Problems

for the course

Beyond the Standard Model and the String Theory Landscape

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1 Electroweak symmetry breaking

Task: Calculate W and Z boson masses as well as the electromagnetic coupling e in terms of v and $g_{1,2}$. Derive the formula for the electric charge $Q = T_3 + Y$, as a function of hypercharge Y and isospin T_3.

Hints: Apply the covariant derivative (for uncolored fields)

$$D_\mu = \partial_\mu - ig_2 A^a_\mu R(T^a) - ig_1 R(Y) B_\mu \tag{1.1}$$

to the Higgs VEV to derive the mass terms for $W^\pm$ and Z. Identify the massless field (the linear combination orthogonal to the massive vectors) as the photon and express the covariant derivative in terms of these fields.

Solution: The Higgs transforms in the fundamental representation of $SU(2)$, hence $R(T^a) = \sigma^a/2$. It has hypercharge $1/2$, hence $R(Y) = 1/2$. It is convenient to work with $W^\pm = (A^1 \mp iA^2)/\sqrt{2}$. Then one has

$$A^1\sigma^1 + A^2\sigma^2 \;=\; (A^1 + iA^2)(\sigma^1 - i\sigma^2)/2 + (A^1 - iA^2)(\sigma^1 + i\sigma^2)/2 \;=\; \sqrt{2}W^-\sigma^- + \sqrt{2}W^+\sigma^+\,, \tag{1.2}$$

where

$$\sigma^+ = (\sigma^1 + i\sigma^2)/2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- = (\sigma^1 - i\sigma^2)/2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{1.3}$$

In the symmetry-broken vacuum, one then finds:

$$D_\mu H = D_\mu \begin{pmatrix} 0 \\ v \end{pmatrix} = -\frac{i}{2}\left(g_2\sqrt{2}W^+_\mu \begin{pmatrix} v \\ 0 \end{pmatrix} + \left[-g_2 A^3_\mu + g_1 B_\mu\right] \begin{pmatrix} 0 \\ v \end{pmatrix}\right). \tag{1.4}$$

This gives rise to the mass term

$$\mathcal{L} \supset |D_\mu H|^2 = \frac{v^2}{4}\left(2g_2^2|W^+_\mu|^2 + (g_1^2 + g_2^2)(Z_\mu)^2\right). \tag{1.5}$$

We have to recall that $W^- = (W^+)*$ and the complex W boson is normalized like a complex scalar field, i.e. without a factor $1/2$ in kinetic and mass term. Moreover, we introduced the canonically normalized massive vector

$$Z_\mu = \frac{1}{\sqrt{g_1^2 + g_2^2}}(g_2 A^3_\mu - g_1 B_\mu)\,. \tag{1.6}$$

Thus, the mass term ris

$$\mathcal{L} \supset m_W^2 |W_\mu^+|^2 + \frac{1}{2} m_Z^2 (B_\mu)^2 \,, \tag{1.7}$$

from which we read off

$$m_W = g_2 v/\sqrt{2} \qquad \text{and} \qquad m_Z = \sqrt{g_1^2 + g_2^2} \cdot v/\sqrt{2}\,. \tag{1.8}$$

Next, we note that the linear combination of A^3 and B orthogonal to Z is

$$A_\mu = \frac{1}{\sqrt{g_1^2 + g_2^2}} (g_1 A_\mu^3 + g_2 B_\mu)\,. \tag{1.9}$$

It is then immediate to express A^3 and B through Z and A:

$$A_\mu^3 = \frac{1}{\sqrt{g_1^2 + g_2^2}} (g_1 A_\mu + g_2 Z_\mu) \qquad \text{and} \qquad B_\mu = \frac{1}{\sqrt{g_1^2 + g_2^2}} (g_2 A_\mu - g_1 Z_\mu)\,. \tag{1.10}$$

Now the covariant derivative for a general field takes the form

$$\begin{aligned} D_\mu &= \partial_\mu - i g_2 \sqrt{2} (W_\mu^+ R(T^+) + W_\mu^- R(T^-)) && (1.11)\\ &\quad - \frac{i}{\sqrt{g_1^2 + g_2^2}} Z_\mu (g_2^2 R(T^3) - g_1^2 R(Y)) - \frac{i g_1 g_2}{\sqrt{g_1^2 + g_2^2}} A_\mu (R(T^3) + R(Y))\,. && (1.12) \end{aligned}$$

It is clear that the transition between A^3, B and Z, A may be interpreted as an SO(2) rotation with a **weak mixing angle** or Weinberg angle θ_W defined by

$$\sin\theta_W = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}\,. \tag{1.13}$$

In terms of this angle, the electromagnetic charge (i.e. the prefactor of A_μ in the covariant derivative) is given by $e = g_2 \sin\theta_W$. The group-theoretic coefficient is $Q = R(T^3) + R(Y)$. One often keeps the necessary use of the appropriate representation implicit, writing simply $Q = T_3 + Y$.

2 The Standard Model is anomaly free

Task: Confirm this statement.

Hints: Famously, in a theory with a single l.h. fermion ψ (or, equivalently, a single Weyl fermion), the anomalous current non-conservation for

$$j_\mu \equiv \overline{\psi} \gamma_\mu \psi \tag{2.1}$$

reads

$$\partial_\mu j^\mu = -\frac{1}{32\pi^2} F\tilde{F}\,. \tag{2.2}$$

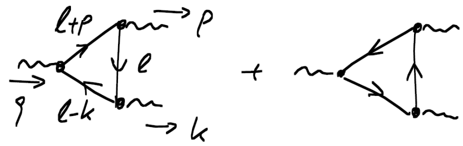

Figure 1: Scattering amplitude interpretation of the expectation value of the axial current. The momentum q is related by Fourier transformation to the argument x of $j^\mu(x)$. In (2.3), x has been set to zero.

A classical way to derive this is to consider the corresponding amplitude relation

$$\langle p,k|\partial_\mu j^\mu(0)|0\rangle = -\frac{1}{32\pi^2}\,\langle p,k|\epsilon^{\alpha\nu\beta\rho}F_{\alpha\nu}F_{\beta\rho}(0)|0\rangle\,. \tag{2.3}$$

Here $\langle p,k|$ stands for a final state with two outgoing gauge bosons with momenta p and k. The l.h. side of this equality is evaluated according to the diagrams in Fig. 1, and the r.h. side simply by expaning the fields in terms of creation and annihilation operators.

Given this diagrammatic understanding, it is very easy to see what the right generalization to the non-abelian case is: At each vertex, the abelian gauge group generator '1' has to be replaced by the corresponding non-abelian generator $(T_a)_{ij}$. As a result, one has

$$\partial_\mu j^\mu_a = -\frac{1}{32\pi^2}\,D_{abc}\epsilon^{\mu\nu\rho\sigma}F^b_{\mu\nu}F^c_{\rho\sigma} \quad \text{with} \quad D_{abc} \equiv \frac{1}{2}\mathrm{tr}[T_a\{T_b,T_c\}]\,. \tag{2.4}$$

It should now be clear how to proceed: Consider the Standard Model fermions as one l.h. fermion field ψ (as above) and let T_a run over all generators of G_{SM}. Clearly, one has to take care of the highly-non-trivial reperesentation in which T_a has to live. But a lot of this is repetitive and can be simplified. For instance, the threefold repetition due to the three generations can be dropped – even a single generation is anomaly free. Furthermore, rather than thinking about a complicated block-diagonal T_a, one can just sum over the different corresponding fermions in the loop. Finally, we clearly only need to show that $D_{abc} = 0$ for all possible different assignements of a, b and c to the factor groups $SU(3)$, $SU(2)$ and $U(1)$. Which particular generator of e.g. $SU(3)$ one choses is immaterial. As a result, the amount of work is actually rather limited.

Solution: As explained above, we need to go through all possible ways to assign the three generators corresponding the three vertices of the triangle to the factors of G_{SM}. Thus, symbolically, we have to consider

$$U(1)^3\,, \qquad U(1)^2\,SU(2)\,, \qquad U(1)\,SU(2)^2\,, \qquad U(1)\,SU(2)\,SU(3)\,,\cdots \tag{2.5}$$

and so on. But the generators of $SU(N)$ groups are all traceless, such that e.g. in the $U(1)^2\,SU(2)$ case we have (for each fermion species or, equivelently, each block)

$$\mathrm{tr}[T^2_{U(1)}T^A_{SU(2)}] = \mathrm{tr}[T^2_{U(1)}]\,\mathrm{tr}[T^A_{SU(2)}] = 0\,. \tag{2.6}$$

Thus, we only need to consider combinations where all three generators come from the same factor or where two come from the same factor and the third from the $U(1)$:

$$U(1)^3\,, \qquad U(1)\,SU(2)^2\,, \qquad U(1)\,SU(3)^2\,, \qquad SU(2)^3\,, \qquad SU(3)^3\,. \tag{2.7}$$

Now let us go through this case by case. In the first case, we simply have to sum the cubes of the charges of all fermions. The anti-commutator is, of course, irrelevant. Using the list at the beginning of Sect. 1.1 of the notes, this gives

$$\begin{aligned} 3\times 2\times \left(\tfrac{1}{6}\right)^3 + 3\times\left(-\tfrac{2}{3}\right)^3 + 3\times\left(\tfrac{1}{3}\right)^3 + 2\times\left(-\tfrac{1}{2}\right)^3 + (1)^3 \\ = \tfrac{1}{36} - \tfrac{8}{9} + \tfrac{1}{9} - \tfrac{1}{4} + 1 = 0\,. \end{aligned} \tag{2.8}$$

Note that the $SU(3)$ and $SU(2)$ representations are only relevant to determine the multiplicities corresponding to each set of fermions.

In the second case, the anti-commutator is again irrelevant. Indeed,

$$\mathrm{tr}[T_{U(1)}\{T^A_{SU(2)}, T^B_{SU(2)}\}] = 2\,\mathrm{tr}[T_{U(1)}]\,\mathrm{tr}[T^A_{SU(2)}T^B_{SU(2)}]\,, \tag{2.9}$$

and

$$\mathrm{tr}[T^A_{SU(2)}\{T_{U(1)}, T^B_{SU(2)}\}] = 2\,\mathrm{tr}[T_{U(1)}]\,\mathrm{tr}[T^A_{SU(2)}T^B_{SU(2)}]\,, \tag{2.10}$$

Since the $SU(2)$-trace always gives $\delta^{AB}/2$, we just need to sum the $U(1)$ charges of all $SU(2)$ doublets:

$$3\times\frac{1}{6} + 0 + 0 - \frac{1}{2} + 0 = 0\,. \tag{2.11}$$

The third case is analogous: We have to sum over the $U(1)$ charges of all $SU(3)$ triplets. (It does not matter whether it is a triplet or anti-triplet since $\mathrm{tr}[T^A_{SU(3)}T^B_{SU(3)}] = \delta^{AB}/2$ holds for both). This gives

$$2\times\frac{1}{6} - \frac{2}{3} + \frac{1}{3} + 0 + 0 = 0\,. \tag{2.12}$$

In the fourth case we have $T^A_{SU(2)} = \sigma^A/2$ and hence

$$\mathrm{tr}[T^A_{SU(2)}\{T^B_{SU(2)}, T^C_{SU(2)}\}] = \frac{1}{8}\mathrm{tr}[\sigma^A\{\sigma^B,\sigma^C\}] = \frac{1}{8}\mathrm{tr}[\sigma^A]\,2\delta^{BC} = 0\,. \tag{2.13}$$

Thus, we see that any theory with only fundamental representations (the antifundamental is equivalent to the fundamental) of $SU(2)$ is *trivially* free of the triangle anomaly. In fact, this extends to *all* representations of $SU(2)$ due to the reality-properties of its representations.

Finally, the fifth and last case is the only one where we need to take into consideration that different representations of the same non-abelian group appear. We write

$$T^A_{SU(3),\,fund.} = T^A_3 \qquad \text{and} \qquad T^B_{SU(3),\,anti-fund.} = T^A_{\bar{3}}\,. \tag{2.14}$$

Now, since for a fundamental field Φ we have

$$\Phi \;\to\; \exp(i\epsilon T)\,\Phi \qquad \text{and} \qquad \Phi^* \;\to\; \exp(-i\epsilon T^*)\,\Phi^* \;=\; \exp(-i\epsilon T^T)\,\Phi^*\,, \tag{2.15}$$

we can conclude that

$$T^A_{\bar{3}} = -(T^A_3)^T\,. \tag{2.16}$$

As a result, we find

$$\mathrm{tr}[T^A_{\bar{3}}\{T^B_{\bar{3}}, T^C_{\bar{3}}\}] = \mathrm{tr}[(-T^A_3)^T\{(-T^B_3)^T, (-T^C_3)^T\}] = -\mathrm{tr}[T^A_3\{T^B_3, T^C_3\}]\,. \tag{2.17}$$

Thus, we have to add the $SU(3)$-triplets and subtract the anti-triplets, each with its multiplicity:

$$2 - 1 - 1 = 0 \,. \tag{2.18}$$

We finally note that triangle anomalies (as considered above) involving different gauge group factors are called ‘mixed’. Without going into details, we also record the fact that a so-called mixed $U(1)$-gravitational anomaly exists. It comes from a triangle diagram involving one gauge-boson and two gravitons. To allow for a consistent coupling of the Standard Model to gravity, this anomaly also has to vanish. The calculation is similar to the $U(1)SU(2)^2$ and the $U(1)SU(3)^2$ case. Since all fermions couple to gravity in the same way, we simply have to add all $U(1)$ charges:

$$6 \times \frac{1}{6} - 3 \times \frac{2}{3} + 3 \times \frac{1}{3} - 2 \times \frac{1}{2} + 1 = 0 \,. \tag{2.19}$$

3 The Standard Model and $SU(5)$

Task: Embed G_{SM} in a natural way in $SU(5)$ and show that the matter content of one generation (with all its gauge charges) follows from the $\mathbf{10} + \overline{\mathbf{5}}$ of $SU(5)$, where $\mathbf{10}$ stands for the antisymmetric second rank tensor and $\overline{\mathbf{5}}$ for the antifundamental representation. Derive the tree-level prediction for the relative strength of the three Standard Model gauge couplings.

Hints: The ‘natural embedding’ corresponds, of course, to identifying the upper-left 3×3 block of 5×5 $SU(5)$ matrices with $SU(3)$ and the lower-right 2×2 block with $SU(2)$. The inverse would be equivalent - this is merely a convention. Hence, when viewed as generators of $SU(5)$, the $SU(3)$, $SU(2)$ and $U(1)$ generators are

$$\begin{pmatrix} \left(T^a_{SU(3)}\right)_{3\times 3} & 0_{3\times 2} \\ 0_{2\times 3} & 0_{2\times 2} \end{pmatrix} , \quad \begin{pmatrix} 0_{3\times 3} & 0_{3\times 2} \\ 0_{2\times 3} & \left(T^a_{SU(2)}\right)_{2\times 2} \end{pmatrix} , \quad \frac{1}{\sqrt{60}} \begin{pmatrix} -2 & & & & \\ & -2 & & & \\ & & -2 & & \\ & & & 3 & \\ & & & & 3 \end{pmatrix} . \tag{3.1}$$

The prefactor of the $U(1)$ generator ensures the standard non-abelian normalization $\mathrm{tr}(T^aT^b) = \delta^{ab}/2$. With this, it is immediate to write down the **branching rule**

$$\mathbf{5} = (\mathbf{3},\mathbf{1})_{-2} + (\mathbf{1},\mathbf{2})_3 \qquad \text{under} \qquad SU(5) \to SU(3) \times SU(2) \times U(1)\,. \tag{3.2}$$

Here we have rescaled the $U(1)$ generator in an obvious way for notational convenience. All one now needs to do is to infer the branching rules for the $\overline{\mathbf{5}}$ and $\mathbf{10}$ and to determine the gauge couplings g_i of the Standard Model in the normalization given in the lecture. (We note that, as is probably well-known, this unification scheme can not work without significant loop corrections.)

Solution: The branching rule for $\overline{\mathbf{5}}$ follows trivially from complex conjugation of the above:

$$\overline{\mathbf{5}} = (\overline{\mathbf{3}},\mathbf{1})_2 + (\mathbf{1},\mathbf{2})_{-3}\,. \tag{3.3}$$

Here we have used the fact that $\overline{\mathbf{2}} = \mathbf{2}$ for $SU(2)$. This is obvious since $Lie(SU(2)) = Lie(SO(3))$ and since, as derived in quantum mechanics, $SO(3)$ has a unique 2-dimensional representation. It can also be demonstrated explicitly by showing that, if

$$\psi_i \to U_{ij}\psi_j \,, \qquad \text{and} \qquad \psi_i^* \to\to U_{ij}^*\psi_i^* \,, \tag{3.4}$$

then the field $\chi_i \equiv \epsilon_{ij}\psi_j^*$ transforms exactly as ψ_i. We leave that to the reader.

