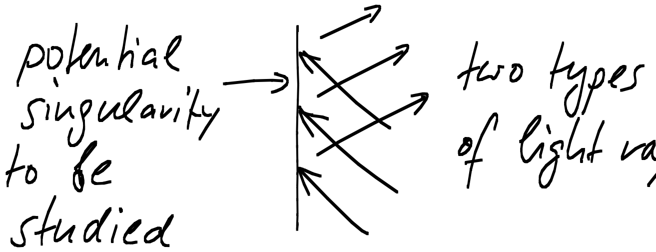
8 Schwarzschild Black Holes

8.1 The Rindler spacetime

- It is defined by $ds^2 = -x^2 dt^2 + dx^2$ ($t \in (-\infty, \infty)$; $x \in (0, \infty)$)
- At the moment, it just provides a useful example for a coordinate singularity of the type encountered in the Schwarzschild geometry. Later on, it will prove important in the study of the Unruh effect.
- Obviously, the metric is singular at $x = 0$.
- Systematic approach to removing this singularity (if this should be possible):
  - consider null geodesics going in / coming out of the singularity and use their affine parameters as coordinates

  (At least in $d=2$ this always works since two null geodesics within one class can never cross. The reason is simply that in $d=2$ there is only one incoming/outgoing light-ray for each point.)

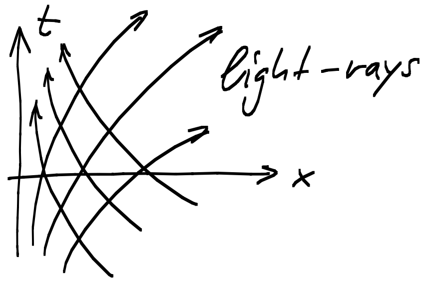


- light rays are easily found from $0 = -x^2\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2$

  $\Rightarrow \quad dt = \pm\, dx/x \quad \Rightarrow \quad t = \pm \ln x + c$
- We label the geodesics by $\pm$ (out/ingoing) and the constants

- This provides our new coordinates u, v:

$$u = t - \ln x$$
$$v = t + \ln x$$



- to get the new metric expression:

$$t = \frac{u+v}{2}\,, \quad x = \exp\frac{v-u}{2}$$

$$dt = \frac{1}{2}(du+dv) \;;\quad dx = \frac{1}{2}e^{\frac{v-u}{2}}(dv-du)$$

$$ds^2 = -x^2dt^2 + dx^2 = e^{v-u}\cdot\frac{1}{4}\left(-(du+dv)^2 + (dv-du)^2\right)$$

$$ds^2 = -e^{v-u}\,du\,dv$$

- Next, we want to reparametrize each geodesic using its affine parameter. We use the fact that our theorem

$u_\mu\xi^\mu = \text{const.}$ only holds if the geodesic defining u^μ is properly (affinely) parametrized.

- As a Killing vector we can use $(1,0)$ (in t, x coords.) since time-translations are obviously a symmetry. Hence

$$g_{\mu\nu}u^\mu\xi^\nu = x^2\frac{dt}{d\tau} = E = \text{const.}$$

affine parameter

- For an outgoing geodesic defined by $u = \text{const.}$, we have
$$d\tau = \frac{1}{E}x^2dt = \frac{1}{E}e^{v-u}\frac{dv}{2}$$

$$\tau = \frac{1}{2E}e^{v-u} + c. \quad (u \text{ is a constant here!})$$

- Since the normalization is arbitrary, we choose e.g.

$$V = e^v$$

- In complete analogy, incoming geodesics are defined by $v = \text{const.}$ $\Rightarrow$

$$d\tau = \frac{1}{E} x^2 dt = \frac{1}{E} e^{v-u} \frac{du}{2}$$

$$\tau = -\frac{1}{2E} e^{v-u} + c$$

$\Rightarrow$ choose, e.g., $U = -e^{-u}$

- Finally: $ds^2 = -dU\,dV$ (Note: $t \in (-\infty, \infty)$, $x \in (0, \infty)$

$$\Downarrow$$

$U \in (-\infty, 0)$, $V \in (0, \infty)$)

