9 More on Black Holes (Penrose diagrams; charged & rotating black holes)

9.1 Conformal transformations

- A conformal trf. is a space-time-dependent rescaling of the metric:
$$g_{\mu\nu}(x) \longrightarrow \tilde{g}_{\mu\nu}(x) = \omega^2(x)\, g_{\mu\nu}(x) .$$
[One may think of a local change of scales.]

Note: This is not a coordinate change. Important physical quantities like the curvature scalar in general change under such a transformation. More formally: Conf. trfs. are not a symmetry of GR.

- Clearly, the transformed quantities can be expressed through the untransformed quantities & ω. For example, in d dimensions
$$\tilde{R} = \omega^{-2} R - 2(d-1)\omega^{-3} g^{\mu\nu} D_\mu D_\nu \omega - (d-1)(d-4)\,\omega^{-4} (\partial_\mu \omega)(\partial^\mu \omega)$$
(and a similar formula for $R = R(\tilde{R}, \omega)$).

- Important fact: - Conformal trfs. do not change the angle between vectors:
$$\frac{g_{\mu\nu} a^\mu b^\nu}{\sqrt{g_{\mu\nu} a^\mu a^\nu}\sqrt{g_{\mu\nu} b^\mu b^\nu}} = \frac{\omega^2 g_{\mu\nu} a^\mu b^\nu}{\sqrt{\omega^2 g_{\mu\nu} a^\mu a^\nu}\sqrt{\omega^2 g_{\mu\nu} b^\mu b^\nu}}$$
- In particular: any light-like vector remains light-like.

- It is sometimes useful to define a fct. $\sigma(x)$ such that
$$\omega = e^\sigma \quad \text{or} \quad \tilde{g}_{\mu\nu} = e^{2\sigma} g_{\mu\nu} .$$

- We then have $g^{\mu\nu} D_\mu D_\nu \omega = g^{\mu\nu} D_\mu (e^\sigma D_\nu \sigma)$
$$= e^\sigma g^{\mu\nu} D_\mu D_\nu \sigma + e^\sigma (\partial_\mu \sigma)(\partial^\mu \sigma).$$

- Hence $\tilde{R} = e^{-2\sigma} \{ R - 2(d-1) g^{\mu\nu} D_\mu D_\nu \sigma - (d-1)(d-2)(\partial_\mu \sigma)(\partial^\mu \sigma) \}.$

- In particular, for $d=2$ we find
$$\tilde{R} = e^{-2\sigma} \{ R - 2 g^{\mu\nu} D_\mu D_\nu \sigma \}.$$

- Thus, in 2 dims. we can always find a conformal trfs. which makes the Ricci scalar vanish.
[$R - 2 D_\mu D^\mu \sigma = 0$ can be viewed as the covariant analogue of $\nabla^2 \varphi = \rho$, which always has a solution, just like in electrostatics.]

- Furthermore, in 2 dims. the curvature tensor only has one independent component: R_{1212}. This means that
$$R = 0 \Rightarrow R_{1212} = 0 \Rightarrow R_{\mu\nu\rho\sigma} = 0.$$

- With this we have shown that in $d=2$ one can always find a conf. trfs. making a given space-time flat, i.e.,

every 2-dim. metric is conformally flat.

9.2 Penrose diagrams

(also known as Penrose-Carter diagrams or conformal diagrams)

- <u>Idea</u>: Find a simple way to represent graphically the <u>causal structure</u> of any 2-dim. spacetime (and of any spacetime whose main features can be reduced to

2 dims. — such as t, r of t, r, θ, φ of the Schwarzschild geometry).

• <u>Procedure:</u>

- given: a certain 2-dim. spacetime (i.e. 1+1 dims.) with metric $ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu \quad (\mu, \nu \in \{0, 1\})$

- choose coordinates in such a way that all variables (in all (finitely many!) coordinate patches vary over a <u>finite range</u>.

  (e.g. $x \in (-\infty, \infty) \rightarrow y = \tanh(x)$ with $y \in (-1, 1)$)

- perform a conformal trf. making this metric flat.

- perform a further coordinate change bringing this flat metric explicitly into Minkowski form, i.e.

$$ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu \quad \text{with} \quad \eta_{\mu\nu} = \text{diag}(-1, 1).$$

[This can always be achieved locally for a flat metric. Non-trivial global features, e.g. if x^1 parametrizes an S^1, will of course remain.]

- The result can be represented as a finite patch of the plane (with trivial metric $ds^2 = -(dx^0)^2 + (dx^1)^2$ implied) on which the whole original spacetime is mapped (unless non-trivial global features require using several patches of this plane).

• <u>Example:</u> Minkowski space

- use light-cone coordinates $u = x^0 - x^1$; $v = x^0 + x^1$

so that $\eta_{\mu\nu} dx^\mu dx^\nu = - du\, dv \quad ; \quad u, v \in (-\infty, \infty)$.

- change coordinates to $\tilde{u} = \tanh u \; ; \; \tilde{v} = \tanh v$ with $\tilde{u}, \tilde{v} \in (-1, 1)$.