Formally speaking, we are claiming that the two representations **2** and its complex conjugate, $\overline{\mathbf{2}}$, are equivalent. This implies an isomorphism between the two vector spaces which commutes with the group action. In our case, the isomorphism is the multiplication with ϵ. We will see a less trivial example of this below, which we will work out and after which it will be even more clear how to finish the $SU(2)$ discussion.

At this point, just looking at the pure $SU(2)$ doublet (there is only one such field in the Standard Model!), we can already identify the $U(1)$ charges with those of the Standard Model. We have the covariant derivative as it follows from the GUT:

$$D_\mu = \partial_\mu - igT^a_{SU(2)}(A_2)^a_\mu - igY_{GUT}(A_1)_\mu \,. \tag{3.5}$$

According to the above,

$$Y_{GUT} = \frac{-3}{\sqrt{60}} \,. \tag{3.6}$$

On the Standard Model side, we have

$$D_\mu = \partial_\mu - ig_2T^a_{SU(2)}(A_2)^a_\mu - ig_Y\, Y\, (A_1)_\mu \tag{3.7}$$

with

$$Y = -1/2 \tag{3.8}$$

for the pure doublet (the lepton doublet). Thus, we learn that

$$g\, Y_{GUT} = g_Y\, Y \qquad \text{or} \qquad \frac{g_Y^2}{g^2} = \frac{3}{5} \,. \tag{3.9}$$

This is the famous normalization change between the Standard Model hypercharge $U(1)$ and the $SU(5)$-normalized $U(1)$. Note that we call the Standard Model gauge couplings g_Y, g_2 and g_3 at this point since, very frequently, the name g_1 is reserved for the hypercharge coupling in GUT normalization, i.e. $g_1 = \sqrt{5/3}g_Y$.

We also see that the down-type r.h. quarks have the correct charge to be the $SU(3)$ anti-triplet coming with this $SU(2)$ doublet. (Their hypercharge differs by a factor -2/3, as it follows from $SU(5)$.)

As for the numerical outcome, we have the GUT prediction that $g_1 = g_2 = g_3$ at the GUT scale. This has to be compared to the observed values of roughly

$$\alpha_1^{-1} \simeq 60\,, \qquad \alpha_2^{-1} \simeq 30\,, \qquad \alpha_3^{-1} \simeq 8 \tag{3.10}$$

at the scale m_Z. Here the first two values follow from $\alpha_{em}^{-1} \simeq 127$, $e = g_2 \sin\theta_W$, $\sin^2\theta_W = g_Y^2/(g_Y^2 + g_2^2)$ and $\sin^2\theta_W \simeq 0.23$ together with $g_1 = \sqrt{5/3}g_Y$ as explained above. Thus, as already noted, significant loop corrections (most plausibly from running over a large energy range) are needed for this unification scheme to work.

Finally, three Standard Model fields are missing and we hope to get them from the **10**. To check this, let us first write down the tensor product

$$\mathbf{5} \times \mathbf{5} = [(\mathbf{3},\mathbf{1})_{-2} + (\mathbf{1},\mathbf{2})_3] \times [(\mathbf{3},\mathbf{1})_{-2} + (\mathbf{1},\mathbf{2})_3] \tag{3.11}$$

and anti-symmetrize:

$$(\mathbf{5}\times\mathbf{5})_A = ((\mathbf{3}\times\mathbf{3})_A, \mathbf{1})_{-4} + (\mathbf{1}, (\mathbf{2}\times\mathbf{2})_A)_6 + (\mathbf{3}, \mathbf{2})_1\,. \tag{3.12}$$

Here the last representation only appears once since the other, equivalent term belongs to the symmetric part of the rank-2 tensor. In giving the $U(1)$ charges we have, as before, suppressed the factor $1/\sqrt{60}$. We learned above that, to get the right Standard Model $U(1)_Y$ charges in this normalization, we need to divide by 6. Given that $(\mathbf{2}\times\mathbf{2})_A$ is clearly a singlet, we recognize the last two terms as r.h. electron and l.h. quark doublet. The first term should then be the r.h. up-type quark.

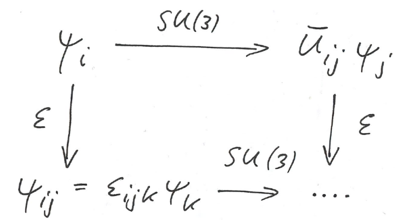

Figure 2: Commuting diagram demonstrating equivalence of representations.

All we need is to establish is that

$$(\mathbf{3}\times\mathbf{3})_A = \overline{\mathbf{3}}\,. \tag{3.13}$$

To do so, we first identify the vector spaces of antisymmetric $SU(3)$ tensor and (anti-)vector by

$$\psi_{ij} = \epsilon_{ijk}\psi_k\,. \tag{3.14}$$

Then we just need to show that they transform consistently, i.e., that the diagram in Fig. 2 commutes. This implies

$$U_{ik}U_{jl}\epsilon_{klm}\psi_m = \epsilon_{ijk}U^*_{km}\psi_m\,. \tag{3.15}$$

To verify this equality, we remove ψ_m and multiply by $(U^T)_{mn}$:

$$U_{ik}U_{jl}U_{nm}\epsilon_{klm} = \epsilon_{ijk}U^*_{km}(U^T)_{mn}\,. \tag{3.16}$$

On the r.h. side the two mutually inverse matrices cancel; the l.h. side is just the epsilon tensor multiplied by the determinant of U, the latter being unity. Thus, we are done.

4 Weyl spinors

Tasks:

(1) Define the canonical map $SL(2,\mathbb{C}) \to SO(1,3)$ using the vector of four sigma matrices $\sigma_\mu = (\mathbb{1}, \sigma_1, \sigma_2, \sigma_3)$. Then go on to show that $(\sigma_\mu)_{\alpha\dot{\alpha}}$ is an invariant tensor of the Lorentz group. Build the Dirac spinor and gamma-matrices from Weyl spinors and sigma matrices and express the transformation of a Dirac spinor under a Lorentz rotation in terms of a given $SL(2,\mathbb{C})$ matrix M.

(2) Rewrite the Dirac spinor invariants

$$\overline{\psi}_D^{(1)}\psi_D^{(2)} \equiv \psi_D^{(1)\dagger}\gamma_0\psi_D^{(2)}\;, \qquad \overline{\psi}_D^{(1)}\gamma_5\psi_D^{(2)}\;, \qquad \overline{\psi}_D^{(1)}\gamma_\mu\psi_D^{(2)}\;, \qquad \overline{\psi}_D^{(1)}\gamma_\mu\gamma_5\psi_D^{(2)} \tag{4.1}$$

in terms of Weyl spinors. Use the upper/lower and lower/upper index summation convention for undotted and dotted Weyl indices respectively:

$$\psi\chi \equiv \psi^\alpha\chi_\alpha\;, \qquad \overline{\psi}\overline{\chi} \equiv \overline{\psi}_{\dot\alpha}\overline{\chi}^{\dot\alpha}\,. \tag{4.2}$$

(3) Check the crucial identity

$$\overline{\sigma}_\mu\sigma_\nu + \overline{\sigma}_\nu\sigma_\mu = -2\eta_{\mu\nu}\mathbb{1} \tag{4.3}$$

and derive the Clifford algebra relation for the γ matrices from it.

Hints:

(1) The first part is a direct generalization of the construction of the map $SU(2) \to SO(3)$ which should be familiar from quantum mechanics. The second step is a straightforward calculation using only the fact that the indices α and $\dot\alpha$ transform with $SL(2,\mathbb{C})$ matrices and with complex conjugate $SL(2,\mathbb{C})$ matrices respectively. In the last step you need to use the convention that the lower two components of a Dirac spinor are given by a Weyl spinor with **upper** dotted index.

A convenient set of conventions is that of the Appendix of the book by Wess and Bagger, in particular

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\;, \qquad \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\;, \qquad \text{such that} \qquad \epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha{}^\gamma\,. \tag{4.4}$$

This allows us to raise and lower Weyl indices with the ϵ tensor. Of course one needs to use the fact – please check if not obvious – that ϵ is an invariant tensor of $SL(2,\mathbb{C}.)$

(2) This is completely straightforward. Deviating from the Wess-Bagger conventions, it may be convenient to define $\gamma^5 \sim \gamma^0\gamma^1\gamma^2\gamma^3$ with a prefactor which ensures that the l.h. projector (i.e. the projector on the undotted Weyl spinor) is $P_L = (\mathbb{1} - \gamma^5)/2$.

(3) Use that $\epsilon^{\alpha\beta} = -i(\sigma_2)^{\alpha\beta}$ together with the familiar commutation relations of the Pauli matrices.

Solution:

(1) Given a 4-vector v, define $\hat v \equiv v^\mu\sigma_\mu$. The matrix $\hat v$ is hermitian, as is the matrix

$$\hat v' = M\hat v M^\dagger\,, \tag{4.5}$$

where $M \in SL(2,\mathbb{C})$. Since the $\{\sigma_\mu\}$ is a basis of hermitian 2×2 matrices, there exists a unique decomposition

$$\hat v' = v'^\mu\sigma_\mu\,, \tag{4.6}$$

which defines the $SL(2,\mathbb{C})$-transformed vector v'. To see that this an $SO(1,3)$ transformation, it suffices to check that v^2 is preserved. This follows immediately from

$$\det\hat v = \begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv_2 & v^0 - v^3 \end{pmatrix} = (v^0)^2 - \overline{v}^2 = -v^2 \tag{4.7}$$

together with the obvious fact that the above $SL(2,\mathbb{C})$ transformation preserves the determinant.

With this, we are ready to check that $(\sigma_\mu)_{\alpha\dot\alpha}$ is an invariant tensor. To do so, let $M \in SL(2,\mathbb{C})$ and let $\Lambda \in SO(1,3)$ be its image under the map defined above. We have

$$\Lambda_\mu{}^\nu M_\alpha{}^\beta \overline{M}_{\dot\alpha}{}^{\dot\beta}(\sigma_\nu)_{\beta\dot\beta} = \Lambda_\mu{}^\nu (M\sigma_\nu M^\dagger)_{\alpha\dot\alpha}\,. \tag{4.8}$$

We also know that

$$M\sigma_\mu M^\dagger v^\mu = \sigma_\mu v'^\mu = \sigma_\mu \Lambda^\mu{}_\nu v^\nu\,. \tag{4.9}$$

for any v and hence

$$M\sigma_\mu M^\dagger = \sigma_\mu \Lambda^\mu{}_\nu\,. \tag{4.10}$$

With this, we return to (4.8) and continue the calculation according to

$$\Lambda_\mu{}^\nu M_\alpha{}^\beta \overline{M}_{\dot\alpha}{}^{\dot\beta}(\sigma_\nu)_{\beta\dot\beta} = \Lambda_\mu{}^\nu(\sigma_\rho)_{\alpha\dot\alpha}\Lambda^\rho{}_\nu = \eta_{\mu\sigma}\Lambda^\sigma{}_\tau\eta^{\tau\nu}\,(\sigma_\rho)_{\alpha\dot\alpha}\Lambda^\rho{}_\nu = \eta_{\mu\sigma}\eta^{\sigma\rho}(\sigma_\rho)_{\alpha\dot\alpha} = (\sigma_\mu)_{\alpha\dot\alpha}\,. \tag{4.11}$$

Thus, we are indeed dealing with an invariant tensor.

Finally, we take

$$\psi_D = \begin{pmatrix} \psi_\alpha \\ \overline{\chi}^{\dot\alpha} \end{pmatrix} \tag{4.12}$$

as a definition of a Dirac spinor. For covariance reasons (and up to possible convention-dependent prefactors, which are however usually not introduced), we have

$$\gamma_\mu = \begin{pmatrix} 0 & (\sigma_\mu)_{\alpha\dot\alpha} \\ (\overline{\sigma}_\mu)^{\dot\beta\beta} & 0 \end{pmatrix}\,. \tag{4.13}$$

The Lorentz transformation matrix is

$$D(M) = \begin{pmatrix} M_\alpha{}^\beta & 0 \\ 0 & \overline{M}^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}\,, \tag{4.14}$$

where the $\overline{M}^{\dot\alpha}{}_{\dot\beta}$ is obtained from $M_\alpha{}^\beta$ by complex conjugation and raising/lowering of the indices.

(2) Using our suggestion to define $\gamma^5 = \mathrm{diag}(-\mathbb{1},\mathbb{1})$, the result follows from the definitions:

$$\overline{\psi}^{(1)}_D\psi^{(2)}_D = \chi^{(1)}\psi^{(2)} + \overline{\psi}^{(1)}\overline{\chi}^{(2)}\,, \qquad \overline{\psi}^{(1)}_D\gamma_5\psi^{(2)}_D = -\chi^{(1)}\psi^{(2)} + \overline{\psi}^{(1)}\overline{\chi}^{(2)} \tag{4.15}$$

$$\overline{\psi}^{(1)}_D\gamma_\mu\psi^{(2)}_D = \overline{\psi}^{(1)}\overline{\sigma}_\mu\psi^{(2)} + \chi^{(1)}\sigma_\mu\overline{\chi}^{(2)}\,, \qquad \overline{\psi}^{(1)}_D\gamma_\mu\gamma_5\psi^{(2)}_D = -\overline{\psi}^{(1)}\overline{\sigma}_\mu\psi^{(2)} + \chi^{(1)}\sigma_\mu\overline{\chi}^{(2)}\,. \tag{4.16}$$

(3) Write

$$(\overline{\sigma}_\mu)^{\dot\alpha\alpha}(\sigma_\nu)_{\alpha\dot\beta} + \{\mu\leftrightarrow\nu\} \;=\; \epsilon^{\dot\alpha\dot\gamma}\epsilon^{\alpha\beta}(\overline{\sigma}_\mu)_{\dot\gamma\beta}(\sigma_\nu)_{\alpha\dot\beta} + \{\mu\leftrightarrow\nu\} \tag{4.17}$$

$$=\;[(-i\sigma_2)\overline{\sigma}_\mu(-i\sigma_2)^T(\sigma_\nu)]^{\dot\alpha}{}_{\dot\beta} + \{\mu\leftrightarrow\nu\} \;=\; [(\sigma_2)\overline{\sigma}_\mu(\sigma_2)(\sigma_\nu)]^{\dot\alpha}{}_{\dot\beta} + \{\mu\leftrightarrow\nu\}\,, \tag{4.18}$$

where in the last two expressions $\overline{\sigma}_\mu$ and σ_ν are assumed to have lower indices. Now use that

$$\sigma_2\overline{\sigma}_0\sigma_2 = \sigma_0 \qquad \text{and} \qquad \sigma_2\overline{\sigma}_i\sigma_2 = -\sigma_i\,. \tag{4.19}$$

Using this minus sign, it becomes clear that the expressions with $\{\mu\nu\} = \{0i\}$ and $\{\mu\nu\} = \{i0\}$ vanish after symmetrization. The case $\{\mu\nu\} = \{00\}$ obviously gives the right answer. For $\{\mu\nu\} = \{ij\}$ one needs to use $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$ to find the result. The Clifford algebra for γ matrices is a direct consequence.

5 Covariant expression for the 1-loop vacuum energy

Task: Derive the covariant expression ($\sim \int d^4k\, \ln(k^2+m^2)$) for the vacuum energy given in the lecture.

Hints: Write down the path integral for gravity and a real scalar and integrate out the scalar, including in particular its vacuum fluctuations. Focus only on the dependence on rescalings of the metric, i.e. on metrics of the form $g_{\mu\nu} = \alpha\eta_{\mu\nu}$.

Solution: The complete partition function (we suppress any source terms for simplicity) reads

$$Z = \int Dg\, D\phi\, \exp\left[i \int d^4x\, \sqrt{g}\left(\frac{1}{2}M_P^2\mathcal{R} - (\partial\phi)^2 - m^2\phi^2\right)\right]. \tag{5.1}$$

Here Dg stands for the integration over all metrics and $\sqrt{g}$ is the square root of the modulus of the determinant of $g_{\mu\nu}$. The ϕ-part of the action can be rewritten as

$$-i\int d^4x\, \sqrt{g}\, \phi M\phi \qquad \text{with} \qquad M \equiv -\partial^2 + m^2\,. \tag{5.2}$$

For our purposes, it will be sufficient to understand how the ϕ part of the partition function changes with α if $g_{\mu\nu} = \alpha\eta_{\mu\nu}$. Under this restriction, we can reparameterize our spacetime such that $g_{\mu\nu} = \eta_{\mu\nu}$ and only keep track of the dependence on the total 4-volume V.

