}^{\rho} \partial_\rho f - \Gamma_{\nu\mu}{}^{\rho} \partial_\rho f$$
$$= (\Gamma_{\mu\nu}{}^{\rho} - \Gamma_{\nu\mu}{}^{\rho}) D_\rho f \equiv T_{\mu\nu}{}^{\rho} D_\rho f$$

- $T_{\mu\nu}{}^{\rho}$ is the torsion tensor (we know that it is a tensor from the transformation properties of the l.h. side).

- Now apply $[D_\mu, D_\nu] - T_{\mu\nu}{}^{\rho} D_\rho$ to a covector field $f \cdot v_\sigma$:

$$([D_\mu, D_\nu] - T_{\mu\nu}{}^{\rho} D_\rho) \cdot f v_\sigma = D_\mu ((D_\nu f) v_\sigma + f D_\nu v_\sigma) - \{\mu \leftrightarrow \nu\}$$
$$- T_{\mu\nu}{}^{\rho} ((D_\rho f) v_\sigma + f D_\rho v_\sigma)$$

$$= (T_{\mu\nu}{}^{\rho} D_\rho f) v_\sigma + (D_\nu f)(D_\mu v_\sigma) - (D_\mu f)(D_\nu v_\sigma) + (D_\mu f)(D_\nu v_\sigma)$$
$$- (D_\nu f)(D_\nu v_\sigma) + f [D_\mu, D_\nu] v_\sigma - T_{\mu\nu}{}^{\rho} (D_\rho f) v_\sigma - T_{\mu\nu}{}^{\rho} f D_\rho v_\sigma$$
$$= f\, ([D_\mu, D_\nu] - T_{\mu\nu}{}^{\rho} D_\rho)\, v_\sigma \, .$$

- We see from this, in complete analogy to our previous discussion of the operator $(D_\mu - D_\mu')$, that the operator $[D_\mu, D_\nu] - T_{\mu\nu}{}^\rho D_\rho$ at a certain point is only sensitive to the value of the covector field $f \cdot \omega_\sigma$ at this point. Thus, it is a linear operator $T_p^* \to T_p^*$. It can be characterized by a matrix $R_{\mu\nu\sigma}{}^\rho$, i.e.

$$\boxed{[D_\mu, D_\nu] v_\rho = T_{\mu\nu}{}^\sigma D_\sigma v_\rho + R_{\mu\nu\rho}{}^\sigma v_\sigma}$$

for any covector field v_ρ.

- Since we are primarily interested in the Riemann connection, $T_{\mu\nu}{}^\rho = 0$ and the only tensor we get from this construction is the $\underline{\text{curvature tensor } R_{\mu\nu\rho}{}^\sigma}$. (= Riemann tensor)

- Explicit calculation:

$$[D_\mu, D_\nu] v_\rho = D_\mu \left(\partial_\nu v_\rho - \Gamma_{\nu\rho}{}^\sigma v_\sigma\right) - \{\mu \leftrightarrow \nu\}$$

$$= \partial_\mu \left(\partial_\nu v_\rho - \Gamma_{\nu\rho}{}^\sigma v_\sigma\right) - \Gamma_{\mu\nu}{}^\tau \left(\partial_\tau v_\rho - \Gamma_{\tau\rho}{}^\sigma v_\sigma\right) - \Gamma_{\mu\rho}{}^\tau \left(\partial_\nu v_\tau - \Gamma_{\nu\tau}{}^\sigma v_\sigma\right)$$
$$- \{\mu \leftrightarrow \nu\}$$

[all terms with derivatives acting on v cancel out; $\Gamma_{\mu\nu}{}^\rho$ is symmetric in $\mu\nu$]

$$= -\partial_\mu \Gamma_{\nu\rho}{}^\sigma v_\sigma + \Gamma_{\mu\rho}{}^\tau \Gamma_{\nu\tau}{}^\sigma v_\sigma - \{\mu \leftrightarrow \nu\}$$

$$\Rightarrow \boxed{R_{\mu\nu\rho}{}^\sigma = -\partial_\mu \Gamma_{\nu\rho}{}^\sigma + \Gamma_{\mu\rho}{}^\tau \Gamma_{\nu\tau}{}^\sigma - \{\mu \leftrightarrow \nu\}}$$

- Recalling that $\Gamma_{\mu\nu}{}^{\rho} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$, we see that $R_{\mu\nu\rho}{}^{\sigma}$ is a tensor constructed only from 1st & 2nd derivatives of the metric. In fact, any other tensor involving only the metric and not more than two partial derivatives can be constructed from $R_{\mu\nu\rho}{}^{\sigma}$ and $g_{\mu\nu}$. In this sense, $R_{\mu\nu\rho}{}^{\sigma}$ is <u>unique</u>.