$u = \operatorname{arctanh} \tilde{u}$

$du = d\tilde{u} / (1 - \tilde{u}^2)$ (and analogously for $\tilde{v}$ & v)

$\Rightarrow ds^2 = d\tilde{u}\, d\tilde{v} \,/\, (1-\tilde{u}^2)(1-\tilde{v}^2)$

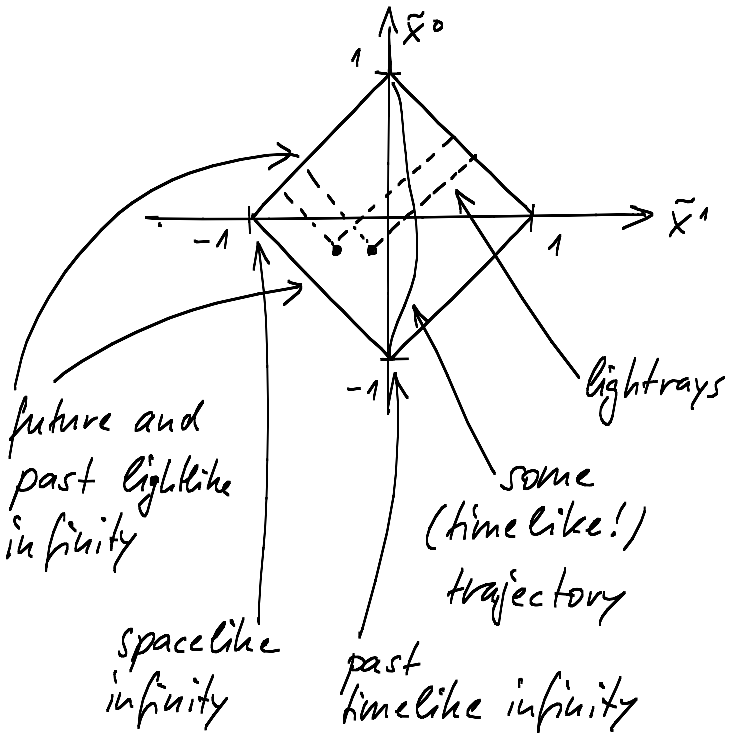
- perform conformal trf. with $\omega^2(\tilde{u}, \tilde{v}) = (1-\tilde{u}^2)(1-\tilde{v}^2)$ to get

$$ds^2 = d\tilde{u}\, d\tilde{v} .$$

(In this case, this was not necessary since the metric was flat anyway. However it is convenient.)

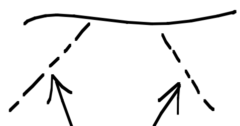
- reparametrize again: $\tilde{u} = \tilde{x}^0 - \tilde{x}^1 \; ; \; \tilde{v} = \tilde{x}^0 + \tilde{x}^1$ to find the metric $ds^2 = -(d\tilde{x}^0)^2 + (d\tilde{x}^1)^2$ (as required)

and $|\tilde{x}^0 - \tilde{x}^1| < 1 \; ; \; |\tilde{x}^0 + \tilde{x}^1| < 1$, i.e.:

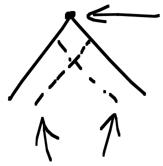


This is obviously strongly distorted compared to the original spacetime. However, lightrays & intersections of lightrays are correctly represented.

- It can be intuitively understood that, e.g., the future timelike infinity is just a point. Imagine a diagram with a spacelike upper boundary:



It is then clearly possible that two lightrays sent in opposite directions never meet. However, this is not the case in Minkowski space. Hence, we <u>had</u> to expect a result like:



- <u>Method of lightrays</u>

  - It is in general complicated to find convenient transformations to follow the above procedure for the construction of a conf. diagram.

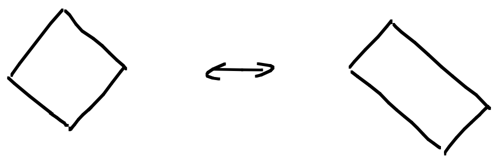
  - A simpler approach (→ arXiv: gr-qc/0503061 by S. Winitzki; also: SPIRES database) uses lightrays to define & construct conf. diagrams:

  - <u>Def:</u> A finite domain of $\mathbb{R}^2$ is a conf. diagram of a given 2d spacetime if $\exists$ 1-1 map between 45°-lines in this domain and lightrays in this spacetime. This map has to respect intersections (i.e. two such lines intersect if and only if the corresponding light-rays do).

(We do not proof equivalence with the earlier definition, but it clearly makes sense at an intuitive level.)

- <u>Note</u>: Obviously, a conformal diagram for a given spacetime is not unique in both definitions. All that matters are boundaries and the way in which 45°-lines intersect and hit the boundaries.

e.g.