After Wick rotation (deformation of the x^0 integration contour from real to imaginary axis by clockwise rotation and subsequent renaming $x^0 = -ix^4$), we have

$$-\int_V d^4x\, \phi M_E\phi \qquad \text{with} \qquad M_E \equiv -\partial^2 + m^2 \qquad \text{and} \qquad \partial^2 = \delta^{\mu\nu}\partial_\mu\partial_\nu\,. \tag{5.3}$$

Now we are dealing with a Gaussian integral with a matrix in the exponent, giving us

$$\int D\phi\, \exp\left[-\int_V d^4x\, \phi M_E\phi\right] = \frac{1}{\sqrt{\det(M_E)}} = \exp\left(-\frac{1}{2}\mathrm{tr}\ln M_E\right). \tag{5.4}$$

Here we have absorbed an infinite constant factor in the defintion of $D\phi$ in the first step und applied the identity $\ln\det = \mathrm{tr}\ln$ in the second step.

Now we note that in Fourier space

$$M_E(k,p) = \delta^4(k-p)\,(k^2+m^2)\,, \tag{5.5}$$

and hence

$$\mathrm{tr}\,\ln\, M_E = \int d^4k\, \delta^4(k-k)\ln(k^2+m^2) = V\int \frac{d^4k}{(2\pi)^4}\, \ln(k^2+m^2)\,. \tag{5.6}$$

Here, in the first step, the δ function has remained outside the log since it only signals that the matrix in question is diagonal. In the second step, we used

$$(2\pi)^4\delta^4(k-p) = \int d^4x\, e^{ix(k-p)} = V \qquad \text{for} \qquad p = k\,. \tag{5.7}$$

Undoing the Wick rotation and reinstating $\int d^4x\,\sqrt{g}$ instead of V, we find

$$Z = \int Dg\,\exp\left[i\int d^4x\,\sqrt{g}\left(\frac{1}{2}M_P^2\mathcal{R}-\lambda\right)\right] \quad \text{with} \quad \lambda = \frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\,\ln(k^2+m^2)\,. \tag{5.8}$$

Note that the intermediate transition to Euclidean space could have been avoided by regularizing the oscillating Gaussian (with the i in the exponent) in some other way.

6 Simple manipulations within the superspace approach

Tasks:

(1) Check that, with our upper-left/lower-right convention for contracting Weyl indices, $\psi\chi = \chi\psi$. Check that consistency requires $\partial_\alpha\partial_\beta = -\partial_\beta\partial_\alpha$. Check that, again for consistency, one must have $(\partial_\alpha)^* = -\overline{\partial}_{\dot\alpha}$.

(2) Check as many of the anticommutation relations between Q, $\overline{Q}$, D and $\overline{D}$ as you need to feel confident.

(3) Derive the transformation rules for the components of the chiral superfield.

Hints: Mostly straightforward manipulations - no hints needed. Recall that $(AB)^* = B^*A^*$ for an abstract algebra with a $*$-operation. When solving (3), it is useful to first work out $\delta_\xi\theta$, $\delta_\xi\theta^2$, $\delta_\xi y^\mu$ and $\delta_\xi f(y)$ for a generic function y.

Solution:

(1) One immediately finds

$$\psi\chi = \psi^\alpha\chi_\alpha = \psi^\alpha\epsilon_{\alpha\beta}\chi^\beta = \chi^\beta\epsilon_{\beta\alpha}\psi^\alpha = \chi^\beta\psi_\alpha = \chi\psi\,. \tag{6.1}$$

Furthermore,

$$\partial_1\partial_2\theta^2\theta^1 = 1 \qquad\qquad \partial_2\partial_1\theta^2\theta^1 = -\partial_2\partial_1\theta^1\theta^2 = 1\,. \tag{6.2}$$

Next, consider $(\partial_\alpha\theta^\beta)^* = \delta_\alpha{}^\beta$. Evaluate it using the rules of an abstract algebra with a '$*$' first. In other words, consider

$$(\partial_\alpha\theta^\beta)^* = \overleftarrow{\overline{\theta}^{\dot\beta}(-\overline{\partial}_{\dot\alpha})}\,, \tag{6.3}$$

where the arrow indicates that the derivative still acts on the variable. Also, we have to impose $\alpha = \dot\alpha$ and $\beta = \dot\beta$. Now, since Grassmann objects always anticommute, we also have

$$\overleftarrow{\overline{\theta}^{\dot\beta}(-\overline{\partial}_{\dot\alpha})} = \overline{\partial}_{\dot\alpha}\overline{\theta}^{\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta} = \delta_\alpha{}^\beta\,, \tag{6.4}$$

as desired. Clearly, the minus sign in the action of the '$*$' on derivatives was needed to get this.

(2) Using the definitions in the lecture, we have

$$\{Q_\alpha,\overline{Q}_{\dot\alpha}\} = \{\partial_\alpha - i(\sigma^\mu)_{\alpha\dot\beta}\overline{\theta}^{\dot\beta}\partial_\mu\,,\,-\overline{\partial}_{\dot\alpha} + i\theta^\beta(\sigma^\nu)_{\beta\dot\alpha}\partial_\nu\} = i(\sigma^\nu)_{\alpha\dot\alpha}\partial_\nu + (-1)(-i)(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu\,. \tag{6.5}$$

Here we used the fact that non-zero contributions only arise from the first term of Q acting on the second term of $\overline{Q}$ and vice versa. The resulting contributions add up giving the overall factor of 2 in the commutator given in the lecture. It is clear that, for two Qs, the result will be zero since each term vanishes separately. Also, for Q and $\overline{D}$ the result is zero on account of the sign flip in the defintion of $\overline{D}$: the analogues of the final two terms in (6.5) cancel in this case.

(3) We need to calculate

$$\delta_\xi \Phi(y,\theta) = [(\xi\partial - i\xi\sigma^\mu\overline{\theta}\partial_\mu) + \text{h.c.}]\,(A(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y))\,. \tag{6.6}$$

We first note that

$$(\delta_\xi\theta)^\alpha = \xi^\beta\partial_\beta\theta^\alpha = \xi^\beta \qquad \text{or} \qquad \delta_\xi\theta = \xi\,. \tag{6.7}$$

Similarly,

$$\delta_\xi\theta^2 = \xi^\alpha\partial_\alpha\theta^\beta\theta_\beta = \xi\theta + \theta^\beta\xi^\alpha\partial_\alpha\theta_\beta = \xi\theta - \theta_\beta\xi^\alpha\partial_\alpha\theta^\beta = 2\xi\theta\,. \tag{6.8}$$

Furthermore, for a generic function $f(y)$, we have

$$\delta_\xi f(y) = (\partial_\mu f(y))\,\delta_\xi y^\mu \tag{6.9}$$

and

$$\begin{aligned} \delta_\xi y^\mu &= [(\xi\partial - i\xi\sigma^\nu\overline{\theta}\partial_\nu) + (-\overline{\xi}\overline{\partial} + i\theta\sigma^\nu\overline{\xi}\partial_\nu)]\,(x^\mu + i\theta\sigma^\mu\overline{\theta}) \\ &= i\xi\sigma^\mu\overline{\theta} - i\xi\sigma^\mu\overline{\theta} + i\theta\sigma^\mu\overline{\xi} + i\theta\sigma^\mu\overline{\xi} = 2i\theta\sigma^\mu\overline{\xi}\,. \end{aligned} \tag{6.10}$$

Note that, to get the sign of the thrid term in the second line right, one needs to take into account that

$$\overline{\xi\partial\theta}^{\dot\beta} = \overline{\xi}_{\dot\alpha}\overline{\partial}^{\dot\alpha}\overline{\theta}^{\dot\beta} = -\overline{\xi}^{\dot\alpha}\overline{\partial}_{\dot\alpha}\overline{\theta}^{\dot\beta} = -\overline{\xi}^{\dot\beta}\,. \tag{6.11}$$

After these preliminaries, one immediately finds

$$\begin{aligned} \delta_\xi\Phi &= (\partial_\mu A)\,(2i\theta\sigma^\mu\overline{\xi}) + \sqrt{2}\xi\psi + \sqrt{2}(\theta\partial_\mu\psi)\,(2i\theta\sigma^\mu\overline{\xi}) + 2(\xi\theta)F \\ &= 1\cdot(\sqrt{2}\xi\psi) \;+\; \sqrt{2}\theta\,(\sqrt{2}i\sigma^\mu\overline{\xi}\partial_\mu A \;+\; \sqrt{2}\xi F) \;+\; \theta^2(-\sqrt{2}i(\partial_\mu\psi)\sigma^\mu\overline{\xi})\,. \end{aligned} \tag{6.12}$$

Here, to derive the last term, we used

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta^2\,. \tag{6.13}$$

The second line of (6.12) is already in a form which allows one to directly read off the quantities $\delta_\xi A$, $\delta_\xi\psi$ and $\delta_\xi F$ as the coefficients of 1, $\sqrt{2}\theta$, and θ^2. To match this with the formula given in the lecture, one also needs to use the relation

$$\psi\sigma^\mu\overline{\xi} = -\overline{\xi}\overline{\sigma}^\mu\psi. \tag{6.14}$$

This relation is easily derived using the definition of $\overline{\sigma}$ in the lectures through complex conjugation. One also needs the hermiticity of Pauli matrices.

7 Deriving component actions

Task: Consider a generic chiral superfield model defined by a Kahler potential $K(\Phi^i, \overline{\Phi}^{\bar{\jmath}})$ and a superpotential $W(\Phi^i)$. The full component lagrangian reads

$$\begin{aligned} \mathcal{L} &= -g_{i\bar{\jmath}}(\partial_\mu A^i)(\partial^\mu \overline{A}^{\bar{\jmath}}) - i g_{i\bar{\jmath}}\overline{\psi}^{\bar{\jmath}}\overline{\sigma}^\mu D_\mu \psi^i + \frac{1}{4}R_{i\bar{\jmath}k\bar{l}}\psi^i\psi^k\overline{\psi}^{\bar{\jmath}}\overline{\psi}^{\bar{l}} \\ & \quad -\frac{1}{2}(D_iD_jW)\psi^i\psi^j + \text{h.c.} \; - \; g^{i\bar{\jmath}}(D_iW)(D_{\bar{\jmath}}\overline{W})\,. \end{aligned} \tag{7.1}$$

Here

$$\begin{gathered} \partial_i = \frac{\partial}{\partial\Phi^i}\,, \qquad \overline{\partial}_{\bar{\imath}} = \frac{\partial}{\partial\overline{\Phi}^{\bar{\imath}}}\,, \qquad g_{i\bar{\jmath}} = \partial_i\overline{\partial}_{\bar{\jmath}}K\,, \qquad \Gamma_{ij}{}^k = g^{k\bar{l}}\partial_i g_{j\bar{l}}\,, \qquad R_{i\bar{\jmath}k\bar{l}} = g_{m\bar{l}}\partial_{\bar{\jmath}}\Gamma_{ik}{}^m\,, \\ D_iW = \partial_iW\,, \qquad D_iD_jW = \partial_i(D_jW) - \Gamma_{ij}{}^k(D_kW)\,, \qquad D_\mu\psi^i = \partial_\mu\psi^i + \Gamma_{jk}{}^i(\partial_\mu A^j)\psi^k\,. \end{gathered} \tag{7.2}$$

Note that Γ and R are exactly the same Christoffel symbols and Riemann tensor that are familiar from general relativity. The formulae only look slightly different since we parameterize the manifold using complex coordinates and they are slightly simpler than usual because the metric is not generic but a Kahler metric. The covariant derivative D_μ has nothing to with spacetime being curved (it is not) but rather related to the fact that ψ lives in a bundle over the scalar manifold and so comparing ψ at two different points in x requires kowledge of the values of A at these points.

Derive the first two and the last term in (7.1). If you wish, try also the others.

Hints: You can save work by shifting x under the integral:

$$x^\mu + i\theta\sigma^\mu\overline{\theta}\;,\; x^\mu - i\theta\sigma^\mu\overline{\theta} \qquad \longrightarrow \qquad x^\mu\;,\; x^\mu - 2i\theta\sigma^\mu\overline{\theta}\,. \tag{7.3}$$

Independently, prove and use the formula

$$(\theta\sigma^\mu\overline{\theta})(\theta\sigma^\nu\overline{\theta}) = -\frac{1}{2}\theta^2\overline{\theta}^2\eta^{\mu\nu}\,. \tag{7.4}$$

Solution: We start with the last formula. It is clear that the r.h. side must be proportional to $\theta^1\theta^2\overline{\theta}^1\overline{\theta}^2$ and hence to $\theta^2\overline{\theta}^2$. The latter is a scalar, so it must be multiplied by an invariant tensor with indices μ and ν, where $\eta^{\mu\nu}$ is the only choice. Thus, one only needs to check normalization. This is done most easily by focussing on $\mu = \nu = 0$:

$$(\theta\sigma^0\overline{\theta})^2 = \left(\theta^1\overline{\theta}^1 + \theta^2\overline{\theta}^2\right)^2 = -2\theta^1\theta^2\overline{\theta}^1\overline{\theta}^2\,. \tag{7.5}$$

We also have

$$\theta^2 = \theta^\alpha\epsilon_{\alpha\beta}\theta^\beta = 2\theta^1\theta^2 \tag{7.6}$$

and hence

$$\theta^2\overline{\theta}^2 = -4\theta^1\theta^2\overline{\theta}^1\overline{\theta}^2\,. \tag{7.7}$$

Recalling that we use the mostly-plus metric, the result follows.

Now we proceed to evaluate the D term of the Kahler potential. Since we are only interested in the kinetic term of A, we can set ψ and F to zero. Thus, with the shift of variables explained above, we have to evaluate

$$K\Big(A^i(x), \overline{A}^{\bar\imath}(x - 2i\theta\sigma\overline{\theta})\Big)\Big|_{\theta^2\overline{\theta}^2}\,. \tag{7.8}$$

This is done by first Taylor expanding $\overline{A}$,

$$K\Big(A^i, \overline{A}^{\bar\imath} - 2i\theta\sigma^\mu\overline{\theta}\partial_\mu\overline{A}^{\bar\imath} + \theta^2\overline{\theta}^2\partial^2\overline{A}^{\bar\imath}\Big)\,, \tag{7.9}$$

where we used (7.4) to simplify the quadratic term in the expansion. Next we Taylor expand K, keeping only what will contribute to the D term:

$$K\Big|_{\theta^2\overline{\theta}^2} = \Big(K_{\bar\imath}(A,\overline{A})\,\theta^2\overline{\theta}^2\partial^2\overline{A}^{\bar\imath} + \frac{1}{2}K_{\bar\imath\bar\jmath}(A,\overline{A})\,(2i\theta\sigma^\mu\overline{\theta}\partial_\mu\overline{A}^{\bar\imath})\,(2i\theta\sigma^\nu\overline{\theta}\partial_\nu\overline{A}^{\bar\jmath})\Big)\Big|_{\theta^2\overline{\theta}^2}\,. \tag{7.10}$$

The second term can again be simplified using (7.4), which gives

$$\begin{aligned} K\Big|_{\theta^2\overline{\theta}^2} &= K_{\bar\imath}(A,\overline{A})\,\partial^2\overline{A}^{\bar\imath} + K_{\bar\imath\bar\jmath}(A,\overline{A})\,(\partial_\mu\overline{A}^{\bar\imath})\,(\partial^\mu\overline{A}^{\bar\jmath}) \\ &= -\partial_\mu(K_{\bar\imath}(A,\overline{A})\,\partial^\mu\overline{A}^{\bar\imath} + K_{\bar\imath\bar\jmath}(A,\overline{A})\,(\partial_\mu\overline{A}^{\bar\imath})\,(\partial^\mu\overline{A}^{\bar\jmath}) \;+\; \text{total derivative} \\ &= -K_{j\bar\imath}(\partial_\mu A^j)(\partial^\mu\overline{A}^{\bar\imath}) \;+\; \text{total derivative}\,. \end{aligned} \tag{7.11}$$

This is our desired result.