- A final (almost trivial) coordinate change

$$T = \tfrac{1}{2}(V + U) \; ; \; X = \tfrac{1}{2}(V - U)$$

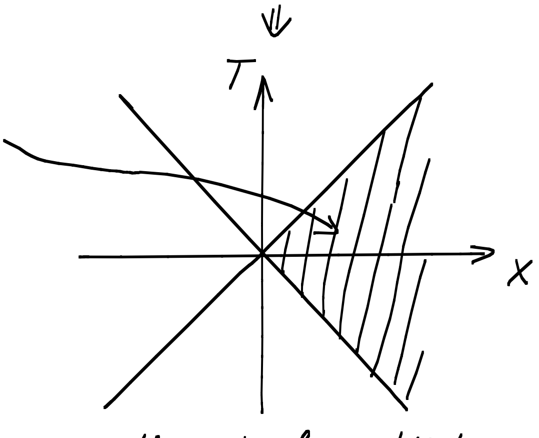
$$(\text{i.e. } V = T + X \; ; \; U = T - X)$$

gives $ds^2 = -dT^2 + dX^2$ with $T + X > 0$; $T - X < 0$

or $X > 0$; $-X < T < X$

$$\Downarrow$$

$\Rightarrow$ Thus, the Rindler spacetime is just a part of Minkowski space. There is no singularity at its boundary ($x^2 = 0$, i.e. $v - u = -\infty$, i.e. $V \cdot U = 0$, i.e. $T = X$ & $T = -X$). We can extend the spacetime to full Minkowski space.



8.2 Kruskal extension of the Schwarzschild solution

- We now repeat the above procedure for the exterior part ($r > 2M$) of the Schwarzschild solution. We ignore θ, φ (i.e. we focus on spacetime points lying on one "ray" going through the origin):

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 .$$

- The (radial) null geodesics are defined by

$$0 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 .$$

$$\Rightarrow \quad \pm dt = \left(1-\frac{2M}{r}\right)^{-1}dr = \frac{r}{r-2M}\,dr$$

$$\pm t = \int \frac{r\,dr}{r-2M} = \int\left(1+\frac{2M}{r-2M}\right)dr$$

$$= \int\left(1+\frac{1}{r/2M-1}\right)dr = r + 2M\ln\left(\frac{r}{2M}-1\right)+C$$

- Defining the "Regge-Wheeler tortoise coordinate" or "tortoise coordinate"

[name: object falling in r^* slows up in terms of r] $\quad r_* = r + 2M\ln\left(\frac{r}{2M}-1\right)$,

we find the out/ingoing geodesics (setting $C=u$ / $C=-v$)

$$t = r_* + u \qquad\qquad u = t - r_*$$
$$\text{or}$$
$$t = -r_* + v \qquad\qquad v = t + r_* .$$

- The metric becomes

$$ds^2 = -\left(1-\frac{2M}{r}\right)du\,dv$$

[For the same reason as in the analysis of the Rindler spacetime, $ds^2 \sim du\,dv$: ds^2 must vanish if either du or dv vanishes, since these are the lightlike geodesics.]

- We check the prefactor: $dr_* = \left(1-\frac{2M}{r}\right)^{-1} dr$

$$du\,dv = dt^2 - dr_*^2 = dt^2 - \left(1-\frac{2M}{r}\right)^{-2} dr^2 \quad \checkmark.$$

- r in $ds^2 = \left(1-\frac{2M}{r}\right) du\,dv$ has to be understood as $r = r(u,v)$, defined by

$$r + 2M \ln\left(\frac{r}{2M} - 1\right) = r_* = \frac{1}{2}(v-u).$$

- Exponentiating this equation (after dividing by $2M$), we get

$$\frac{r}{2M} - 1 = e^{-r/2M}\, e^{(v-u)/4M}$$

$$\text{or} \quad \underbrace{\left(1-\frac{2M}{r}\right)}_{\text{This is just the prefactor of } du\,dv \text{ in } ds^2 \text{, i.e.}} = \frac{2M}{r} e^{-r/2M}\, e^{(v-u)/4M}.$$

$$ds^2 = -\underbrace{\frac{2M}{r} e^{-r/2M}}_{\substack{\text{not singular} \\ \text{at } r = 2M}} \underbrace{\left(e^{-u/4M} du\right)\left(e^{v/4M} dv\right)}_{\substack{\text{analogous to} \\ ds^2 = -e^{v-u} du\,dv \\ \text{of the Rindler spacetime}}}$$