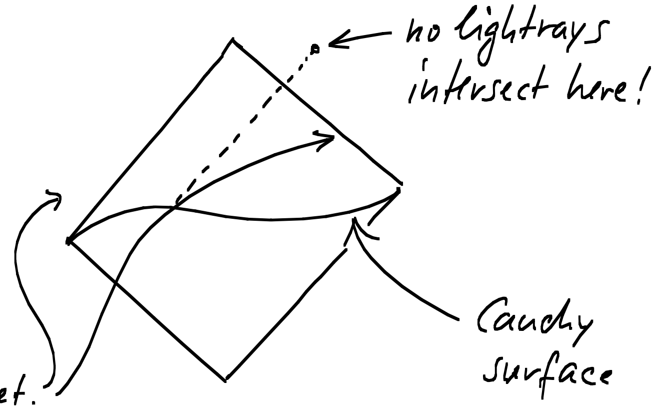
both O.K. for Minkowski space!

- <u>Useful practical procedure:</u>

  - start with a Cauchy surface (i.e. a spacelike hypersurface intersected by every time-like or light-like, maximally extended curve exactly once; intuitively, this is where one has to define the initial data for dynamically solving Einstein's eqs. for all future).

  - analyse light rays coming from this surface (or ending on it) and their intersections

  - E.g. Minkowski space:
    - any two light rays coming from the same Cauchy surface & going in opposite directions will meet. (in particular those from spacelike infinity)



  - there is nothing outside this rectangle since no light rays could intersect there (while lightrays intersect everywhere in Mink. space).

More examples:

• <u>Minkowski space in spherical coordinates</u>

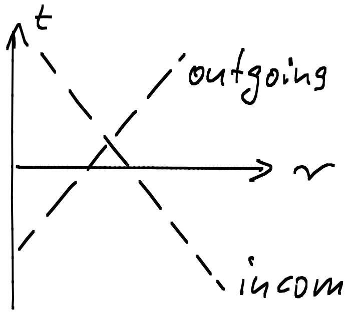
$$ds^2 = -dt^2 + dr^2 + \underbrace{r^2 d\Omega^2}$$

we suppress these S^2's

$t \in (-\infty, \infty)$; $r \in (0, \infty)$

$ds^2 = -dt^2 + dr^2$; Cauchy surface: e.g. $t = 0$.

light rays: $t = \pm r + \text{const.}$

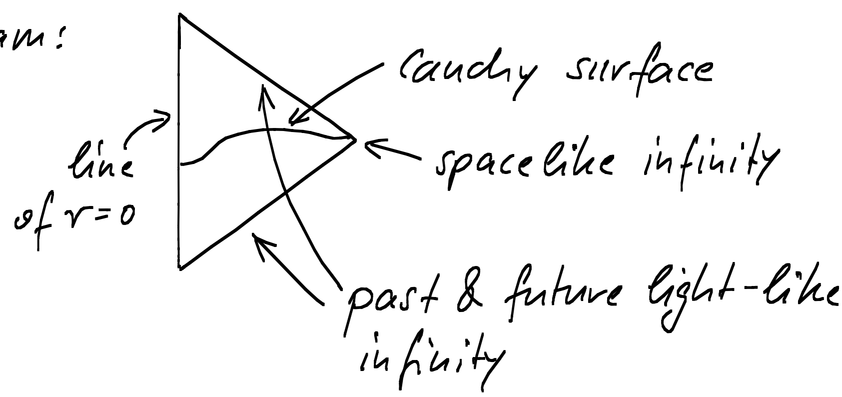


in the future of the Cauchy surface:

- any outgoing light-ray intersects any incoming light ray from larger r at Cauchy surface

in the past of the Cauchy surface: analogously

⇒ Penrose diagram:



• <u>FRW-universe</u>

$$ds^2 = -dt^2 + t^{2q}\left(dr^2 + \underbrace{r^2 d\Omega^2}\right)$$

again, we suppress the S^2's

(We had specifically derived the cases $q = 2/3$ (matter domination) & $q = 1/2$ (radiation domination).)

- parameter range: $t \in (0, \infty)$; $r = (0, \infty)$
  ↑ curvature singularity at $t = 0$

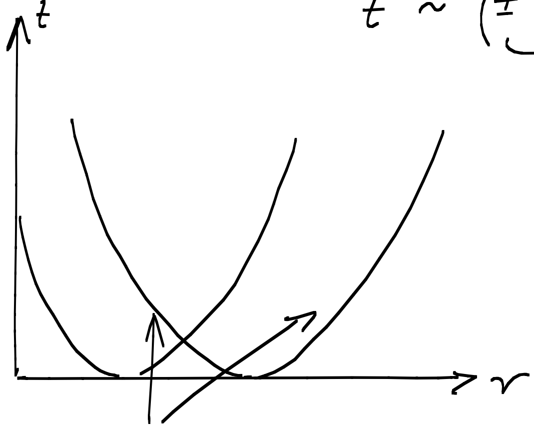
- light rays: $dt = \pm t^q dr$

$$t^{-q} dt = \pm dr$$

$$\frac{1}{1-q} t^{1-q} = \pm r + \text{const.}$$

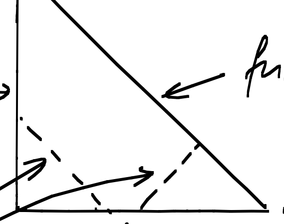
$$t \sim (\underbrace{\pm r + \text{const.}}_{\text{in the range where this is positive}})^{1/(1-q)}$$

(for $q = 1/2$, i.e. $t \sim (\pm r + \text{const.})^2$)



(The intersections are the same as for the part of Minkowski space above the Cauchy surface analyzed in the previous example.)