To derive the last term in (7.1), we only need to consider the terms involving F. It is clear that the Taylor expansion in θ^2 and $\overline{\theta}^2$ gives

$$K\Big|_{\theta^2\overline{\theta}^2} \supset K_{i\bar\jmath}F^i\overline{F}^{\bar\jmath} \qquad \text{and} \qquad W\Big|_{\theta^2} \supset W_iF^i + \text{h.c.} \tag{7.12}$$

Varying w.r.t. $\overline{F}^{\bar\jmath}$ one finds

$$\overline{W}_{\bar\jmath} + K_{i\bar\jmath}F^i = 0 \qquad \text{and hence} \qquad F^i = -g^{i\bar\jmath}\overline{W}_{\bar\jmath}\,. \tag{7.13}$$

Inserting this in the three terms of (7.12), the result

$$\mathcal{L} \supset -g^{i\bar\jmath}W_i\overline{W}_{\bar\jmath} \tag{7.14}$$

eventually follows.

Let us finally consider the fermion kinetic term. It will be convenient to shift the variable such that we have to deal with

$$K(\Phi(x + 2i\theta\sigma\overline{\theta}), \overline{\Phi}(x))\Big|_{\theta^2\overline{\theta}^2}\,. \tag{7.15}$$

Now, suppressing the spacetime arguments and the projection on the highest component for brevity, we expand the chiral superfields in the fermionic components:

$$2K_{i\bar\jmath}\,(\overline{\theta\psi})\,(\theta\psi)\,. \tag{7.16}$$

Then we expand ψ to linear order in the quantity $2i\theta\sigma^\mu\overline{\theta}$:

$$4K_{i\overline{\jmath}}\,(\overline{\theta}\overline{\psi}^{\overline{\jmath}})\,(\theta\partial_\mu\psi^i)\,(i\theta\sigma^\mu\overline{\theta}) \tag{7.17}$$

At this point we have to employ (6.13) and the hermitian conjugate relation

$$\overline{\theta}^{\dot\beta}\overline{\theta}^{\dot\alpha} = -\frac{1}{2}\epsilon^{\alpha\beta}\overline{\theta}^2\,. \tag{7.18}$$

Thus, we have

$$(\overline{\theta}_{\dot\alpha}\overline{\psi}^{\dot\alpha})\,(\theta^\alpha\partial_\mu\psi_\alpha)\,(i\theta^\beta\sigma^\mu_{\beta\dot\beta}\overline{\theta}^{\dot\beta}) = -\frac{1}{2}\,i(\overline{\theta}_{\dot\alpha}\overline{\psi}^{\dot\alpha})\,(\partial_\mu\psi^\beta\sigma^\mu_{\beta\dot\beta}\overline{\theta}^{\dot\beta})\,\theta^2 = \frac{1}{4}\,i\partial_\mu\psi\sigma^\mu\overline{\psi}\,\theta^2\overline{\theta}^2\,. \tag{7.19}$$

Now we use the relation

$$\psi\sigma^\mu\overline{\chi} = -\overline{\chi}\overline{\sigma}^\mu\psi\,, \tag{7.20}$$

which follows from the hermiticity of σ matrices and the anticommutation of spinors. Morover, we implement the $\theta^2\overline{\theta}^2$ projection. This gives

$$-i\,K_{i\overline{\jmath}}\,\overline{\psi}^{\overline{\jmath}}\overline{\sigma}^\mu\partial_\mu\psi^i\,. \tag{7.21}$$

With the renaming $K_{i\overline{\jmath}} \to g_{i\overline{\jmath}}$ this is the partial-derivative part of our kinetic term.

We still have to find the term responsible for its covariantization. For this, we note that we obtained the term $2i\theta\sigma^\mu\overline{\theta}$ from expanding ψ. But we could equally well have expanded A in $K_{i\overline{\jmath}}$ to obtain this term. The calculation proceeds precisely as above, but in final formula ∂_μ acting on ψ is dropped. Instead, one has to replace $K_{i\overline{\jmath}}$ by

$$\partial_k K_{i\overline{\jmath}}\partial_\mu A^k\,. \tag{7.22}$$

Thus, we finally have the term

$$-i\partial_k g_{i\overline{\jmath}}\,\partial_\mu A^k\overline{\psi}^{\overline{\jmath}}\overline{\sigma}^\mu\psi^i\,. \tag{7.23}$$

To see that this is what we want, we work backward from (7.1) and rewrite the corresponding term:

$$-ig_{i\overline{\jmath}}\overline{\psi}^{\overline{\jmath}}\overline{\sigma}^\mu\Gamma_{jk}{}^i(\partial_\mu A^j)\psi^k = -ig_{i\overline{\jmath}}\overline{\psi}^{\overline{\jmath}}\overline{\sigma}^\mu g^{i\overline{l}}\partial_j g_{k\overline{l}}(\partial_\mu A^j)\psi^k\,. \tag{7.24}$$

Now the agreement is apparent.

8 Fierz identities for Weyl spinors

Task: Derive the covariant orthonormality condition for σ matrices

$$(\sigma_\mu)_{\alpha\dot\alpha}(\overline{\sigma}_\nu)^{\dot\alpha\alpha} = -2\eta_{\mu\nu}\,. \tag{8.1}$$

Use it to simplify expressions like $(\sigma_\mu)_{\alpha\dot\alpha}(\sigma^\mu)^{\beta\dot\beta}$ and $(\sigma_\mu)_{\alpha\dot\alpha}(\overline{\sigma}^\mu)^{\dot\beta\beta}$. From this, Fierz identities like

$$(\phi\sigma^\mu\overline{\chi})(\psi\sigma_\mu\overline{\eta}) = -2(\psi\phi)(\overline{\chi}\overline{\eta}) \qquad \text{and} \qquad (\phi\sigma^\mu\overline{\chi})(\overline{\eta}\overline{\sigma}_\mu\psi) = 2(\psi\phi)(\overline{\chi}\overline{\eta}) \tag{8.2}$$

immediately follow. One can use those to replace bi-spinors within some longer expressions according to

$$(\cdots \overline{\chi}\psi \cdots) = \frac{1}{2}(\cdots \overline{\sigma}^\mu \cdots)(\psi\sigma_\mu\overline{\chi}) \qquad \text{and} \qquad (\cdots \psi\overline{\chi} \cdots) = \frac{1}{2}(\cdots \sigma^\mu \cdots)(\overline{\chi}\overline{\sigma}_\mu\psi)\,. \tag{8.3}$$

Hints and background: Fierz identities are probably familiar in the context of Dirac spinors, where they are also used to rewrite expressions with four spinors in such a way that the pairs connected by index contraction (possibly through γ matrices) change. The basic underlying idea making this possible is the completeness of $\{\mathbb{1}, \gamma_\mu, \gamma_5, \gamma_\mu\gamma_5, [\gamma_\mu, \gamma_\nu]\}$ in the space of 4×4 matrices. In our context, things are much simpler since the 4 σ-matrices already provide a basis of the space of 2×2 matrices.

Solution: Let us start by rewriting the second matrix on the l.h. side of (8.1) according to

$$(\overline{\sigma}_\nu)^{\dot\alpha\alpha} = (\sigma_\nu)^{\alpha\dot\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}(\sigma_\nu)_{\beta\dot\beta} = [(i\sigma_2)\sigma_\nu(-i\sigma_2)]^{\alpha\dot\alpha} = [\{\sigma_0, -\sigma_1, \sigma_2, -\sigma_3\}]^{\alpha\dot\alpha} = [\{\sigma_0, -\sigma_i\}]^{\dot\alpha\alpha}\,. \tag{8.4}$$

With this and the usual orthonormality relations between the Pauli matrices and the unit matrix, the r.h. side of (8.1) immediately follows.

Now we recall that the σ matrices form a basis of 2×2 hermitian matrices. In fact, over the complex numbers they are a basis of all 2×2 matrices. Hence we have

$$M_{\alpha\dot\alpha} = M^\mu(\sigma_\mu)_{\alpha\dot\alpha} \tag{8.5}$$

for generic $M_{\alpha\dot\alpha}$ and appropriate coefficients M^μ. Multiplying by $(\overline{\sigma}_\nu)^{\dot\alpha\alpha}$ and using (8.1), one finds

$$M_{\alpha\dot\alpha}(\overline{\sigma}_\nu)^{\dot\alpha\alpha} = -2M^\mu\eta_{\mu\nu}\,. \tag{8.6}$$

Solving this for M^μ and inserting in (8.5) gives

$$-\frac{1}{2}M_{\beta\dot\beta}(\overline{\sigma}^\mu)^{\dot\beta\beta}(\sigma_\mu)_{\alpha\dot\alpha} = M_{\alpha\dot\alpha} = M_{\beta\dot\beta}\delta_\alpha{}^\beta\delta_{\dot\alpha}{}^{\dot\beta}\,. \tag{8.7}$$

or, since M was generic,

$$(\sigma_\mu)_{\alpha\dot\alpha}(\overline{\sigma}^\mu)^{\dot\beta\beta} = -2\delta_\alpha{}^\beta\delta_{\dot\alpha}{}^{\dot\beta}\,. \tag{8.8}$$

Using the hermiticity of σ matrices and lowering the indices, one then also has

$$(\sigma_\mu)_{\alpha\dot\alpha}(\sigma^\mu)_{\beta\dot\beta} = -2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\,. \tag{8.9}$$

From this, the identities in (8.2) follow straightforwardly by multiplication and contraction with four spinors, where one has of course to be very careful with the spinor ordering and signs. Finally, (8.3) provides two different ways for reinterpreting (8.2) as a method for replacing two spinors within a longer string of Weyl spinor expressions.

9 SUSY in components

Task: Demonstrate that the SUSY algebra is represented on the scalar (or chiral) multiplet, without using superspace. Realize SUSY without the auxiliary field (just on A and ψ) by allowing yourself to use the equations of motion (i.e. working on-shell).

Hints: As explained, while SUSY is very conveniently derived in superspace, it can also be discussed entirely at the level of component fields. This is important since in many cases (in higher dimensions, in many supergravity theories, or in situations with more than the minimal set of Qs, also known $\mathcal{N}=2$ or $\mathcal{N}=4$ SUSY), no superspace description exists or is not efficient. To discuss this component description, one focuses on the bosonic generators

$$\delta_\xi = \xi Q + \overline{\xi Q}\,. \tag{9.1}$$

Their algebra, defined with commutators, is equivalent ot the SUSY algebra. Start by calculating $[\delta_\xi, \delta_\eta]$ using the known algebra of the Qs. Then check that the algebra is represented on the components by using the explicit expressions for $\delta_\xi A$, $\delta_\xi \psi$ and $\delta_\xi F$ that were given in the lecture and that have already been derived in a previous exercise. Show also that the algebra still 'closes' (a common synonym for being represented) if $\delta_\xi F$ is dropped and, in the other expressions, F is replaced using the (for simplicity free) equations of motion. Note that in this latter case one has to use equations of motion 'to close the algebra'. One also says that the algebra is only realized 'on-shell'.

Use the Fierz identities and try not get lost in the many spinors and indices, especially when evaluating the algebra on ψ.

Solution: First, one has

$$[\xi Q, \overline{\eta}\overline{Q}] = \xi^\alpha Q_\alpha \overline{Q}_{\dot\alpha}\overline{\eta}^{\dot\alpha} - \overline{\eta}_{\dot\alpha}\overline{Q}^{\dot\alpha} Q^\alpha \xi_\alpha = \xi^\alpha \{Q_\alpha, \overline{Q}_{\dot\alpha}\}\overline{\eta}^{\dot\alpha} = 2\xi\sigma^\mu\overline{\eta}\, P_\mu = -2i\,\xi\sigma^\mu\overline{\eta}\,\partial_\mu \tag{9.2}$$

and hence

$$[\delta_\xi, \delta_\eta] = [\xi Q, \overline{\eta}\overline{Q}] + [\overline{\xi Q}, \eta Q] = [\xi Q, \overline{\eta}\overline{Q}] - (\xi \leftrightarrow \eta) = -2i(\xi\sigma^\mu\overline{\eta} - \eta\sigma^\mu\overline{\xi})\,\partial_\mu\,. \tag{9.3}$$

To see that this explicitly holds for the scalar multiplet, we start with the scalar component that gives this multiplet its name:

$$\begin{aligned}
[\delta_\xi, \delta_\eta]A &= \delta_\xi\delta_\eta A - (\xi \leftrightarrow \eta) = \delta_\xi \sqrt{2}\eta\psi - (\xi \leftrightarrow \eta) = \sqrt{2}\eta(i\sqrt{2}\eta\sigma^\mu\overline{\xi}\partial_\mu A + \sqrt{2}\xi F) - (\xi \leftrightarrow \eta) \\
&= 2i\eta\sigma^\mu\overline{\xi}\,\partial_\mu A - (\xi \leftrightarrow \eta) = -2i\xi\sigma^\mu\overline{\eta}\,\partial_\mu A - (\xi \leftrightarrow \eta)\,.
\end{aligned} \tag{9.4}$$

This is the desired result.

The analogous calculation for the fermion is slightly more involved:

$$\begin{aligned}
[\delta_\xi, \delta_\eta]\psi &= \delta_\xi(i\sqrt{2}\sigma^\mu\overline{\eta}\partial_\mu A + \sqrt{2}\eta F) - (\xi \leftrightarrow \eta) \\
&= i\sqrt{2}\sigma^\mu\overline{\eta}\,\partial_\mu(\sqrt{2}\xi\psi) + \sqrt{2}\eta\, i\sqrt{2}\,\overline{\xi}\overline{\sigma}^\mu\partial_\mu\psi - (\xi \leftrightarrow \eta) \\
&= 2i(\sigma^\mu\overline{\eta})(\xi\partial_\mu\psi) + 2i\eta(\overline{\xi}\overline{\sigma}^\mu\partial_\mu\psi) - (\xi \leftrightarrow \eta)\,.
\end{aligned} \tag{9.5}$$

Here in the last line we have introduced (formally superfluous) brackets to emphasize where the consecutive contraction of Weyl indices is interrupted. Now we focus only on the first term of the last line and make use of the first Fierz-type identity in (8.3):

$$2i(\sigma^\mu\overline{\eta})(\xi\partial_\mu\psi) = i(\sigma^\mu\overline{\sigma}^\nu\partial_\mu\psi)(\xi\sigma_\nu\overline{\eta}) = -i(\sigma^\nu\overline{\sigma}^\mu\partial_\mu\psi)(\xi\sigma_\nu\overline{\eta}) - 2i(\xi\sigma_\nu\overline{\eta})\,\partial^\nu\psi\,. \tag{9.6}$$

Here in the second step we also employed the σ matrix analogue of the Clifford algebra relation. Now we rewrite the last σ matrix in term one as a $\overline{\sigma}$ matrix and apply the second Fierz-type identity in (8.3):

$$2i(\sigma^\mu\overline{\eta})(\xi\partial_\mu\psi) = 2i\xi(\overline{\eta}\overline{\sigma}^\mu\partial_\mu\psi) - 2i(\xi\sigma^\mu\overline{\eta})\,\partial_\mu\psi\,. \tag{9.7}$$

We see that now, taking also into account the ξ-η-antisymmetrization, the first term cancels the second term of last line in (9.5). The last term of (9.7) is our desired result, so we are done.

The calculation for the auxiliary field is again simpler:

$$\begin{aligned}[]
[\delta_\xi, \delta_\eta]F = \delta_\xi i\sqrt{2}\overline{\eta}\overline{\sigma}^\mu \partial_\mu \psi - (\xi \leftrightarrow \eta) &= i\sqrt{2}\overline{\eta}\overline{\sigma}^\mu \partial_\mu \left[i\sqrt{2}\sigma^\nu \overline{\xi}\partial_\nu A + \sqrt{2}\xi F\right] - (\xi \leftrightarrow \eta) \\
&= -2\overline{\eta}\overline{\sigma}^\mu \sigma^\nu \overline{\xi}\, \partial_\mu \partial_\nu A + 2i\overline{\eta}\overline{\sigma}^\mu \xi\, \partial_\mu F - (\xi \leftrightarrow \eta)\,.
\end{aligned} \tag{9.8}$$

Here, the first term in the second line simplifies if one uses the symmetry of $\partial_\mu\partial_\nu$ to replace the product of σ matrices by $\eta_{\mu\nu}\mathbb{1}$. After this, the expression is proportional to $\overline{\xi}\overline{\eta}$ and vanishes upon ξ-η-antisymmetrization. The second term in the last line of (9.8) provides, after rewriting in terms of σ^μ, our desired result.