- Thus, in analogy to the treatment of Rindler spacetime, we can introduce

$$U = -e^{-u/4M} \; ; \; V = e^{v/4M}$$

$$dU = \frac{1}{4M} e^{-u/4M} du \; ; \; dV = \frac{1}{4M} e^{v/4M} dv$$

finding
$$ds^2 = -\frac{32M^3}{r} e^{-r/2M}\, dU\,dV.$$

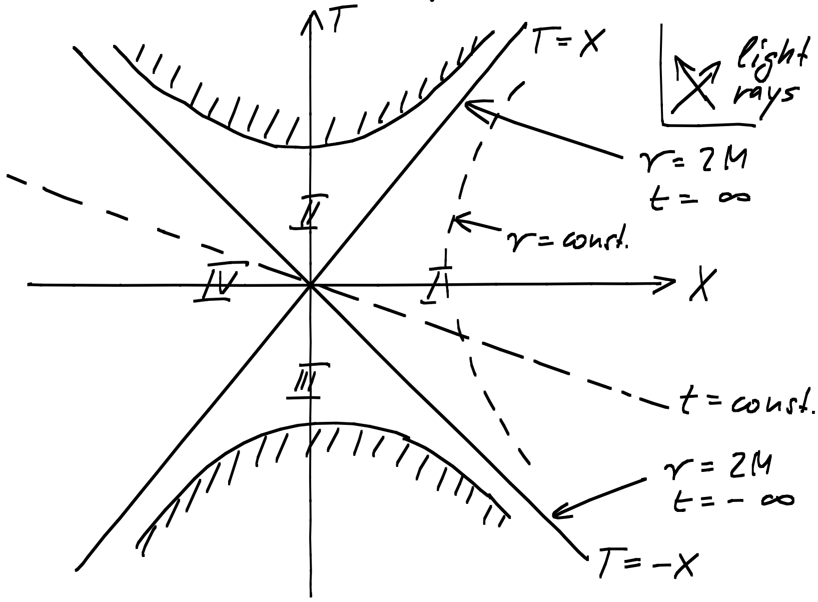
- Finally, we introduce $T = \frac{1}{2}(V+U)$; $X = \frac{1}{2}(V-U)$

and find $$ds^2 = \frac{32M^3}{r} e^{-r/2M} \left(-dT^2 + dX^2\right).$$

• It is now easy to include the S^2-part, giving the final result
$$\underline{\underline{ds^2 = \frac{32M^3}{r} e^{-r/2M} \left(-dT^2 + dX^2\right) + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right).}}$$

• As before, r has to be understood as $r = r(X,T)$, defined by
$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = e^{(v-u)/4M} = -U \cdot V = -T^2 + X^2.$$

• We can now draw the Schwarzschild geometry in Kruskal coordinates:



– The exterior of the Schwarzschild solution, $r > 2M$, implies $X^2 > T^2$.

– In addition,
$$X = \frac{1}{2}(V - U)$$
$$= e^{v/4M} + e^{-u/4M}$$ is clearly always positive in this region.

– Thus, the exterior now corresponds to $X > 0$ & $-X < T < X$. ("region I" in the figure).

– The horizon obviously is $T = \pm X$ $(X > 0)$.
(It is now obvious that there is no singularity at the horizon.)

– To get more intuition, it is useful to express t through X, T:

$t = \frac{1}{2}(u+v)$; recall: $U = -e^{-u/4M}$; $V = e^{v/4M}$

$$\Downarrow \qquad \Downarrow$$

$$u = -4M \ln(-U) \quad v = 4M \ln V$$

$$t = 2M \ln\left(-\frac{V}{U}\right)$$

$$= 2M \ln\left(\frac{X+T}{X-T}\right) = 2M \ln\left(\frac{1-T/X}{1+T/X}\right)$$