- Action of $[D_\mu, D_\nu]$ on vector fields:

$$[D_\mu, D_\nu](v^\rho w_\rho) = 0 \quad \text{(torsion freeness)}$$

$$[D_\mu, D_\nu](v^\rho w_\rho) = D_\mu\left((D_\nu v^\rho) w_\rho + v^\rho D_\nu w_\rho\right) - \{\mu \leftrightarrow \nu\}$$

$$= (D_\mu D_\nu v^\rho) w_\rho + (D_\nu v^\rho)(D_\mu w_\rho) + (D_\nu v^\rho)(D_\mu w_\rho) + v^\rho D_\mu D_\nu w_\rho$$
$$- \{\mu \leftrightarrow \nu\}$$

$$= \left([D_\mu, D_\nu] v^\rho\right) w_\rho + v^\rho [D_\mu, D_\nu] w_\rho$$

$$\Rightarrow \left([D_\mu, D_\nu] v^\rho\right) w_\rho = - R_{\mu\nu\rho}{}^{\sigma} v^\rho w_\sigma \quad , \text{ for any } w!$$

$$\Rightarrow [D_\mu, D_\nu] v^\sigma = - R_{\mu\nu\rho}{}^{\sigma} v^\rho$$ (in complete analogy to the action of $\Gamma_{\mu\nu}{}^{\rho}$ on vectors and covectors)

- The generalization to arbitrary tensors is obvious:

$$[D_\mu, D_\nu]\, t^{\rho_1 \cdots \rho_m}{}_{\sigma_1 \cdots \sigma_n} = -R_{\mu\nu\tau}{}^{\rho_1} t^{\tau \cdots \rho_m}{}_{\sigma_1 \cdots \sigma_n} - \cdots - R_{\mu\nu\tau}{}^{\rho_m} t^{\rho_1 \cdots \tau}{}_{\sigma_1 \cdots \sigma_n}$$
$$+ R_{\mu\nu\sigma_1}{}^{\tau} t^{\rho_1 \cdots \rho_m}{}_{\tau \cdots \sigma_n} + \cdots + R_{\mu\nu\sigma_n}{}^{\tau} t^{\rho_1 \cdots \rho_m}{}_{\sigma_1 \cdots \tau}$$

- Since $g_{\mu\nu}$ is covariantly constant, we have

$$0 = [D_\mu, D_\nu] g_{\rho\sigma} = R_{\mu\nu\rho}{}^{\tau} g_{\tau\sigma} + R_{\mu\nu\sigma}{}^{\tau} g_{\rho\tau} = R_{\mu\nu\rho\sigma} + R_{\mu\nu\sigma\rho}$$

- Thus $R_{\mu\nu\rho\sigma}$ is antisymm. in ρ, σ.
- $R_{\mu\nu\rho\sigma}$ is also antisymm. in μ, ν (by its definition)
- Thus, if we want to construct a tensor with fewer indices from $R_{\mu\nu\rho\sigma}$, we have to multiply with $g^{\nu\sigma}$ $\underbrace{(\text{or } g^{\mu\rho} \text{ or } g^{\nu\rho} \text{ or } g^{\mu\sigma})}$

  These choices are equivalent to the first choice (possibly up to a sign)

- We define the Ricci tensor $R_{\mu\rho} = R_{\mu\nu\rho\sigma} g^{\nu\sigma} = R_{\mu\nu\rho}{}^{\nu}$.
- Useful fact: $R_{\mu\nu} = R_{\nu\mu}$

Derivation: $R_{\mu\nu\rho\sigma}$ has the properties

$$
\begin{aligned}
&(1)\quad R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} \qquad &&\text{(obvious)}\\
&(2)\quad R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} \qquad &&\text{(see above)}\\
&(3)\quad R_{[\mu\nu\rho]\sigma} = 0
\end{aligned}
$$

"Antisymmetrization", i.e. $\frac{1}{3!}$ (Sum of all permutations, with odd permutations getting a minus sign)

(Property (3) can be read off from the explicit expression in terms of $\Gamma_{\mu\nu}{}^{\rho}$ using $\Gamma_{\mu\nu}{}^{\rho} = \Gamma_{\nu\mu}{}^{\rho}$; an abstract proof will be given later using diff. forms.)