Finally, we want to repeat the calculations for $[\delta_\xi, \delta_\eta]$ on A and on ψ with the auxiliary replaced according to the equations of motion. Specifically for the free theory, that means

$$F = -m\overline{A}\,, \tag{9.9}$$

such that we now work with the SUSY transformation rules

$$\delta_\xi A = \sqrt{2}\xi\psi \tag{9.10}$$
$$\delta_\xi \psi = i\sqrt{2}\sigma^\mu \overline{\xi}\partial_\mu A - m\sqrt{2}\xi\overline{A}\,. \tag{9.11}$$

In the analysis of $[\delta_\xi, \delta_\eta]A$ we do not even need term with m which formerly involved F. As we can see be revisiting our calculation above, this term simply drops out under ξ-η-antisymmetrization. By contrast, in the fermion case we now find instead of the last line of (9.5)

$$[\delta_\xi, \delta_\eta]\psi = 2i(\sigma^\mu \overline{\eta})(\xi \partial_\mu \psi) - \delta_\xi\, m\sqrt{2}\eta\overline{A} - (\xi \leftrightarrow \eta)\,. \tag{9.12}$$

Now, treating the first term as before, one finds

$$[\delta_\xi, \delta_\eta]\psi = 2i\xi(\overline{\eta}\overline{\sigma}^\mu \partial_\mu \psi) - 2i(\xi\sigma^\mu \overline{\eta})\,\partial_\mu \psi - 2m\eta(\overline{\xi\psi}) - (\xi \leftrightarrow \eta)\,. \tag{9.13}$$

Making use of ξ-η-antisymmetrization, this can be rewritten as

$$[\delta_\xi, \delta_\eta]\psi = -2\eta[\overline{\xi}(i\overline{\sigma}^\mu \partial_\mu \psi + m\overline{\psi})] - 2i(\xi\sigma^\mu \overline{\eta})\,\partial_\mu \psi - (\xi \leftrightarrow \eta)\,. \tag{9.14}$$

We recognize the equation of motion for ψ in the first term and our desired result in the second. Thus, if one is prepared to use the equations of motion, one can indeed live without the auxiliary field (on-shell SUSY).

The reader may want to continue this exercise independently by also checking the invariance of the free lagrangian, off-shell and on-shell.

10 Gauge coupling unification

Task: Demonstrate that precision gauge coupling unification in the $SU(5)$ scheme does not work well in the Standard Model but, by contrast, works extremely well with low-scale supersymmetry.

Hints: Recall that the beta function of a gauge theory with coupling g is commonly defined as

$$\beta(g) = \frac{dg}{d\ln\mu} = \frac{b\,g^3}{16\pi^2} + \cdots\,. \tag{10.1}$$

Here in the last expression we gave the leading-order result with the widely used 'beta-function-coefficient' b encoding the numerical prefactor. For a $U(1)$ gauge theory and for matter with charge q, one explicitly finds

$$b = \frac{q^2}{6}\,c \quad \text{with} \quad c = 2\,/\,4\,/-22 \quad \text{for} \quad \text{complex scalar / Weyl fermion / real vector} \tag{10.2}$$

respectively. Here the last option is somewhat formal: Indeed, while a charged scalar or Weyl fermion is easy to add to an abelian gauge theory, a charged complex vector (the combination of two real vectors) only arises as part of a non-abelian structure, in which the $U(1)$ gauge theory must also be included. Nevertheless, formally it is useful to know the above numerical value of '-22'. The derivation of these three numbers needs only the calculation of the log-divergence in the familiar self-energy diagram and can be found in many QFT textbooks, e.g. Peskin/Schroeder.

Obviously, the non-abelian case requires the substitution

$$q^2 \quad \to \quad \mathrm{tr}(T_R^a T_R^b) \equiv T_R\,\delta^{ab} \tag{10.3}$$

in the relevant self-energy diagram, where R stands for the representation in which the matter in the loop transforms. Here T_R is the so-called Dynkin-index of the representation R. The corresponding substitution in the beta function coefficient hence reads $q^2 \to T_R$. Concretely, one has $T_F = 1/2$ and $T_A = N$ for the fundamental and adjoint of $SU(N)$. One sometimes also refers to $T_A = T(A) = C_2(A)$ as the quadratic Casmir of the adjoint representation.

It is now straightforward to obtain the values of $b_{1,2,3}$ and $b'_{1,2,3}$ for the running of the couplings of $U(1)$, $SU(2)$ and $SU(3)$ in the Standard Model and the MSSM. It is convenient to work with quantities like α_i^{-1}, in particular solving explicitly and analytically the renormalization group equation for these inverse couplings. Moreover, it is useful to work with $\Delta\alpha_{12} \equiv \alpha_1^{-1} - \alpha_2^{-1}$ etc. Also, please use $SU(5)$-normalization for the $U(1)$ gauge coupling. Calculate the values of the mass scales M_{12}, M_{23} and M_{13} at which the various gauge couplings meet in the Standard Model and the MSSM (with initial values and SUSY breaking at m_Z, to keep things simple). Also, turn the logic around and derive the predicted value of α_3 at m_Z as it follows from the GUT hypothesis and the values of $\alpha_{1,2}$ at m_Z.

Solution: Let us start with the Standard Model and with b_3. We have contributions from the triplets (or equivalently anti-triplets) corresponding to l.h. and r.h. up and down-type quarks as well as from the gluons:

$$b_3 = \frac{1}{6}\left(4\cdot 2\cdot 2\cdot N_f\cdot\frac{1}{2} - 22\cdot 3\right) = \frac{4}{3}N_f - 11 = -7\,. \tag{10.4}$$

Here, in the first term, the 4 comes from the Weyl fermion nature of our matter, the 2·2 from l.h./r.h. and up/down, the $N_f = 3$ from the three families, and the 1/2 from $T_F = 1/2$. In the second term we have the -22 from the vector nature of the gluons and the 3 from $T_A = N = 3$.

Next, we consider $SU(2)$:

$$b_2 = \frac{1}{6}\left(4 \cdot (3+1) \cdot N_f \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} - 22 \cdot 2\right) = \frac{4}{3}N_f - \frac{43}{6} = -\frac{19}{6}\,. \tag{10.5}$$

Here, in the first term we have again a 4 from the Weyl fermion nature, a $(3+1)$ from the 3 colors of the quark doublet and the 1 lepton doublet, as well as $N_f/2$ as above. In the second term we have a 2 from the scalar nature of the Higgs as well as $T_F = 1/2$. The third term is self explanatory, with $T_A = N = 2$.

Finally, for $U(1)$ we have:

$$\begin{aligned} b_1 &= \frac{1}{6}\left(4\left[6 \cdot \left(\frac{1}{6}\right)^2 + 3\left(\frac{2}{3}\right)^2 + 3\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{2}\right)^2 + 1^2\right] N_f + 2 \cdot 2\left(\frac{1}{2}\right)^2\right)\frac{3}{5} \\ &= \frac{4}{3}N_f + \frac{1}{10} = \frac{41}{10}\,. \end{aligned} \tag{10.6}$$

Here the 5 terms inside the square bracket correspond to the contributions from quark doublet, up and down-quark, lepton doublet and r.h. electron. The additional contribution outside the square bracket comes from the Higgs, with a factor 2 because it is a complex scalar and another 2 because it is a doublet. Finally, the charges are given in Standard Model hypercharge normalization, which is corrected by the explicit factor of 3/5 to bring us to the right normalization for the beta function coefficient of the $U(1)$ subgroup of $SU(5)$.

The reader will not be surprised to note that the matter contribution to all b_i is the same since, as we already know, matter comes in complete $SU(5)$ multiplets.

To get the SUSY version of the above, one needs to add the effects of gauginos, extra Higgs and Higgsino fields, and sfermions. The gauginos give

$$\Delta b_1^g = 0\,, \qquad \Delta b_2^g = \frac{1}{6} \cdot 4 \cdot 2 = \frac{4}{3}\,, \qquad \Delta b_3^g = \frac{1}{6} \cdot 4 \cdot 3 = 2\,. \tag{10.7}$$

Here the 4 comes from the gauginos being Weyl fermions and the $(0, 2, 3)$ are the relevant values of T_A.

The Higgs effect is doubled because we have now have two doublets. In addition, we have to replace $2 \to 2 + 4$, since instead of a complex scalar we now have a complex scalar and a Weyl fermion. This amounts to a total factor of 6 or, equivalently, an additional term worth 5 times the Standard Model Higgs effect. Using the Higgs part of the previous analysis, this gives

$$\Delta b_1^h = \frac{1}{2}\,, \qquad \Delta b_2^h = \frac{5}{6}\,, \qquad \Delta b_3^h = 0\,. \tag{10.8}$$

Finally, the matter part suffers the substitution $4 \to 4 + 2$, i.e., an additional term worth one half of the previous value. This means

$$\Delta b_1^m = \Delta b_2^m = \Delta b_3^m = 2\,. \tag{10.9}$$

Adding everything up and also displaying the Standard Model coefficients again for easier reference, we now finally have

$$b_i = \left(\frac{41}{10}, -\frac{19}{6}, -7\right) \qquad \text{and} \qquad b_i' = \left(\frac{33}{5}, 1, -3\right) \tag{10.10}$$

for the Standard Model and the MSSM respectively.

For the rest of the excercise, our basic numerical input is

$$\frac{2\pi}{\alpha_1} = 370.7\,, \qquad \frac{2\pi}{\alpha_2} = 185.8\,, \qquad \frac{2\pi}{\alpha_3} = 53.2\,. \tag{10.11}$$

The first two values are known with much more precision, the last corresponds to $\alpha_3 = 0.118$ at m_Z, by now also very well measured. We have already derived these numbers very roughly in a previous exercise, but here want to be a bit more precise. The reader may consult to Review of Particle Properties of the Particle Data Group (PDG).

On the analytic side, our main input are the three equations

$$\alpha_i^{-1}(\mu) = -\frac{b_i}{2\pi}\,\ln(\mu) + (\text{const.})_i \tag{10.12}$$

or, applied to our case of interest,

$$\alpha_i^{-1}(m_Z) = \alpha_i^{-1}(M) + \frac{b_i}{2\pi}\,\ln\left(\frac{M}{m_Z}\right)\,. \tag{10.13}$$

Here M is some high scale and if, for example, we assume that the two couplings α_1 and α_2 become equal at the scale $M = M_{12}$, then we deduce

$$\Delta\alpha_{12}(m_Z) = \frac{\Delta b_{12}}{2\pi}\ln\left(\frac{M_{12}}{m_Z}\right)\,, \tag{10.14}$$

where $\Delta\alpha_{12} \equiv \alpha_1^{-1} - \alpha_2^{-1}$ and $\Delta b_{12} \equiv b_1 - b_2$. We find

$$M_{12} = m_Z\,\exp\left(\frac{2\pi\Delta\alpha_{12}(m_Z)}{\Delta b_{12}}\right) = 90\,\text{GeV}\,\exp\left(\frac{370.7 - 185.8}{41/10 - (-19/6))}\right) \tag{10.15}$$

and, using analogous formulae for the other ‘unification scales’, we find

$$M_{12} = 1.0\times 10^{13}\,\text{GeV}\,, \qquad M_{23} = 9.5\times 10^{16}\,\text{GeV}\,, \qquad M_{13} = 2.4\times 10^{14}\,\text{GeV} \tag{10.16}$$

in the Standard Model. The running of inverse gauge couplings that corresponds to thses results is sketched in Fig. 3. We see that gauge couplings do not really unify and the so-called grand unification scale M_G remains somewhat vague, with a value in the range of $10^{13}\cdots 10^{17}$ GeV. Alternatively, one may define M_G by the unification of α_1 and α_2, and attempt to predict α_3 at the weak scale by running it backwards from that point using b_3. This is illustrated in the figure by the dashed line, and it is apparent that this prediction will not be very good.

By contrast, as one now immediately verifies using the formulae above, the same analysis in the MSSM with SUSY breaking at m_Z gives

$$M'_{12} = 2.0\times 10^{16}\,\text{GeV}\,, \qquad M'_{23} = 2.2\times 10^{16}\,\text{GeV}\,, \qquad M'_{13} = 2.1\times 10^{16}\,\text{GeV}\,. \tag{10.17}$$

This has been celebrated as a great success of the SUSY-GUT idea, the scale of which is hence quantitatively fixed: $M_G \simeq 2\times 10^{16}$ GeV. However, to a certain extent this perfection is accidental, as we will explain after turning the argument around to predict $\alpha_3(m_Z)$.

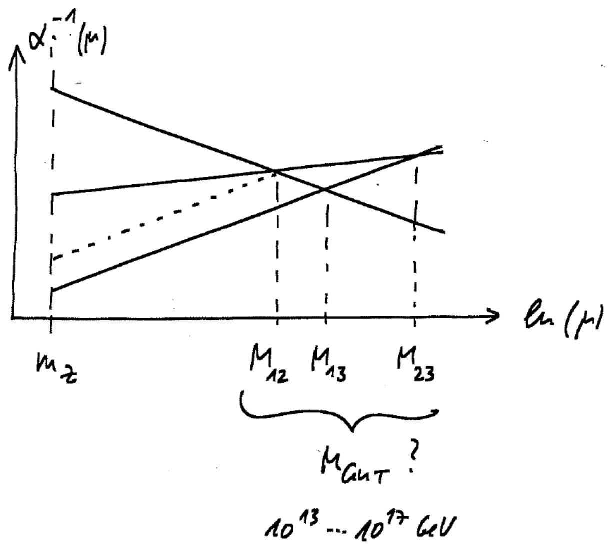

Figure 3: One-loop running of inverse gauge couplings in the Standard Model.

To derive this prediction, one combines (10.14) with its analogue for α_{13}, under the assumption that $M_{12} = M_{13} = M_G$. Eliminating M_G, one finds

$$\Delta\alpha_{13}(m_Z)/b_{12} = \Delta\alpha_{13}(m_Z)/b_{13} \tag{10.18}$$

or

$$\alpha_3^{-1}(m_Z) = \alpha_1^{-1}(m_Z) - \frac{b_{13}}{b_{12}}\,\Delta\alpha_{12}(m_Z)\,, \tag{10.19}$$

implying the predicted value $\alpha_3^{pred.}(m_Z) \simeq 0.117$. The corresponding non-SUSY prediction would be 0.071, i.e. completely off.

But one should not overstate the prefection of the result above: There are 2-loop corrections to the running, which are very well understood and lift the prediction to $\alpha_3^{pred.}(m_Z) \simeq 0.129$, which is about 10% too large. This becomes slightly better but still not perfect if one takes into account that SUSY is broken not at m_Z but at least at about a TeV. Finally, there are threshold corrections both at the SUSY breaking and the GUT scale, which also affect unification. By this we mean effects arising because not all SUSY partners and not all new GUT scale particles are degenerate at the respective scales m_{soft} and M_G. Thus, SUSY unification works well but not as prefectly as the naive 1-loop analysis suggests. It does in fact become even slightly better if the SUSY breaking scale is raised above 1 TeV. However, one has to be honest and admit that, once one gives up on the SUSY solution of the hierarchy problem, the SUSY breaking scale could be anywhere and one can not really claim any more that one predicts $\alpha_3(m_Z)$. A few more details and references to many much more detailed analyses can be found in the PDG review section on Grand Unification.

11 Graviton spin (helicity)

Task: Show that, under transverse rotations by an angle ϕ, a linear superposition of the two physical photon states can be represented by a complex number rotating with a phase $\exp(i\phi)$. Show that, analogously, the general physical graviton state rotates twice as fast (i.e., that 'the graviton has spin 2').

Hints: Let the photon momentum be $k \sim (1,1,0,0)^T$. Then transverality $\epsilon \cdot k = 0$ together with the gauge choice $\epsilon^0 = 0$ leaves the two basis polarizations

$$\epsilon_{(1)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \qquad \epsilon_{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} . \tag{11.1}$$

The general state can be characteized by $\alpha\epsilon_{(1)} + \beta\epsilon_{(2)}$ or, equivalently, by

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{R}^2 \qquad \text{or} \qquad \alpha + i\beta \in \mathbb{C} . \tag{11.2}$$

Similarly, under the constraints of transversality and tracelessness ($\epsilon^{\mu\nu}\eta_{\mu\nu} = 0$), the graviton polarization basis for $k \sim (1,1,0,0)^T$ is

$$\epsilon_{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \qquad \epsilon_{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \tag{11.3}$$

Again, the general state is $\alpha\epsilon_{(1)} + \beta\epsilon_{(2)}$ and the real or complex reperesentation is provided by (11.2).