Useful rewriting: $\tanh a = \frac{e^a - e^{-a}}{e^a + e^{-a}} = \frac{1-e^{-2a}}{1+e^{-2a}}$

$$\Rightarrow e^{-2a} = \frac{1-\tanh a}{1+\tanh a}$$

$$a = \frac{1}{2}\ln\left(\frac{1+\tanh a}{1-\tanh a}\right) \quad ; \quad \tanh a \equiv b$$

$$a = \operatorname{arctanh} b$$

$$\Downarrow$$

$$\operatorname{arctan} b = \frac{1}{2}\ln\left(\frac{1+b}{1-b}\right)$$

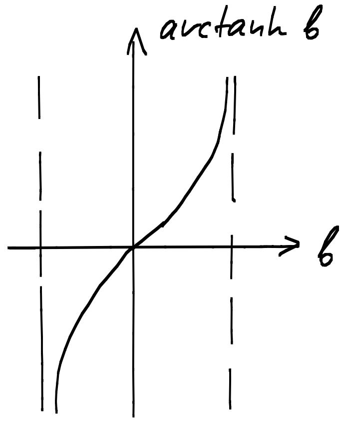
$$\Rightarrow t = 2M \ln\left(\frac{X+T}{X-T}\right) = 4M \operatorname{arctanh}(T/X)$$

- This shows that

  $T = X \ (X>0) \rightarrow t = \infty$

  $T = -X \ (X>0) \rightarrow t = -\infty$

  (see figure)



- $t =$ const. corresponds to rays $T = \alpha X$
- $r =$ const. corresponds to hyperbola $X^2 - T^2 = \alpha$
- The whole horizon ($r = 2M$, t finite) collapses to the

point $T=X=0$ in T,X-space. (Of course, this means that it is actually an S^2 since have suppressed θ & φ.)

- We now come to the actual extension (i.e the discussion of the regions outside ($X>0$; $-X<T<X$)):

- The metric $ds^2 = \frac{32M^3}{r} e^{-r/2M}(-dT^2 + dX^2) + r^2 d\Omega_2$

is non-singular at $T = \pm X$ ($X>0$) and can be analytically continued beyond this curve (i.e. beyond the horizon). In practical terms this simply means taking the above formula for ds^2 seriously for all values of T, X which can be reached without crossing the lines where $r=0$.

- At $r=0$, there is a true (curvature) singularity. ($R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ blows up)

The location of these singularities is specified by $T^2 - X^2 = 1$, i.e. two hyperbola in regions I & III (see figure). In the whole region between these lines the given metric represents a solution of the vacuum Einstein equations.

- We first note the interiour region of our solution in r, t had only one singularity (at $r=0$) while we now have two disconnected ones. This is not surprising since

$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = -T^2 + X^2 \quad ; \quad t = 4M \operatorname{arctanh}(T/X)$$

obviously does not allow to unambiguously fix T, X if t, r are given.

- In fact, we should not take the inner part of our solution in r, t seriously at all since, coming from the physically motivated outer region, we had to go through a coordinate singularity to get there. [Alternatively, we can say that by going from X, T to r, t, we find identical r, t for different X, T. Thus, the true spacetime (in the inner region) is a double cover of the r, t-spacetime.

- Our old r, t inner solution describes either region II or region III, and we don't know which. We should simply not think of our old r, t-inner region any more. It is misleading.

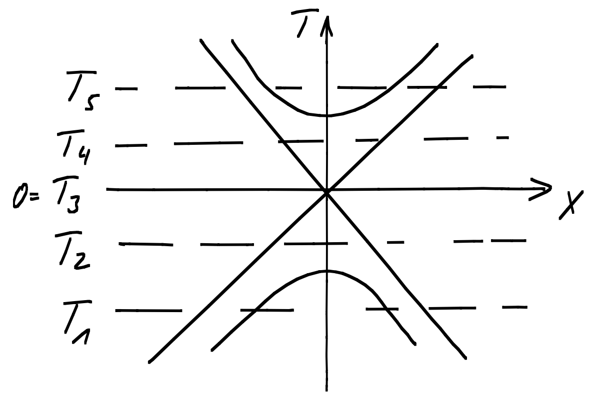
- The singularity is obviously spacelike (see figure). This explains why falling into the singularity is unavoidable for an observer in region II: The singularity is simply in his entire future (rather than at some location in space).

- Since only region II can be reached from the outer region I, we will call this region "the black hole" (and the corresponding singularity the black hole singularity).

- By analogy, region III is called the "white hole". Every observer in this region must have originated in the white hole singularity.

- Region IV is a mirror image of "our" outer region. It is not possible to communicate with any observer in this region. (Unless both observers are prepared to meet in region II and then fall into the singularity.)