⇒ Penrose diagram:

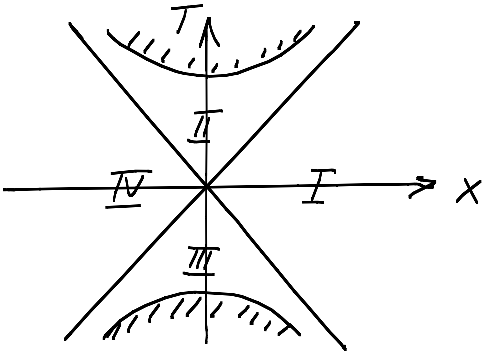


These two light-rays have never met in the past! ⇒ famous homogeneity problem of universe.

9.3 Penrose diagram of black hole

- For us, this important example is almost trivial since we already have a coordinate system where lightrays are 45° - lines:

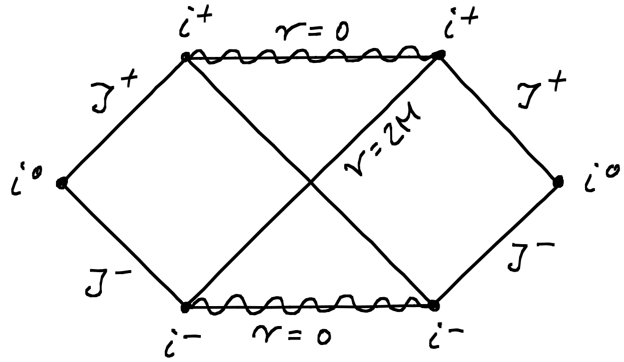
$$ds^2 = \frac{32M^3}{r} e^{-r/2M}(-dT^2 + dX^2) + \underbrace{r^2 d\Omega^2}_{\text{to be suppressed}}$$



We just need to map this to a finite domain. The intersection structure of light rays is already obvious.

In particular: - The "infinity part" of regions I & IV is just like that of Minkowski space in spherical coordinates
- All lightrays in regions II / III go to / come from the singularity

⇒ Penrose diagram:



- We use this example to introduce the standard notation:

i^+ = futere timelike infinity
i^0 = spatial infinity
i^- = past timelike infinity
$\mathcal{J}^+$ = future null infinity
$\mathcal{J}^-$ = past null infinity

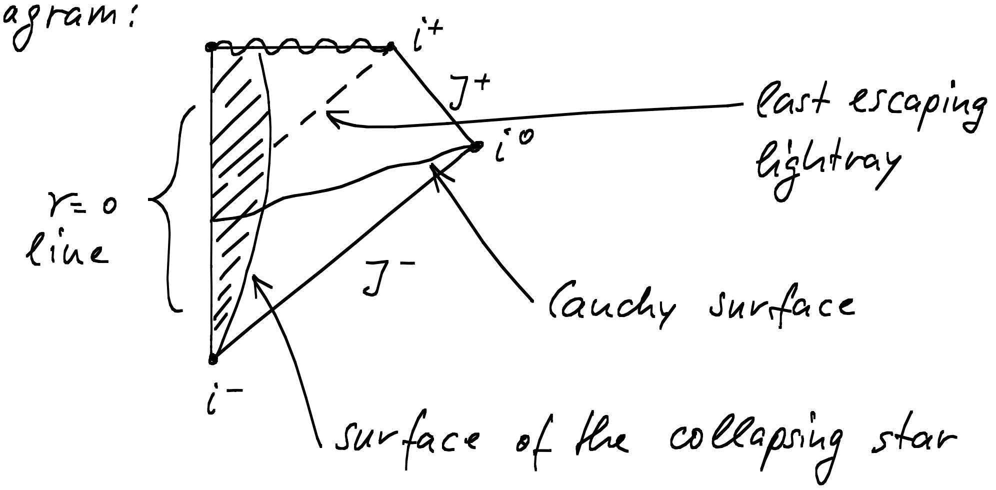
also: (wavy line) = singularity

- Another important example is that of the <u>spherical collapse of matter creating a black hole.</u>

- We choose spherical coordinates and ignore the S^2's.
- We choose the Cauchy surface before the collapsing matter significantly changes the structure of space-time.
- This implies that the past of the Cauchy surface will have the same structure as in Minkowski space.

- In the future of the Cauchy surface the line $r=0$ will end in the singularity.
- There will be a last outgoing lightray from $r=0$ which can escape the singularity.
- There will be a last incoming lightray hitting $r=0$ before the singularity. All following incoming lightrays will end in the singularity.

This information is sufficient to draw the following Penrose diagram:



9.4 Reissner-Nordström black holes
(= charged black holes)

- We can still have static, spherically symm. solutions (same ansatz for ds^2 as in Schwarzschild case).