Solution: The relevant Lorentz transormation reads

$$\epsilon^\mu \to \Lambda^\mu{}_\nu \epsilon^\nu \qquad \text{with} \qquad \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{pmatrix} \qquad \text{and} \qquad \begin{matrix} c = \cos\phi \\ s = \sin\phi \end{matrix} . \tag{11.4}$$

The vector $(\alpha,\beta)^T$ transforms by a ϕ-rotation by definition. Elementary complex algebra then implies that

$$\alpha + i\beta \;\to\; \alpha' + i\beta' = e^{i\phi}(\alpha + i\beta) . \tag{11.5}$$

For the graviton, the general state can be represented by

$$\begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \tag{11.6}$$

and the transformed state is

$$\begin{aligned}
\begin{pmatrix} \alpha' & \beta' \\ \beta' & -\alpha' \end{pmatrix} &= \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} c\alpha - s\beta & c\beta + s\alpha \\ s\alpha + c\beta & s\beta - c\alpha \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix} \\
&= \begin{pmatrix} c^2\alpha - sc\beta - cs\beta - s^2\alpha & sc\alpha - s^2\beta + c^2\beta + sc\alpha \\ cs\alpha + c^2\beta - s^2\beta + cs\alpha & s^2\alpha + sc\beta + sc\beta - c^2\alpha \end{pmatrix} \\
&= \begin{pmatrix} c'\alpha - s'\beta & s'\alpha + c'\beta \\ s'\alpha + c'\beta & -(c'\alpha - s'\beta) \end{pmatrix}
\end{aligned} \tag{11.7}$$

with $c' = c^2 - s^2 = \cos 2\phi$ and $s' = 2sc = \sin 2\phi$. Hence

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} c' & -s' \\ s' & c' \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{11.8}$$

and, in the complex plane,

$$\alpha + i\beta \;\to\; \alpha' + i\beta' = e^{2i\phi}(\alpha + i\beta)\,. \tag{11.9}$$

12 Polyakov vs. Nambu-Goto

Task: Derive the Nambu-Goto from the Polyakov action of the string following the lecture notes. (This is an entirely non-creative 'reading assignment'. If you want to make it more creative, do not use the lecture notes.)

13 Point particle action

Task: Guess the 'Polyakov action' for the point particle and derive the 'Nambu-Goto action' given in the lecture.

Hints: Introduce a worldline metric $h_{\tau\tau} = h$, such that $ds^2 = h_{\tau\tau} d\tau^2$. Allow a worldline cosmological constant term (which is forbidden in the string case by Weyl invariance, but permitted for the point particle).

Solution: The natural guess is

$$S_P = c \int d\tau \sqrt{h}(h^{-1}\dot{X}^2 - \lambda)\,, \tag{13.1}$$

from which one derives the equations of motion for h:

$$0 = \frac{1}{2\sqrt{h}}(h^{-1}\dot{X}^2 - \lambda) + \sqrt{h}(-h^{-2}\dot{X}^2) = -\frac{1}{2}(h^{-3/2}\dot{X}^2 + h^{-1/2}\lambda)\,. \tag{13.2}$$

It follows that

$$h = -\dot{X}^2/\lambda \tag{13.3}$$

and hence

$$S_P = c \int d\tau \sqrt{h}(-h^{-1}h\lambda - \lambda) = -2c\lambda \int d\tau \sqrt{h} = -2c\sqrt{\lambda} \int d\tau\, \sqrt{-\dot{X}^2}\,. \tag{13.4}$$

This reproduces the original (Nambu-Goto-type) point particle action for, e.g., $c = m/2$ and $\lambda = 1$, such that the final result is

$$S_P = \frac{m}{2} \int d\tau \sqrt{h_{\tau\tau}}\, \left(h_{\tau\tau}^{-1} \dot{X}^\mu \dot{X}_\mu - 1\right)\,. \tag{13.5}$$

14 Commutation relations of oscillator modes

Task: Demonstrate the consistency of the commutation relations of p^μ, x^μ, α_n^μ, $\tilde{\alpha}_n^\mu$ with those of the X^μs and Π^μs at equal time.

Hint: It is efficient to first calculate the commutator of X^μ with Π^μ, of X^μ with itself etc. using the mode expansion and then apply a Fourier transformation to both sides.

Solution: Collecting formulae from the lecture notes we have

$$\begin{aligned} X^\mu(\tau,\sigma) &= x^\mu + l^2 p^\mu \tau + \frac{il}{2}\sum_{n\neq 0}\frac{1}{n}\left[\tilde{\alpha}_n^\mu e^{-2in\sigma^+} + \alpha_n^\mu e^{-2in\sigma^-}\right], && (14.1)\\ \Pi^\nu(\tau,\sigma') &= \frac{1}{2\pi\alpha'}\left\{l^2 p^\nu + l\sum_{n\neq 0}\left[\tilde{\alpha}_n^\nu e^{-2in\sigma'^+} + \alpha_n^\nu e^{-2in\sigma'^-}\right]\right\}. && (14.2)\end{aligned}$$

When writing the commutator, we may right away focus on those pairings of terms from the mode expansion which have a chance of being non-zero:

$$\begin{aligned} [\Pi^\nu(\tau,\sigma'),X^\mu(\tau,\sigma)] &= -\frac{l^2}{2\pi\alpha'} i\eta^{\mu\nu} - \frac{il^2}{4\pi\alpha'}\sum_{n\neq 0}\eta^{\mu\nu}\left[e^{-2in(\sigma^+-\sigma'^+)} + e^{-2in(\sigma^--\sigma'^-)}\right]\\ &= -\frac{i}{\pi}\eta^{\mu\nu} - \frac{i}{2\pi}\eta^{\mu\nu}\sum_{n\neq 0}\left[e^{-2in(\sigma-\sigma')} + e^{2in(\sigma-\sigma')}\right].\\ &= -\frac{i}{\pi}\eta^{\mu\nu} - \frac{i}{\pi}\eta^{\mu\nu}\sum_{n\neq 0} e^{-2in(\sigma-\sigma')} \;=\; -\frac{i}{\pi}\eta^{\mu\nu}\sum_{n=-\infty}^{\infty} e^{-2in(\sigma-\sigma')}. && (14.3)\end{aligned}$$

To get the sign right, it is crucial to note that $[\alpha_m^\mu,\alpha_n^\nu]/n = m\delta_{m+n}\eta^{\mu\nu}/n = -\delta_{m+n}\eta^{\mu\nu}$. One may finish here by recognizing the δ function in $\sigma-\sigma'$ on the r.h. side.

But let us be fully explicit by finally appling a Fourier transformation in σ' and σ to both sides of our result. Using also the canonical commutation relations, the l.h. side gives

$$\int_0^\pi d\sigma'\, e^{2im\sigma'}\int_0^\pi d\sigma\, e^{2ik\sigma}\big(-i\eta^{\mu\nu}\delta(\sigma-\sigma')\big) = -i\eta^{\mu\nu}\int_0^\pi d\sigma\, e^{2i(m+k)\sigma} = -i\pi\eta^{\mu\nu}\delta_{m+k}\,. \qquad (14.4)$$

Analogously, on the r.h. side one finds

$$-\frac{i}{\pi}\eta^{\mu\nu}\sum_n(\pi\delta_{n-k})(\pi\delta_{m+n}) = -i\pi\eta^{\mu\nu}\delta_{m+k}\,. \qquad (14.5)$$

Thus, both sides agree.

The commutators $[X^\mu(\tau,\sigma),X^\nu(\tau,\sigma')]$ (and similarly for Π^μ) vanish since the relevant sums contain explicit factors of the mode indes n. For example, dropping all prefactors and the manifestly vanishing x^μ/p^μ contribution, one encounters expressions like

$$\sum_{n\neq 0}\frac{1}{n^2}n\left(e^{-2in(\sigma-\sigma')} + e^{2in(\sigma-\sigma')}\right). \qquad (14.6)$$

But this is zero by antisymmetry in n.

15 Trace of the energy momentum tensor

Task: Use a symmetry argument to show that the trace of the energy-momentum tensor of the string vanishes identically (no hint needed).

Solution: By Weyl invariance,

$$0 = S_P[h_{ab} + \epsilon h_{ab}] - S_P[h_{ab}] \simeq \epsilon h_{ab} \frac{\delta S_P}{\delta h_{ab}} = \epsilon h_{ab} \left(-\frac{\sqrt{-h}}{4\pi} T^{ab} \right) = \epsilon \left(-\frac{\sqrt{-h}}{4\pi} \right) T^a{}_a \,. \tag{15.1}$$

Hence, $T^a{}_a = 0$.

16 Virasoro algebra

Task: Derive the classical part of the Virasoro algebra using the mode expansion of the generators and the canonical commutation relations (or equivalently Poisson brackets) of the oscillator modes. Then also derive the anomaly under the assumption that the operator-ordering ambiguity in L_0 is resolved by normal ordering, i.e. that $\langle 0,0|L_0|0,0\rangle$.

Hints: For the first part, use the derivation or Leibniz rule for commutators: $[A, BC] = [A, B]C + B[A, C]$. For the second part, argue that only expressions with L_0 on the r.h. side are affected by the ordering ambiguity. Thus, the anomaly must take the form $A(m)\delta_{m+n}$. Then evaluate the commutator $[L_1, [L_m, L_{-m-1}]$ directly and with the Jacobi identity (in derivation form). Derive from this a recursive formula for the $A(m)$. Show that $A(m) = am^3 + bm$ satisfies this relation and hence determines $A(m)$ unambiguously. Fix a, b by evluating $[L_m, L_n]$ with (m, n) being $(1, -1)$ and $(2, -2)$ in the zero-momentum vacuum $|0, 0\rangle$.

A very similar derivation can be found in Green/Schwarz/Witten, but try to succeed on your own before consulting the book.

Solution: We focus on $D = 1$ and find from the known mode expansion

$$[L_m, L_n] = \frac{1}{4} \sum_{k,l} [\alpha_{m-k}\alpha_k \,,\, \alpha_{n-l}\alpha_l] \,. \tag{16.1}$$

Applying the derivation or Leibniz rule for commutators once gives

$$[L_m, L_n] = \frac{1}{4} \sum_{k,l} \left\{ [\alpha_{m-k}\alpha_k \,,\, \alpha_{n-l}]\, \alpha_l \,+\, \alpha_{n-l}\, [\alpha_{m-k}\alpha_k \,,\, \alpha_l] \right\} \,. \tag{16.2}$$

The second application together with the standard commutation relations gives

$$\begin{aligned} [L_m, L_n] &= \frac{1}{4} \sum_{k,l} \{ \alpha_{m-k}[\alpha_k, \alpha_{n-l}]\alpha_l + [\alpha_{m-k}, \alpha_{n-l}]\alpha_k\alpha_l + \alpha_{n-l}\alpha_{m-k}[\alpha_k, \alpha_l] + \alpha_{n-l}[\alpha_{m-k}, \alpha_l]\alpha_k \} \\ &= \frac{1}{4} \sum_{k} \{ k\alpha_{m-k}\alpha_{n+k} + (m-k)\alpha_k\alpha_{m+n-k} + k\alpha_{n+k}\alpha_{m-k} + (m-k)\alpha_{m+n-k}\alpha_k \} \,. \end{aligned} \tag{16.3}$$

If we were allowed to change the order of the αs on the r.h. side and to shift the summation index according to $k \to k-n$ (in terms one and three), we would obtain

$$[L_m, L_n] = \frac{1}{2}\sum_k \{(k-n)\alpha_{m+n-k}\alpha_k + (m-k)\alpha_{m+n-k}\alpha_k\} = \frac{m-n}{2}\sum_k \alpha_{m+n-k}\alpha_k = (m-n)L_{m+n}\,. \tag{16.4}$$

It is clear that the above operations are only questionable in situations with an ordering ambiguity on the r.h. side, i.e. for $m+n=0$. Hence, we have shown that

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n} \tag{16.5}$$

with some so far unknown function A.

Now we evaluate the commutator given in the hints directly,

$$[L_1, [L_m, L_{-m-1}]] = (2m+1)[L_1, L_{-1}] = (2m+1)(2L_0 + A(1))\,, \tag{16.6}$$

and through the derivation rule,

$$\begin{aligned}
[L_1, [L_m, L_{-m-1}]] &= [L_m, [L_1, L_{-m-1}] + [[L_1, L_m], L_{-m-1}] \\
&= (2+m)[L_m, L_{-m}] + (1-m)[L_{m+1}, L_{-m-1}] \\
&= (2+m)(2mL_0 + A(m)) + (1-m)((2m+2)L_0 + A(m+1))\,.
\end{aligned} \tag{16.7}$$

Comparing both results gives the recursion relation

$$(m-1)A(m+1) = (2+m)A(m) - (2m+1)A(1)\,. \tag{16.8}$$

Given also that $A(m) = -A(-m)$ by its definition, it is clear that $A(1)$ and $A(2)$ are sufficient to determine all $A(m)$ unambiguously. Moreover, it is easy to check that $A(m) = am^3 + bm$ solves the recursion:

$$(m-1)(a(m+1)^3 + b(m+1)) = (2+m)(am^3 + bm) - (2m+1)(a+b) \tag{16.9}$$

for all a, b. Thus, if we can fix a, b, we have found the unique solution.

This is easy to achieve: Note first that each term in L_{-1} (and even more so in L_1) contains either an annihilator or a p. Hence

$$\langle 0,0|[L_1, L_{-1}]|0,0\rangle = 0\,, \tag{16.10}$$

implying $A(1) = 0$. By contrast, L_{-2} contains a single term without annihilators, hence

$$\langle 0,0|[L_2, L_{-2}]|0,0\rangle = \langle 0,0|L_2 L_{-2}|0,0\rangle = \frac{1}{4}\langle 0,0|\alpha_1\alpha_1\alpha_{-1}\alpha_{-1}|0,0\rangle = \frac{1}{2}\,. \tag{16.11}$$

This implies $A(2) = 1/2$. Thus, we have to solve

$$a + b = 0 \qquad \text{and} \qquad 8a + 2b = \frac{1}{2}\,, \tag{16.12}$$

giving $a = -b = 1/12$. Clearly, if we generalize from one to D bosons, nothing changes except that, in the very last step, one gets a factor of $D/2$ on the r.h. side of (16.11). Thus, the result given in the lecture follows.

17 Normal ordering constant as Casimir energy

Task: Finish the calculation of the normal ordering constant a of the open string as the Casimir energy of 2d field theory on a strip,

$$-a = \lim_{\Lambda\to\infty}\left[\frac{D-2}{2}\left\{\sum_{n=1}^{\infty} n\right\}_{\Lambda} + \pi R^2\lambda(\Lambda)\right]. \tag{17.1}$$

For hints see lecture notes.

Solution: As explained in the lecture, we think of the sum as of a sum over modes with physical momenta $k_n = n/R$, suggesting a regularization by a suppression factor $\exp(-k_n/\Lambda)$. The sum S then reads

$$\begin{aligned}
S(\Lambda) &= \sum_{n=1}^{\infty} e^{-n/\Lambda R} = -\frac{d}{d\alpha}\sum_{n=1}^{\infty} e^{-\alpha n} \qquad (\text{ with } \ \alpha = 1/\Lambda R) \qquad (17.2)\\
&= -\frac{d}{d\alpha}\left(\frac{1}{1-e^{-\alpha}}\right) = \frac{e^{-\alpha}}{(1-e^{-\alpha})^2} = \frac{1}{(1-e^{-\alpha})(e^{\alpha}-1)}\\
&= \frac{1}{(\alpha-\alpha^2/2+\alpha^3/6)(\alpha+\alpha^2/2+\alpha^3/6)} + \mathcal{O}(\alpha)\\
&= \frac{1}{\alpha^2(1-\alpha/2+\alpha^2/6)(1+\alpha/2+\alpha^2/6)} + \mathcal{O}(\alpha)\\
&= \frac{1}{\alpha^2(1+\alpha^2/12)} + \mathcal{O}(\alpha) = \frac{1}{\alpha^2}\left(1-\frac{1}{12}\alpha^2\right) + \mathcal{O}(\alpha)\\
&= \Lambda^2R^2 - \frac{1}{12} + \mathcal{O}(1/\Lambda)\,.
\end{aligned}$$

This gives rise to

$$-a = \lim_{\Lambda\to\infty}\left[\frac{D-2}{2}\left(\Lambda^2R^2 - \frac{1}{12} + \mathcal{O}(1/\Lambda)\right) + \pi R^2\lambda(\Lambda)\right]. \tag{17.3}$$

The cosmological-constant counterterm is unambiguously determined to be $\lambda(\Lambda) = \Lambda^2(D-2)/(2\pi)$, such that finally

$$a = \frac{D-2}{24}\,. \tag{17.4}$$

18 Kalb-Ramond field from the worldsheet perspective.

Task: Work out the formal expression

$$\int_{\Sigma_2} B_2\,, \tag{18.1}$$

such that it becomes a standard Riemann double integral in $d\sigma^1 d\sigma^2$ with an integrand depending on the functions $X^\mu(\sigma)$ and $B_2(X)$.