- To understand how regions I & IV are connected, it is useful to consider spacelike slices $T = \text{const.}$:



- Let us re-introduce the coordinate φ of the S^2, but continue to suppress θ (setting, e.g., $\theta = \pi/2$, i.e. focussing on the equator of the S^2).

- Let us then consider the submanifold parameterized by X and φ at every $T = \text{const}$.

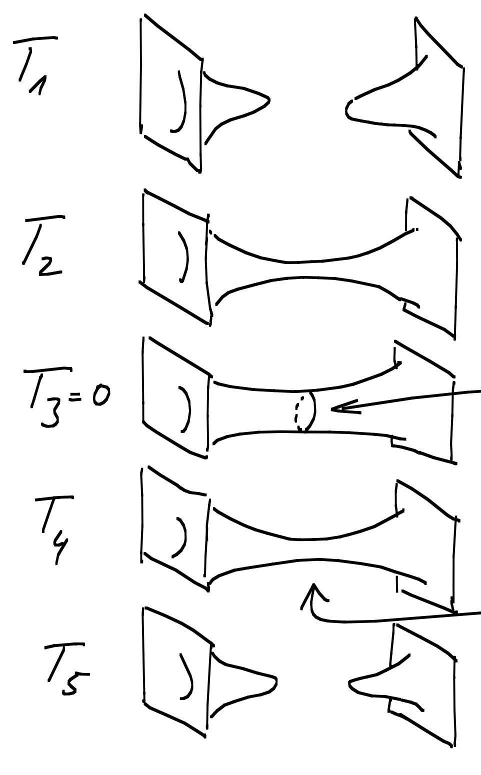
- For every X *), there is an S^1 (parameterized by $\varphi \in (0, 2\pi]$), the radius of which is given by

$$\left(\frac{r}{2M} - 1\right) e^{r/2M} = -T^2 + X^2 .$$

[*) with $X^2 > 1 - T^2$]

this is a monotonically growing fct. of r

- This gives rise to the following set of submanifolds at $T = \text{const.}$:



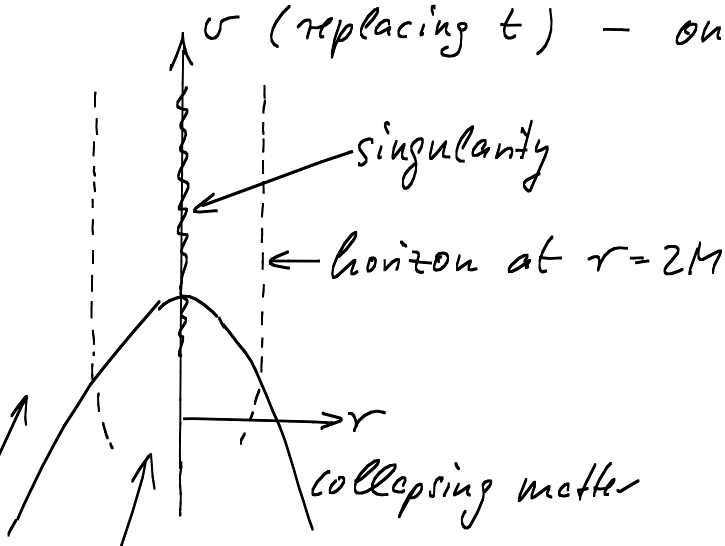
$r = r(T = 0, X = 0) = 2M$
(corresponds to center of previous figure)

This region of spacetime connecting I & IV is called the <u>Einstein-Rosen bridge.</u>

- If black holes exist, the physical reality of region I is obvious. The same is presumably also true for region II since it can be reached from I by observers (although they can't come back to report).

- The reality of regions III & IV is less clear. In particular, it is doubtful that true Schwarzschild black holes (i.e. eternal or static black holes, from the point of view of region I) exist in the universe.

- The physical black holes for which there is evidence (at the galactic centre or as remnants of supernovae) have formed from the spherical collapse of matter. In such situations, regions III & IV do not exist:



– one of the possible coordinate choices (v,r) or (u,r) ("mixed" with respect to our previous use of (t,r) or (u,v))

[also called Eddington-Finkelstein coordinates, → see problems]

the behaviour in this region depends on the details of the dynamical collapse

[It would be inconvenient to think in terms of (t,r) since then, once the horizon has formed, all the outside matter will stay outside.]

the region outside the matter distribution is exactly Schwarzschild since it is the unique spherically symmetric solution