- We now need to solve $M^2 G_{\mu\nu} = T_{\mu\nu}$

  ↑ energy-mom. tensor of spheric. symm. field config. of electrodynamics

- At the same time, the el.magn. field solves Maxwell's eqs. without sources (covariant!)

- We can assume either electric or magnetic charge for the black hole (or both).

- The charge Q is fixed by applying Gauss' law to $\bar{E}$ (or, completely analogously, the magnetic charge P is related in the same way to $\bar{B}$)

- The result (→ problems) reads

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2$$

with $\Delta = 1 - \frac{2M}{r} + \frac{Q^2+P^2}{r^2}$

and $\text{"}E_1 = E_r\text{"} = F_{rt} = \frac{Q}{r^2}$ ← This is, of course, what follows from Gauss' law in flat space, but it is, in fact, also true for the BH (in these coordinates).

$$\text{"}B_1 = B_r\text{"} = \frac{F_{\theta\varphi}}{r^2 \sin\theta} = \frac{P}{r^2}$$

(Recall: $\{x^\mu\} = \{t, r, \theta, \varphi\}$)

↑ to understand that these are the quantities to be integrated over the S^2, recall that

$$d * F = j_{el.}$$
$$dF = j_{mag.}$$

and apply Stokes' theorem (in forms) using also stationarity.

<u>Comment:</u> The spherically symmetric form of F for a magnetic charge in the centre can be derived as follows:

$$dF = j_{mag.}$$

$$\int_V dF = \int_V j_{mag.} = 4\pi P \leftarrow \text{mag. charge}$$

$$\int_{S^2(r)} F = 4\pi P \qquad \left[ds^2 = r^2 d\Omega^2 = r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \right]$$

spherical symmetry $\Rightarrow$ $F_{ij} = A \cdot \sqrt{g_{S^2(r)}}\, \varepsilon_{ij}$; $i,j \in \{2,3\}$

$$F_{ij} = A \cdot r^2 \sin\theta\, \varepsilon_{ij} = A r^2 \cdot \underbrace{Vol(S^2(1))}_{\text{volume form of unit } S^2}$$

$$\int_{S^2(r)} F = A r^2 \cdot 4\pi = 4\pi P$$

$$\Rightarrow A = P/r^2 \quad \Rightarrow \quad F_{23} = F_{\theta\varphi} = -F_{\varphi\theta} = \underline{\underline{P \cdot \sin\theta}}$$

(an analogous discussion applies to the electric case)

- We now continue with the analysis of the metric

$$ds^2 = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2$$

$$\text{with } \Delta = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

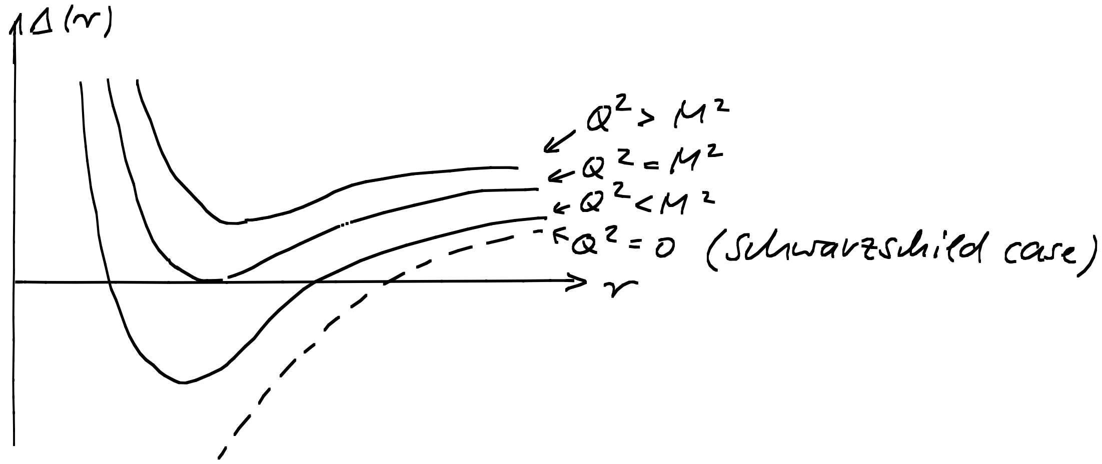
(We suppress the P^2 of $Q^2 + P^2$ for brevity.)

- $\Delta \to +\infty$ for $r \to 0$ & $\Delta \to 1$ for $r \to \infty$
- extremum of Δ: $\Delta' = \frac{2M}{r^2} - \frac{2Q^2}{r^3} \Rightarrow r_0 = Q^2/M$

$$\Delta(r_0) = 1 - M^2/Q^2$$

$\Rightarrow$ $\Delta(r_0) > 0$ for $M^2 < Q^2$, $\ldots = 0$ for $M^2 = Q^2$, $\ldots < 0$ for $M^2 > Q^2$

$\Rightarrow$



① $M^2 < Q^2$ (unphysical!)

- Δ is always positive; the metric defining Minkowski space at infinity is well-defined all the way to $r = 0$, where one finds a curvature singularity.

- The singularity is timelike (in contrast to the Schwarzschild case, where the sign change of $f(r)$ exchanges the roles of t & r).