Hints: Interpet and apply the equality (in two dimensions)

$$\int dx \wedge dy = \int dx\, dy\ (dx \wedge dy)(\partial_x, \partial_y) \tag{18.2}$$

between a form integral and a Riemann double integral.

Solution: In analogy to the formula for translating a form integral in a Riemann integral given in the hint, we have (suppressing the index Σ_2 always coming with the integral sign)

$$\int B_2 = \int d\sigma^1\, d\sigma^2\, B_2(\partial_1, \partial_2)\,. \tag{18.3}$$

Since B_2 is originally defined in target space rather than on the worldsheet, we need to push-forward the vectors ∂_a to the target space using the embedding map $X^\mu(\sigma)$ before we can explicitly insert them in B_2:

$$\int B_2 = \int d\sigma^1\, d\sigma^2\, B_2 \left(\partial_1 X^\mu \frac{\partial}{\partial X^\mu}, \partial_2 X^\mu \frac{\partial}{\partial X^\mu} \right)\,. \tag{18.4}$$

With

$$B_2 = \frac{1}{2!}\, B_{\mu\nu}\, dX^\mu \wedge dX^\nu \tag{18.5}$$

one now finds

$$\int B_2 = \int d\sigma^1\, d\sigma^2\, B_{\mu\nu}(X(\sigma))\, (\partial_1 X^\mu(\sigma))\, (\partial_2 X^\nu(\sigma))\,, \tag{18.6}$$

where σ stands for $\{\sigma^1, \sigma^2\}$. The factor $1/2!$ disappeared since we dropped a second term, where ∂_1 and ∂_2 would have been exchanged.

19 Euler number and genus of Riemann surfaces

Task: Calculate explicitly the Ricci scalar $\mathcal{R}$ of a 2-sphere of radius R and use this result to derive the formula

$$\chi(\Sigma) = 2 - 2g \tag{19.1}$$

for the Euler number

$$\chi(\Sigma) \equiv \frac{1}{4\pi} \int_\Sigma d^2\sigma\, \sqrt{\det(g_{ab})}\, \mathcal{R}\,. \tag{19.2}$$

Here g is the 'number of holes' or 'number of handles" of the Riemann surface.

Hints: Recall that the Riemann tensor in 2d is highly symmetric and that you hence do not need to calculate all components to obtain the Ricci scalar. In the second part of the problem, it will be sufficient if you give a 'physicist's derivation', drawing lots of pictures and taking the existence of intuitively obvious limits for granted.

Solution: We will use the standard parameterization of the unit sphere by azimuthal and polar angle, such that

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2\,. \tag{19.3}$$

Recalling our general 2d result

$$\mathcal{R}_{abcd} = \frac{1}{2}(g_{ac}g_{bd} - g_{ad}g_{bc})\mathcal{R}\,, \tag{19.4}$$

from the discussion of the symmetries of the bosonic string, we have

$$\mathcal{R}_{\theta\phi\theta\phi} = \frac{1}{2}\, g_{\theta\theta}g_{\phi\phi}\,\mathcal{R} = \frac{1}{2}\,\sin^2\theta\,\mathcal{R}\,. \tag{19.5}$$

Thus,

$$\mathcal{R} = \frac{2}{\sin^2\theta}\,\mathcal{R}_{\theta\phi\theta\phi} = \frac{2}{\sin^2\theta}\,\mathcal{R}_{\theta\phi\theta}{}^{\phi}\, g_{\phi\phi} = 2\,\mathcal{R}_{\theta\phi\theta}{}^{\phi}\,. \tag{19.6}$$

The required curvature coefficient can be obtained from the standard formula

$$\mathcal{R}_{abc}{}^{d} = -\partial_a\Gamma_{bc}{}^{d} + \Gamma_{ac}{}^{e}\Gamma_{be}{}^{d} - \{\,a \leftrightarrow b\,\}\,. \tag{19.7}$$

It is explicitly given by

$$\mathcal{R}_{\theta\phi\theta}{}^{\phi} = -\partial_\theta\Gamma_{\phi\theta}{}^{\phi} + \Gamma_{\theta\theta}{}^{e}\Gamma_{\phi e}{}^{\phi} + \partial_\phi\Gamma_{\theta\theta}{}^{\phi} - \Gamma_{\phi\theta}{}^{e}\Gamma_{\theta e}{}^{\phi}\,. \tag{19.8}$$

Using the standard formula

$$\Gamma_{ab}{}^{c} = \frac{1}{2}g^{cd}(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}) \tag{19.9}$$

we calculate the Christoffel symbols

$$\begin{aligned}
\Gamma_{\phi\theta}{}^{\phi} &= \frac{1}{2}g^{\phi\phi}(\partial_\phi g_{\theta\phi} + \partial_\theta g_{\phi\phi} - \partial_\phi g_{\phi\theta}) = \frac{1}{2}g^{\phi\phi}\partial_\theta g_{\phi\phi} = \frac{\cos\theta}{\sin\theta}\,, && (19.10)\\
\Gamma_{\theta\theta}{}^{\phi} &= \frac{1}{2}g^{\phi\phi}(2\partial_\theta g_{\theta\phi} - \partial_\phi g_{\theta\theta}) = 0\,, && (19.11)\\
\Gamma_{\theta\theta}{}^{\theta} &= 0\,, && (19.12)\\
\Gamma_{\phi\theta}{}^{\theta} &= \frac{1}{2}g^{\theta\theta}(\partial_\phi g_{\theta\theta} + \partial_\theta g_{\phi\theta} - \partial_\theta g_{\phi\theta}) = 0\,. && (19.13)
\end{aligned}$$

Here the zero result in the third line is obvious since the only non-zero derivative $\partial_\theta g_{\phi\phi}$ can not appear.

With this, we finally obtain

$$\mathcal{R}_{\theta\phi\theta}{}^{\phi} = -\partial_\theta\left(\frac{\cos\theta}{\sin\theta}\right) - \frac{\cos^2\theta}{\sin^2\theta} = 1 + \frac{\cos^2\theta}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta} = 1\,, \tag{19.14}$$

hence $\mathcal{R} = 2$ for the unit sphere and

$$\mathcal{R} = 2/R^2 \tag{19.15}$$

for a sphere of radius R.

Since the surface is $4\pi R^2$, we obtain the Euler number

$$\chi(S^2) = 2\,, \tag{19.16}$$

consistent with (19.1) and the absence of handles on a sphere.

Now let us move on to the case of a torus, i.e. a sphere with one handle, $g = 1$. On the one hand, it is clear that

$$\chi(T^2) = 0 \tag{19.17}$$

since an explicit geometry with everywhere vanishing curvature can easily be given. On the other hand, one can deform the geometry to a 'pancake' with a handle attached in its upper flat region, cf. Fig. 4. If there were no handle, the curvature integral in (19.2) would give $\chi = 2$, with the only contribution coming from the edge of the pancake. With the handle, we know we get zero. Thus, the hatched regions where the handle is attached give a negative contribution of -2 to the curvature integral defining χ. Obviously, further handles will give an identical negative contribution, demonstrating the correctness of the term $-2g$ in (19.1).

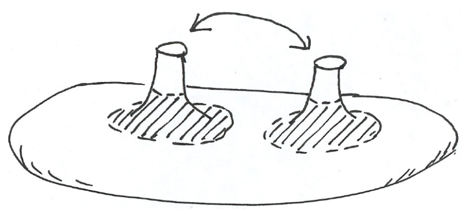

Figure 4: A torus deformed to a 'pancake' with a handle attached. The handle is realised by two 'smokestacks', to be identified at their edges. The only regions with non-zero curvature are at the edge of the pancake and in the hatched areas where the handle is attached.

20 Dilaton vs. String Coupling

Task: Give an argument for identifying the dilaton in the $\exp(-2\phi)$ prefactor of the 26d Einstein-Hilbert term with the dilaton defined as the coefficient of the Einstein-Hilbert term on the worldsheet.

Hint: Think of the loop expansion parameters in 26d-field-theory and on the worldsheet.

Solution: Think of graviton-graviton scattering in 26d quantum gravity as a low-energy effective field theory. Work in the string frame and treat the dilaton as fixed to some background VEV ϕ_0, plus small fluctuations which we will not be interested in. Expanding the metric as $\eta_{\mu\nu} + h_{\mu\nu}$ and rescaling $h_{\mu\nu} \to h_{\mu\nu}\kappa e^{\phi_0}$, we see that 3-vertices are proportional to κe^{ϕ_0} and 4-vertices to $\kappa^2 e^{2\phi_0}$. Hence, for example, a 1-loop contribution to a given process is suppressed relative to the tree level by $\kappa^2 e^{2\phi_0}$. (Draw a few example diagrams to be certain.) Morover, we can set $\kappa^2 \sim M_s^{-24}$, with $M_s \sim 1/l_s \sim 1/\sqrt{\alpha'}$ the string scale without loss of generality, as explained in the lecture. Finally, we assume that the string scale provides the cutoff Λ for the UV divergent loop diagrams. Thus, the ratio of 1-loop to tree level is

$$\kappa^2 e^{2\phi_0}\Lambda^{24} \sim e^{2\phi_0} \sim g_s^2\,, \tag{20.1}$$

consistently with the expectation from the amplitude formula of the worldsheet analysis, under the assumption that ϕ_0 governs the worldsheet Einstein-Hilbert term. Up to an additive redefinition of ϕ, this identifies the two a priori different definitions of the dilaton.

21 Elementary exercises with 2d spinors

Task: Make the action of SO(1,1) and SO(2) on vectors and spinors completely explicit, paying particular attention to how the transformations of spinors and vectors differ in the Lorentz case.

Hints: Fix the normalization of generators by analogy to the higher-dimensional case. Recall what you know from undergraduate special relativity.

Solution: In the non abelian case, the normalization of the generators J_{ab} is unambiguously fixed by the non-trivial Lie algebra relations

$$[J_{ab}, J_{cd}] = i(\eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc})\,, \tag{21.1}$$

which implies

$$(J_{ab})_{cd} = i(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bd})\,. \tag{21.2}$$

In the $SO(1,1)$ (and similarly in the $SO(2)$)) case, the Lie algebra is trivial and does not fix the normalization. We still use the gnereral-d definition, such that

$$(J_{01})_{ab} = i\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)_{ab} \qquad \text{and} \qquad (J_{01})^a{}_b = i\left(\begin{array}{cc} 0 & -1 \\ -1 & 0 \end{array}\right)^a{}_b\,. \tag{21.3}$$

Hence a boost specified by $\epsilon^{01} = -\epsilon_{10} = \alpha/2$ explicitly reads

$$\exp(i\epsilon^{ab}J_{ab}) = \exp\left(\begin{array}{cc} 0 & \alpha \\ \alpha & 0 \end{array}\right) = \left(\begin{array}{cc} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{array}\right)\,. \tag{21.4}$$

The last equality follows, e.g., from its obvious infinitesimal version together with the group property, which in turn follows from the well-known formulae for $\cosh(\alpha+\beta)$ and $\sinh(\alpha+\beta)$. This is where remembering undergraduate special relativity is useful.

Moreover, it is convenient to switch from the coordinates $x^{0,1}$ underlying the above formulae to light-cone coordinates, $x^\pm = x^0 \pm x^1$. One has $x'^+ = x'^0 + x'^1 = x^0\cosh\alpha + x^1\sinh\alpha + x^0\sinh\alpha + x^1\cosh\alpha = x^+\exp\alpha$ and similarly for x'^-. Hence,

$$\exp(i\epsilon^{ab}J_{ab}) = \left(\begin{array}{cc} e^\alpha & 0 \\ 0 & e^{-\alpha} \end{array}\right) \tag{21.5}$$

in that basis.

Next, we have

$$\frac{1}{4}[\gamma_0, \gamma_1] = \frac{1}{2}\left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right) \tag{21.6}$$

and hence

$$S(\alpha) = \exp(i\epsilon^{ab}\{i[\gamma_0,\gamma_1]/4\}) = \left(\begin{array}{cc} e^{\alpha/2} & 0 \\ 0 & e^{-\alpha/2} \end{array}\right)\,. \tag{21.7}$$

We now see explicitly how $SO(1,1)$, here realized as $\mathbb{R}$ with addition as the group operation, is represented in two different ways on vectors and spinors.

Repeating the analysis for $SO(2)$, we now label the coordinates by $1,2$ rather than $0,1$ since no special role is played by $x^0 = t$. The lower-index version of J, now called $(J_{12})_{ab}$, remains unchanged. The upper-lower version reads

$$(J_{12})^a{}_b = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^a{}_b \tag{21.8}$$

and hence

$$\exp(i\epsilon^{ab} J_{ab}) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} . \tag{21.9}$$

The correct Clifford algebra is obtained if γ_0 is multiplied by 'i', such that dilation by $e^{\alpha/2}$ becomes a phase rotation by half of the $SO(2)$ rotation angle:

$$S(\alpha) = \exp(i\epsilon^{ab}\{i[\gamma_0, \gamma_1]/4\}) = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} . \tag{21.10}$$

This was, of course, expected.

22 SUSY algebra in 2d

Task: Check the 2d SUSY algebra given in the lecture using the explicit definitions of Q and $\overline{Q}$.

Hints: Check explicitly that $\overline{\psi}\chi = \overline{\chi}\psi$ if we impose a Majorana condition on our 2d spinors. Then work out explicitly what $\overline{Q}^\alpha$ is in terms of $\overline{\theta}$ and $\partial/\partial\theta$. It is convenient to think of the action of bilinears like $(\partial/\partial\theta_\alpha)\epsilon_\alpha$. After these preliminaries, write down the commutator of $\bar{\epsilon}Q$ and $\overline{Q}\eta$, which is equivalent to the SUSY algebra (as you already learned in 4d).

Solution: Let us start with checking that $\overline{\psi}\chi = \overline{\chi}\psi$ for Majorana spinors:

$$\overline{\psi}\chi = \psi^\dagger\gamma^0\chi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}^T \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \chi_- \\ \chi_+ \end{pmatrix} = i(\psi_+\chi_- - \psi_-\chi_+) = i(\chi_+\psi_- - \chi_-\psi_+) = \overline{\chi}\psi \, . \tag{22.1}$$

Next, we want to understand how the formal $*$-operation must act on $\partial/\partial\overline{\theta}$ for Q to be Majorana. For this purpose, consider

$$\left(\bar{\epsilon}^\alpha \frac{\partial}{\partial\overline{\theta}^\alpha}\right)(\overline{\theta}\psi) = \bar{\epsilon}\psi \tag{22.2}$$

and

$$\left(\frac{\partial}{\partial\theta_\alpha}\epsilon_\alpha\right)(\overline{\psi}\theta) = \epsilon_\alpha\overline{\psi}^\beta\delta_\beta{}^\alpha = -\overline{\psi}\epsilon = -\bar{\epsilon}\psi \, . \tag{22.3}$$

We see that, with the definition

$$\overline{\left(\frac{\partial}{\partial\overline{\theta}}\right)} = -\left(\frac{\partial}{\partial\theta}\right) , \tag{22.4}$$

which is consistent with a similar minus-sign we encountered in 4d SUSY, the spinor $\partial/\partial\overline{\theta}$ is a Majorana spinor. Here Majorana is defined by the relation $\overline{\psi}\chi = \overline{\chi}\psi$.

Also, we have

$$\overline{i(\partial\!\!\!/)\theta)} = -i\theta^\dagger\gamma^0\gamma^0\partial\!\!\!/^\dagger\gamma^0 = -i\overline{\theta}\partial\!\!\!/\,. \tag{22.5}$$

Using the definition of Q_α from the lecture, this gives

$$\overline{Q}^\alpha = -\frac{\partial}{\partial\theta_\alpha} - i(\overline{\theta}\partial\!\!\!/)^\alpha = -\frac{\partial}{\partial\theta_\alpha} - i(\overline{\theta}\gamma^a)^\alpha\,\partial_a\,. \tag{22.6}$$

The first term in Q is Majorana due to (22.2) and (22.3) above. The second term is Majorana because θ is Majorana, ∂_a is a real operator, and $i\gamma^a$ is real.