Explicitly, property (3) reads

$$R_{\mu\nu\rho\sigma} + R_{\nu\rho\mu\sigma} + R_{\rho\mu\nu\sigma} - R_{\nu\mu\rho\sigma} - R_{\mu\rho\nu\sigma} - R_{\rho\nu\mu\sigma} = 0$$

Using (1): $R_{\mu\nu\rho\sigma} + R_{\nu\rho\mu\sigma} + R_{\rho\mu\nu\sigma} = 0 \qquad (3')$

Now we can easily obtain the desired result:

$$R_{\mu\nu\rho\sigma} \overset{(3')}{=} -R_{\nu\rho\mu\sigma} - R_{\rho\mu\nu\sigma} \overset{(2)}{=} R_{\nu\rho\sigma\mu} + R_{\rho\mu\sigma\nu}$$

$$\overset{(3')}{=} -R_{\rho\sigma\nu\mu} \underbrace{- R_{\sigma\nu\rho\mu} - R_{\mu\sigma\rho\nu}}_{\overset{(2)}{=} R_{\sigma\nu\mu\rho} + R_{\mu\sigma\nu\rho} \overset{(3')}{=} -R_{\nu\mu\sigma\rho}} - R_{\sigma\rho\mu\nu}$$

$$\overset{(1),(2)}{=} 2R_{\rho\sigma\mu\nu} - R_{\mu\nu\rho\sigma}$$

$$\Rightarrow 2R_{\mu\nu\rho\sigma} = 2R_{\rho\sigma\mu\nu}$$

Thus we have found a further useful property of $R_{\mu\nu\rho\sigma}$:

$$(4) \quad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \quad (\text{consequence of } (1) \text{--} (3))$$

$$\Rightarrow R_{\mu\nu} = R_{\mu\rho\nu\sigma} g^{\rho\sigma} = R_{\nu\sigma\mu\rho} g^{\rho\sigma} = R_{\nu\sigma\mu\rho} g^{\sigma\rho} = R_{\nu\mu} \quad \square .$$

- Comment:

- Property (3) is also known as the 1st Bianchi identity
- The 2nd Bianchi identity (or simply Bianchi identity) is

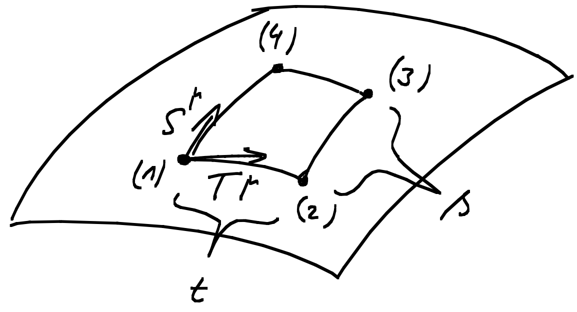
$$D_{[\mu} R_{\nu\rho]\sigma}{}^{\tau} = 0 .$$

(Problem: Proof this using the definition of $R_{\mu\nu\rho}{}^{\sigma}$ via a commutator of D_μ's. Apply the Jacobi identity $[A,[B,C]]$ + cycl. permutations $= 0$. (Derive also the Jacobi identity.))

- Finally, having defined Riemann tensor $R_{\mu\nu\rho\sigma}$ and the Ricci tensor $R_{\mu\nu}$, we define the Ricci scalar or curvature scalar $R = R_\mu{}^\mu = R_{\mu\nu} g^{\mu\nu}$.

3.6 "Physical" interpretation of the curvature

- Curvature characterizes the non-triviality of parallel transport around closed loops:



- Consider a manifold with a point (1) and two vectors T^μ & S^μ in the tangent space $T_{(1)}$ at this point.

- Consider some curve through (1) with tangent vector T^μ. Move a distance t (in the parametrization used to define T^μ) along this curve to define the point (2). (We say that is a motion of distance t in the direction of T^μ.)
- Parallel transport S^μ to (2). Then move a distance s in the direction of $S^\mu_{(2)}$ to define (3).
- Then move a distance t in the direction of $-T^\mu_{(3)}$ to define (4).
- Finally, move a distance s in the direction of $-S^\mu_{(4)}$ to define (1)'.

[If the torsion vanishes, (1)' will agree with (1) at leading order in t, s for $t \to 0$ and $s \to 0$ ($\to$ problems).]

- Next, parallel transport a vector v^μ at (1) along the closed loop defined above to define v'^μ, also at (1). Define $\delta v^\mu = v'^\mu - v^\mu$.

- Claim: $\delta v^\mu = t \cdot s \, T^\nu S^\rho R_{\nu\rho\sigma}{}^\mu v^\sigma +$ higher orders in t & s.

(Problem: Derive this by an explicit calculation.)

- Intuitive motivation of this result:

If there exists a covariantly constant vector field $v^\mu(x)$ with value v^μ at (1), then $\delta v^\mu = 0$. At the same time $[D_\mu, D_\nu] v^\rho = 0$, i.e. $R_{\mu\nu\sigma}{}^\rho v^\sigma = 0$. Thus $R_{\mu\nu\sigma}{}^\rho v^\sigma$ characterizes the impossibility to extend v^μ to a covar. constant vector field (in the μ-ν submanifold).

Thus: $R_{\mu\nu\rho}{}^\sigma$ characterizes the development of a σ-component of a vector in the ρ-direction upon parallel transport around an infinitesimal closed loop in the μ-ν-submanifold.