- It is a <u>naked singularity</u> (not hidden behind a horizon).

- Such singularities are forbidden by the

  <u>Cosmic censor conjecture</u>:

  Complete gravitational collapse always leads to a black hole rather than a naked singularity; i.e., all singularities that are created dynamically are hidden behind a horizon and can not be seen by distant observers.

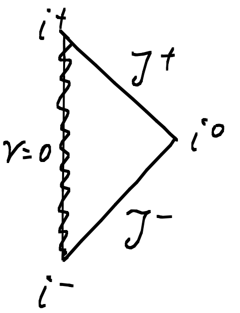
  [A more precise mathematical formulation can be given. It requires the specification of which initial data for the collapse is allowed. However, the proof remains an unresolved issue.]

- Rough argument why such black holes are impossible:

- In the collapse of charged matter, both mass and electric field of the charges contribute to the final mass. To achieve $M^2 < Q^2$, one would need to start with matter with negative mass.

- A more precise but less general argument can be given as follows:
  - Start with a black hole with $Q^2 < M^2$ and keep adding charge until $Q^2 > M^2$. This turns out to be impossible since by adding charged particles (even with very small mass) one always also adds energy and hence M^2 also grows, avoiding the creation of black holes with $Q^2 > M^2$. (→ problems for detailed calculation)

- conf. diagram:



(just like Minkowski space, but with the line $r=0$ replaced by a singularity)

② $M^2 > Q^2$

- $\Delta(r)$ has two zeros:

$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 0$$

$$r^2 - 2Mr + Q^2 = 0$$

$$r_\pm = M \pm \sqrt{M^2 - Q^2}$$

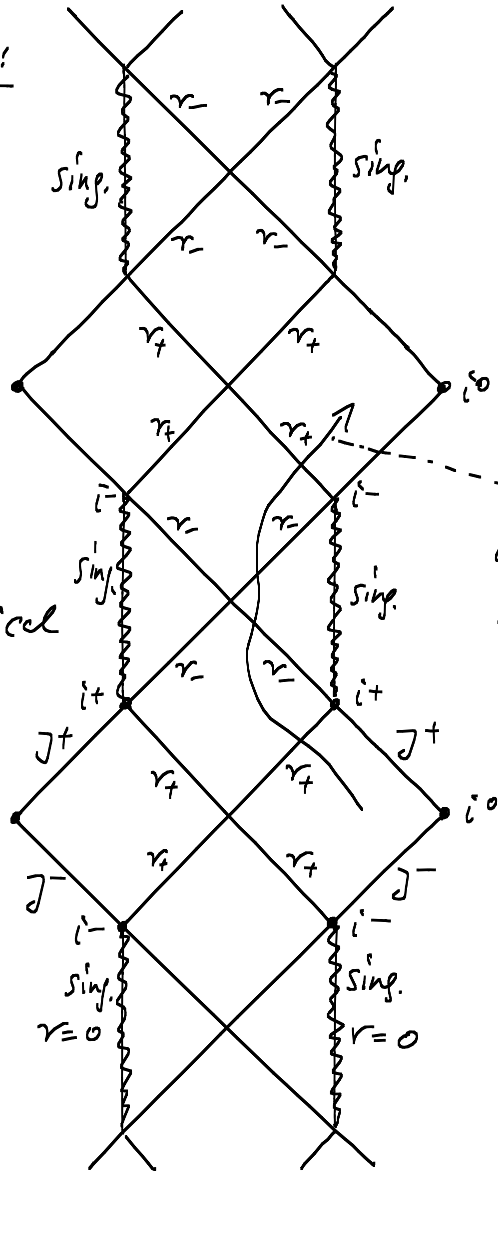
- At r_+, one finds the same behaviour as at the Schwarzschild horizon:
  - Objects can fall through but can never come back.
  - The outside observer sees though objects approaching the horizon forever, but they get more and more redshifted

- After passing r_+, objects must keep falling to smaller r (r is now the time coordinate, so smaller r are in their future)

- After passing r_-, r & t trade roles again. The singularity at $r=0$ is timelike and <u>not</u> in the future of the falling object. There is a choice of going to $r=0$ or return to r_-.

- If an object or observer returns and passes r_- again, he finds himself in the same situation as an observer escaping from a white hole: he <u>must</u> go to r_+ and will be ejected in the outside space. However, this is a different space than the one he came from:

- <u>Penrose diagram:</u>
  (without proof)