3.7 The Einstein-Hilbert action

In Minkowski space, an action for a field theory with field $\varphi(x)$ are generally given as

$$S[\varphi] = \int d^4x \, \mathcal{L}[\varphi] = \int d^4x \, \mathcal{L}(\varphi, \partial_\mu \varphi),$$

where $\mathcal{L}$ is a scalar (a function) on Mink. space. The simplest example is

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2} m^2 \varphi^2$$

for a scalar field with mass m. After quantization, this gives rise to spinless particles with mass m.

- An example are pions which, however, are bound states of quarks and gluons. (Thus, in this case the above action is only an effective description at low energies.)

- Another well-known action of this type is given by
$$\mathcal{L}[A_\mu] = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$
of electrodynamics, characterizing free photons.

- In general relativity, we need to make these actions diffeomorphism-invariant. It is known from analysis that
$$\int d^4x\, f(x) = \int d^4x'\, \det\left(\frac{\partial x}{\partial x'}\right) f'(x'),$$
where $f'(x)$ is defined by $f(x) = f'(x'(x))$ and
$$\left(\frac{\partial x}{\partial x'}\right)^\mu{}_\nu = \frac{\partial x^\mu}{\partial x'^\nu},$$
with $x = x(x')$ is the inverse function of $x' = x'(x)$.

<u>Claim:</u> $\int d^4x \sqrt{-g}\, f(x)$ is diffeomorphism invariant.
(Here $g = \det(g_{\mu\nu})$. The "$-$" is necessary since, given that $g_{\mu\nu}$ can locally be brought to the form $\mathrm{diag}(-1,1,1,1)$, $\det(g_{\mu\nu}) < 0$.)

<u>Demonstration:</u> $g'_{\mu\nu} = g_{\rho\sigma} \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} = M_\mu{}^\rho g_{\rho\sigma} (M^T)^\sigma{}_\nu$,

where $M_\mu{}^\rho = \frac{\partial x^\rho}{\partial x'^\mu}$.

Thus, $\sqrt{-g'} = \sqrt{-(\det M)(\det g_{\mu\nu})(\det M)} = \sqrt{-g}\, \det\left(\frac{\partial x}{\partial x'}\right)$.

$$\Rightarrow \int d^4x' \sqrt{-g'}\, f'(x') = \int d^4x \, \det\left(\frac{\partial x'}{\partial x}\right) \sqrt{-g} \cdot \det\left(\frac{\partial x}{\partial x'}\right) f(x)$$

$$= \int d^4x \sqrt{-g}\, f(x) \quad \square .$$

- We are thus naturally lead to the action

$$S = \int d^4x \sqrt{-g} \left(\mathcal{L}_{gravity} + \mathcal{L}_{matter} \right)$$

$$= \int d^4x \sqrt{-g} \left(\frac{M^2}{2} R + \mathcal{L}_{matter} \right),$$

where $\mathcal{L}_{matter}$ is, e.g., $\mathcal{L}_{el.dyn.}$ or $\mathcal{L}_{scalar}$ (see above). The gravity part is also known as the Einstein-Hilbert term. M^2 is the reduced planck mass, related to the usual planck mass $M_P = \frac{1}{G}$ by $M = M_P / \sqrt{8\pi}$ (for the origin of the factor $\sqrt{8\pi}$ see later).

<u>Comments</u>

- We could, already at this point, focus only on the R-term, make the ansatz $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ($h_{\mu\nu}$ small), and derive the quadratic action in $h_{\mu\nu}$

$$\left(R[\eta_{\mu\nu} + h_{\mu\nu}] = O(h^2) \right).$$

This would give rise to EOMs similar to the Maxwell-eqs. for A_μ and describe the propagation of gravity waves (see later for details).

- We could consider terms with more than two derivatives (e.g. R^2 or contractions of two $R_{\mu\nu\rho\sigma}$-terms). However, at low curvature (= small derivatives of $g_{\mu\nu}$ = large-distance-physics) these terms are unimportant unless the coefficients are very large. Experimental bounds suggest that such contributions to the action are indeed unimportant for all practical purposes.

- If one is prepared to give up analyticity in $h_{\mu\nu}$, one can also consider terms like R^{-1} etc. Such terms are sometimes considered in cosmological models (for very large scales, i.e., small curvature).

- It is natural to add a term $\mathcal{L} = -\Lambda = \text{const.}$ (i.e. $S_\Lambda = -\int d^4x \sqrt{-g}\, \Lambda$), the so-called cosmological constant. (It can be viewed either as part of $\mathcal{L}_{grav.}$ or $\mathcal{L}_{matter}$; we view the second option as more natural.) Such a term is probably responsible for the observed accelerated expansion of the universe. It is irrelevant at small (including galactic) scales.