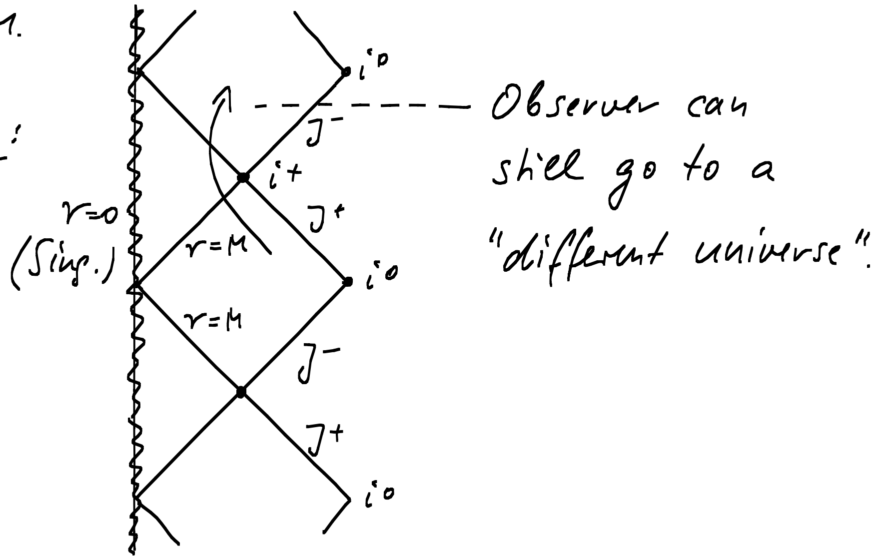
Observer falling into a Reissner-Nordström black hole and emerging a different exterior space.

<u>Note:</u>
This exciting story is probably unphysical because small non-spherical perturbations will strongly affect the geometry.

③ $M^2 = Q^2$ (extremal black hole)

$$\Delta = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = 1 - \frac{2M}{r} + \frac{M^2}{r^2}$$ has only one zero at $r = M$.

Penrose diagram:



- This case is particularly interesting since two extremal black holes are attracted by their mass and repelled by their charge in such a way that the net force is precisely zero.

- The metric can be rewritten as $(\rho = r - M)$

$$ds^2 = -H^{-2}(\rho)\,dt^2 + H^2(\rho)\underbrace{\left(d\rho^2 + \rho^2 d\Omega^2\right)}_{\text{flat 3d metric}}$$

with $H(\rho) = 1 + \frac{M}{\rho}$.

- This solution generalizes to several extremal black holes with $Q_a = M_a$ by simply writing

$$ds^2 = -H^{-2}(\bar{x})\,dt^2 + H^2(\bar{x})\,d\bar{x}^2$$

$$\text{with} \quad H = 1 + \sum_a \frac{M_a}{|\bar{x} - \bar{x}_a|}.$$

- In supersymmetric theories such black holes play a crucial role since they leave a certain part of the supersymmetry unbroken. They are known as <u>BPS black holes</u>.

(After related extremal solutions in gauge theories discussed by Bogomol'nyi, Prasad and Sommerfield.)

- These BPS black holes can be generalized to <u>BPS black branes</u> in supergravity or string theory (instead of point-like objects, extended only in time, they now fill some of the spatial dimensions). In this framework, they form one of the possible descriptions of <u>D-branes</u> (which are behind the so-called "2nd superstring revolution").

<u>9.5 Kerr black holes</u> (rotating black holes)

- For a rotating black hole, the spherical symmetry is broken.
- The Kerr metric reads

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{2Mar\sin^2\theta}{\rho^2}(dt\,d\varphi + d\varphi\,dt)$$

$$+ \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\cdot\Delta\cdot\sin^2\theta\right]d\varphi^2$$

with

$$\Delta(r) = r^2 - 2Mr + a^2$$

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta$$

$$a = J/M$$

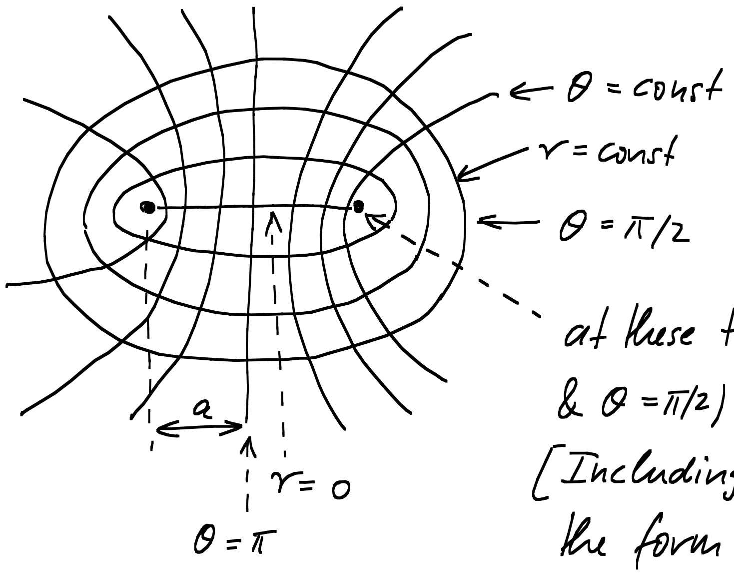
↑ GR-generalization of angular momentum ("Komar angular momentum").

- Note that the metric is stationary but not static!

- It is possible to include charge by

$$2Mr \quad \rightarrow \quad 2Mr - (Q^2+P^2) \quad [\text{Kerr-Newman metric}].$$

- Graphical representation of inner region (r-θ-plane)



∂_t & ∂_φ are Killing vector fields

at these two points (where $r=0$ & $\theta = \pi/2$) there is a curv. singularity. [Including φ, this singularity has the form of a ring.]

- Horizons: $\quad \Delta = 0 \quad \Rightarrow \quad r^2 - 2Mr + a^2 = 0$

$$r_\pm = M \pm \sqrt{M^2 - a^2} \qquad (M^2 > a^2)$$

- There is a further subtlety here related to the fact that the metric is stationary but not static. To understand this, it is useful to introduce the notion of a

- <u>Killing horizon:</u>

  If a Killing vector field ξ^μ is null along some null hypersurface Σ, one says that <u>Σ is a Killing horizon of ξ^μ.</u>

  (<u>Recall:</u> A hypersurface is called null if its normal vector is null.)

- Fact: Every event horizon in a stationary, asymptotically flat spacetime is a Killing horiz. for some Kill. vect. ξ^μ.

(An <u>event horizon</u> is, roughly speaking, a surface from behind which you can not return, once you passed it)

- In the static case, the event horizon is at the same time the Killing horizon of $\xi^\mu = (\partial/\partial t)^\mu$.

- The physical importance of Killing horizons is obvious:

  As long as ξ^μ is timelike, one can have observers moving with constant velocity parallel to ξ^μ. Such observers are "stationary with respect to that Killing field". If $\xi^\mu \xi_\mu$ changes sign at a Killing horizon, no such observers are possible beyond that surface.

- For the Kerr solution, the situation is more tricky:

  The event horizons are Killing horizons with respect to some linear combination of ∂_t & ∂_φ. The surface where $\xi^\mu \xi_\mu$ vanishes for $\xi^\mu = (\partial_t)^\mu$ is not null and hence not a Killing horizon. It is called the

<u>stationary limit surface</u>

(beyond that surface, no observers can exist which are stationary with respect to $\xi^\mu = (\partial_t)^\mu$, i.e. stationary with respect to the <u>asymptotically time-translation Killing vector field</u>, or simply "stationary" for short.)

- To be explicit, the stationary limit surface is defined by ($\xi^\mu = (\partial_t)^\mu$)

$$0 = \xi^\mu \xi_\mu = g_{\mu\nu} \xi^\mu \xi^\nu = g_{tt} = -\left(1 - \frac{2Mr}{r^2 + a^2 \cos^2\theta}\right)$$

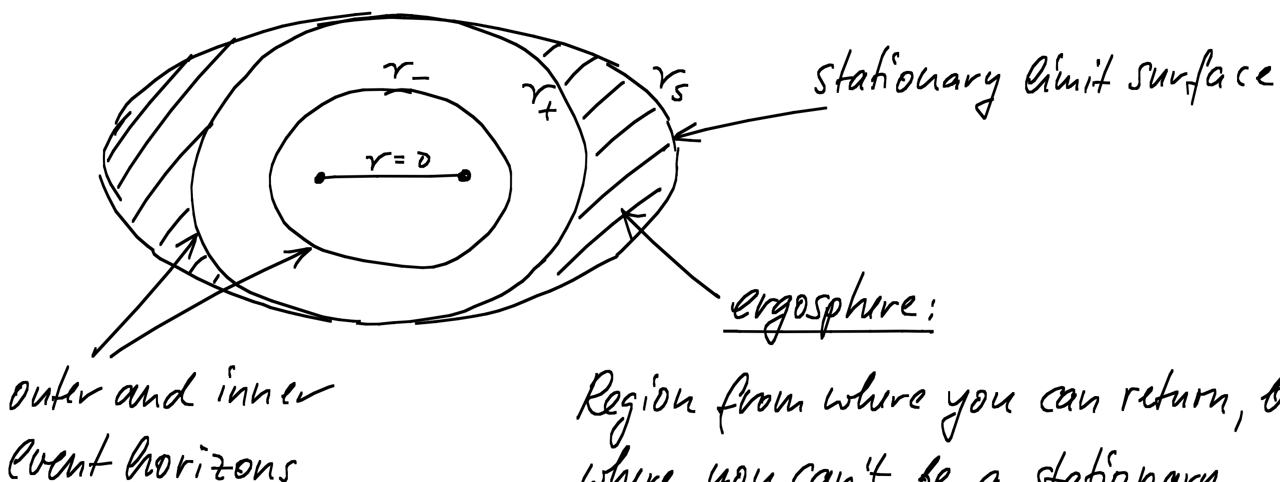
or $2r = r^2 + a^2 \cos^2\theta$

or $$\boxed{(r_s - M)^2 = M^2 - a^2 \cos^2\theta}$$

- The outer event horizon can similarly be defined by

$$\boxed{(r_+ - M)^2 = M^2 - a^2}$$

$\Rightarrow$ These surfaces touch at $\theta = 0$ & $\theta = \pi$:



Region from where you can return, but where you can't be a stationary observer (you are forced to move along the φ-direction with the rotating black hole).

- Note:
  - As in the Reissner-Nordström case, there are the extremal case ($M^2 = a^2$) and the unphysical case with a naked singularity ($M^2 < a^2$).
  - As in the Reissner-Nordström case, the singularity is timelike and one can go to different outer regions avoiding the singularity.