Now the actual calculation is easy. Using two Majorana SUSY parameters ϵ and η, we have

$$[\overline{\epsilon}Q, \overline{Q}\eta] = \left[\overline{\epsilon}^\alpha\frac{\partial}{\partial\overline{\theta}^\alpha} + i\overline{\epsilon}\partial\!\!\!/\theta\,,\, -\frac{\partial}{\partial\theta_\beta}\eta_\beta - i\overline{\theta}\partial\!\!\!/\eta\right] = -2i\epsilon\partial\!\!\!/\eta\,. \tag{22.7}$$

Crucially, since Q and $\overline{Q}$ are not independent, there are no additional QQ or $\overline{QQ}$ relations.

23 Explicit state-operator mapping in the free case

Task: Calculate explicitly the operators which, if inserted at $z = 0$ in the radial description of the closed string, define the single-particle excited states α^μ_{-m} with $m \geq 1$.

Hints: Work with the euclidean (Wick-rotated) version of the theory, defining e.g. $(\sigma^1, \sigma^2) = (\sigma^1, i\sigma^0)$. Write $w = \sigma^1 + i\sigma^2$, such that the worldsheet cylinder corresponds to a vertical strip with width π in the complex w plane. Define $z = \exp(-2iw)$, such that constant-time cuts of the cylinder are mapped to circles in the w-plane. The origin of the z plane now corresponds to the infinite past of the cylinder, $\sigma^0 = -i\infty$.

Express our mode expansion of $\partial_- X$ (we suppress the index μ for brevity) in terms of the variable z. Invert the result, expressing the oscillator modes in terms of integrals of ∂X over a closed contour in the z plane.

Finally, use the expression obtained for a creation operator α_n under a path integral over fields on the z plane. Assuming that the fields X can be Tayor expanded in z and $\overline{z}$ at the origin, obtain the desired expression for the vertex operators. Start by arguing why the vacuum state with momentum $p = 0$ corresponds to the unit operator.

Solution: Start by rewriting our formula from the lecture (with $l = 1$) as

$$X_R = \frac{1}{2}x + \frac{1}{2}p\sigma_- + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n\,e^{-2in\sigma_-} \tag{23.1}$$

as

$$\partial_- X = \sum_n \alpha_n\,e^{-2in\sigma_-}\,, \tag{23.2}$$

where we also set $p/2 = \alpha_0$. Using $\sigma_- = \sigma^0 - \sigma^1 = -i\sigma^2 - \sigma^1 = -w$, this becomes

$$-\partial_w X = \sum_n \alpha_n e^{2inw}\,. \tag{23.3}$$

Next, with $2w = i \ln z$, we have

$$-2iz\partial_z X = \sum_n \alpha_n z^{-n} \qquad \text{or} \qquad \partial_z X = \frac{i}{2} \sum_n \frac{\alpha_n}{z^{n+1}} \,. \tag{23.4}$$

The coefficient of $1/z^{n+1}$ is extracted, using the residue theorem, by performing a counter-clockwise contour integral with the measure $dz\, z^n/(2\pi i)$:

$$\alpha_n = -2 \oint \frac{dz}{2\pi}\, z^n\, \partial_z X \,. \tag{23.5}$$

In the above, α_n is an operator acting on a state, defined one of the circles in the z plane. Similarly, X is a local field operator integrated over this circle. All of this is to be interpreted at the fixed radial time corresponding to this circle. Obviously, such an operator identity can be used under the path integral, with some operator inserted at $z = 0$ to define the initial state and with the boundary conditions at $|z| \to \infty$ defining the final state. The latter will not be relevant for us and we will ignore them.

Start by inserting the unit operator at $z = 0$ and calling the corresponding (so far unknown) state $|\Psi\rangle$:

$$\lim_{t_i \to -\infty} e^{-(t_f - t)H}\, \alpha_n\, e^{-(t-t_i)H}\, |\Psi\rangle \quad \sim \quad \int^{X_f(r_f)} DX\, e^{-S_P[X]} \oint \frac{dz}{2\pi}\, z^n\, \partial_z X \,. \tag{23.6}$$

Here the r.h. side is to be interpreted as a functional depending on the boundary conditions $X = X_f$ at some largest circle of radius r_f. The l.h. side is defined by evolving an initial state $|\Psi\rangle$ to the time t, corresponding to the radius r of the contour integral on the r.h. side, applying α_m, and then evolving to the final time t_f corresponding to r_f. The tilde means that we are not keeping track of normalizations.

It is immediately clear that, assuming that we integrate over well-behaved functions X, the r.h. side vanishes for $n \geq 0$ since there are no appropriate poles inside the contour. But the state annihiliated by all α_n with non-negative n is, by definition, the vacuum: $|\Psi\rangle = |0, 0\rangle$.

Next, we consider creation operators, α_{-n} with $n > 0$. We also use that the vacuum corresponds to the unit operator and repeat the step from (23.5) to (23.6) for this case:

$$\alpha_{-n}|0, 0\rangle \quad \sim \quad \int^{X_f(r_f)} DX\, e^{-S_P[X]} \oint \frac{dz}{2\pi\, z}\, \frac{1}{z^{n-1}}\, \partial_z X \,, \tag{23.7}$$

where now $n > 0$. We have simplified the l.h. side since, as noted, we do not keep track of the normalization. Finally, we may Taylor expand $X(z, \overline{z})$ keeping only the term which will provide a non-zero contribution to the contour integral:

$$\alpha_{-n}|0, 0\rangle \quad \sim \quad \int^{X_f(r_f)} DX\, e^{-S_P[X]}\, \frac{(\partial_z)^n X(0)}{(n-1)!} \,. \tag{23.8}$$

Thus, up to normalization, $(\partial_z)^n X(0)/(n-1)!$ is our final result for the operator corresponding to the creation operator α_{-n}.

24 Dimensional reduction

Task: Perform the KK reduction of the 5d lagrangian

$$\mathcal{L}_5 = \overline{\Psi} i \partial\!\!\!/ \Psi - M \overline{\Psi}\Psi \tag{24.1}$$

to 4d on S^1. Give your result in a compact, standard 4d notation as appropriate for a theory with towers of Dirac fermions.

Now gauge the fermion in the lagrangian above, adding also a standard gauge-kinetic term. Perform again the dimensional reduction, but disregard the higher modes of the gauge field (to avoid dealing with towers of massive vectors, which is interesting but not essential in our context). It appears that 5d gauge invariance, which should be manifest as a discrete shift symmetry of A_5, is broken. Resolve this puzzle.

Hints: It is essential to use

$$\gamma^5 = i \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \tag{24.2}$$

which differs by a prefactor from standard 4d conventions. This is clear since otherwise the 5d Clifford algebra relations[1]

$$\{\gamma^M, \gamma^N\} = -\eta^{MN} \tag{24.3}$$

would have an incorrect sign for index choice $(MN) = (55)$. The rest is a straightforward analysis following the scalar case presented in the lecture. It is more convenient to use exponentials rather than sines and cosines when dimensionally reducing the 5d fields.

Solution: Let us make the ansatz

$$\Psi(x,y) = \sum_{n=-\infty}^{+\infty} \psi_n^L(x)\, e^{iny/R} + \sum_{n=-\infty}^{+\infty} \psi_n^R\, e^{iny/R}, \tag{24.4}$$

where $x \equiv \{x^\mu\}$ and the indices L/R denote left and right-handed 4d fermions. After a straightforward calculation, using in particular manipulations like

$$\overline{\psi}_n^L(x)\, e^{-iny/R}\, i\gamma^5 \partial_5\, \psi_n^R(x)\, e^{iny/R} \;=\; \overline{\psi}_n^L(x)(-in/R)\, \psi_n^R(x)\,, \tag{24.5}$$

one arrives at

$$\begin{aligned} S \;&=\; 2\pi R \int d^4x \Bigg[\overline{\psi}_0^L i\partial\!\!\!/ \psi_0^L + \overline{\psi}_0^R i\partial\!\!\!/ \psi_0^R - M\overline{\psi}_0^L \psi_0^R + \text{h.c.} \\ &\quad + \sum_{n\neq 0} \Big\{ \overline{\psi}_n^L i\partial\!\!\!/ \psi_n^L + \overline{\psi}_n^R i\partial\!\!\!/ \psi_n^R - M\overline{\psi}_n^L \psi_n^R + \text{h.c.} \\ &\qquad (-in/R)\overline{\psi}_n^L \psi_n^R + (in/R)\overline{\psi}_n^R \psi_n^L \Big\} \Bigg]. \end{aligned} \tag{24.6}$$

[1]Recall that we use the mostly-plus convention.

We can absorb the volume factor in a field redefinition and write this as a tower of pairs of l.h. and r.h. fermions,

$$S = \int d^4x \sum_{n=-\infty}^{+\infty} \left[\overline{\psi}_n^L i\partial\!\!\!/ \psi_n^L + \overline{\psi}_n^R i\partial\!\!\!/ \psi_n^R - M_n \overline{\psi}_n^L \psi_n^R + \text{h.c.} \right] \tag{24.7}$$

with Dirac-type mass terms, but with complex mass parameters

$$M_n = M + in/R \,. \tag{24.8}$$

Of course, the complex phases of the M_n can be absorbed in a phase rotation of, for example, the right handed parts. The mass parameters now become real and the two terms with M_n and $\overline{M}_n$ can be combined in Dirac mass terms. Thus, we obtain

$$S = \int d^4x \sum_{n=-\infty}^{+\infty} \overline{\psi}_n (i\partial\!\!\!/ - M_n)\psi_n \tag{24.9}$$

with

$$M_n = \sqrt{M^2 + (n/R)^2} \,. \tag{24.10}$$

Introducing the gauging, one gets a 4d gauge theory and a real scalar coming from A_5, as explained in the lecture. Crucially, one also finds a coupling of the scalar to the fermions,

$$\overline{\Psi} i\gamma^5 iA_5 \Psi \quad \to \quad -i\phi\, \overline{\psi}_n^L \psi_n^R + \text{h.c.} \tag{24.11}$$

The fermions can again be rescaled to absorb the volume prefactor ($2\pi R$) of the fermionic part of the action. If $M \ll n/R$, it is natural to focus on the zero-mode level of this Kaluza-Klein theory:

$$S = \int d^4x \left(-\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2g^2}(\partial\phi)^2 + \overline{\psi}_0^L \, iD\!\!\!\!/ \, \psi_0^L + \overline{\psi}_0^R \, iD\!\!\!\!/ \, \psi_0^R - \psi_0^L (M + i\phi)\psi_0^R + \text{h.c.} \right) . \tag{24.12}$$

It is clear however that, to make the apparently broken shift symmetry $\phi \to \phi + 1/R$ manifest, one needs to include higher fermion modes. Indeed, when the modulus ϕ continuously changes its value from zero to $1/R$, the mode with $n = -1$ takes the place of the former zero mode. Thus, the model as a whole returns to a physically equivalent situation, as it should be given that $\phi = 0$ and $\phi = 1/R$ are related by a gauge transformation.

25 SO(2n) vs. U(n)

Task: In the lecture we used the fact that, if $R_{\alpha\beta}{}^{\gamma}{}_{\delta}$ is pure in the second index pair, then the holonomy is reduced to $U(n)$. (Here we use greek rather than latin indices to symbolize that, e.g., α may stand for either i or $\bar{\imath}$.) This fact is in principle obvious and does not require any demonstration. Still, to make the simple underlying techniques more manifest, consider the following simple problem:

Let v^α, w^β specify a vector pair such that $v^\alpha w^\beta R_{\alpha\beta}{}^\gamma{}_\delta \equiv Q^\gamma{}_\delta$ describes an infinitesimal $SO(2n)$ rotation in the complex basis, corresponding to the appropriate parallel transport along an infinitesimal loop. According to the pure index structure, Q takes the form

$$Q = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} . \tag{25.1}$$

In other words: $v^\alpha w^\beta R_{\alpha\beta}{}^i{}_j \equiv M^i{}_j$ and $v^\alpha w^\beta R_{\alpha\beta}{}^{\bar{\imath}}{}_{\bar{\jmath}} \equiv N^i{}_j$ etc.

Which properties of M and N follow from the fact that Q corresponds to an infinitesimal $SO(2n)$ transformation? (Of course, the transformation described by Q is in $Lie(SO(2n))$ by the very definition of R, such that these properties could also be derived from elementary differential geometry. But we here want a purely algebraic derivation.)

Hint: Use the notation $z'^i = M^i{}_j z^j$ and $z = x + iy$ such that

$$\begin{pmatrix} x \\ y \end{pmatrix} \tag{25.2}$$

is the column vector transforming under $SO(2n)$.

Solution: The matrix Q characterizes a linear transformation in the $(z, \overline{z})$-basis. To translate this into the (x, y) basis, write

$$\begin{aligned} x'^i &= (z'^i + \overline{z}'^i)/2 = (M^i{}_j z^j + N^j{}_j \overline{z}^j)/2 = (M^i{}_j x^j + iM^i{}_j y^j + N^i{}_j x^j - iN^i{}_j y^j)/2 && (25.3) \\ y'^i &= (z'^i - \overline{z}'^i)/2i = (M^i{}_j z^j - N^j{}_j \overline{z}^j)/2i = (M^i{}_j x^j + iM^i{}_j y^j - N^i{}_j x^j + iN^i{}_j y^j)/2i \, . && (25.4) \end{aligned}$$

From this, the real-basis form Q_r of the transformation Q is easily read off:

$$Q_r = \frac{1}{2} \begin{pmatrix} M + N & i(M - N) \\ -i(M - N) & M + N \end{pmatrix} . \tag{25.5}$$

Our requirement $Q_r \in Lie(SO(2n))$ implies that Q_r is real antisymmetric, i.e.

$$\overline{M} + \overline{N} = M + N \qquad M^T + N^T = -M - N \; , \tag{25.6}$$

$$\overline{M} - \overline{N} = -M + N \qquad M^T - N^T = M - N \; . \tag{25.7}$$

Adding the first and the third equation gives $N = \overline{M}$. The other two equations imply $M^T = -N$. Thus,

$$N = \overline{M} \quad , \qquad M = -M^\dagger \, , \tag{25.8}$$

and

$$Q = \begin{pmatrix} M & 0 \\ 0 & \overline{M} \end{pmatrix} \qquad \text{with} \qquad M \in Lie(U(n)) \, . \tag{25.9}$$

We see that Q does indeed describe an infinitesimal $U(n)$ rotation in the complex basis.

26 Complex projective spaces

Task: Consider $\mathbb{C}P^n$ with charts as defined in the lecture and obtain explicitly the transition functions $\phi_i \circ \phi_j^{-1}$. Give a general formula for the components $g_{i\bar{\jmath}}$ of the Fubini-Study metric in some chart ϕ_k. Show consistency between different charts. In the special case of $\mathbb{C}P^1 \cong S^2$, show agreement with the round metric on the sphere (up to normalization).

Hints: Deriving the transition functions is completely straightforward, but some care is needed concerning the indexing of the variables in the two charts. Getting the Fubini-Study metric in one chart requires just differentiation. To show that the Fubini-Study metric is well-defined, it is useful to first investigate how the Kahler potential transforms between coordinate patches. Try to make use of the (multi-variable generalization) of the fact that $\partial_z \overline{\partial}_{\overline{z}} \ln(z\overline{z}) = 0$ for $z \neq 0$. If you get stuck use, e.g., the lecture notes by Candelas (see refs. of lecture course) or the Wikipedia page for 'Fubini-Study metric'. In the last part, think of the stereographic projection.

Solution: The two sets of local coordinates in ϕ^i and ϕ^j may be chose as

$$(x^1, \cdots, x^n) \;=\; \left(\frac{z^0}{z^i}, \cdots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i}, \cdots, \frac{z^n}{z^i}\right) \tag{26.1}$$

and

$$(y^1, \cdots, y^n) \;=\; \left(\frac{z^0}{z^j}, \cdots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \cdots, \frac{z^n}{z^j}\right) . \tag{26.2}$$

The coordinate change is found by explicitly rewriting each of the x^k in terms of the y-coordinates. For definiteness, let us assume $i < j$. Then we find for $k \leq i$:

$$x^k = \frac{z^{k-1}}{z^i} = \frac{z^{k-1}}{z^j} \cdot \frac{z^j}{z^i} = y^k \cdot \frac{1}{y^{i+1}} . \tag{26.3}$$

For $i + 1 \leq k < j$:

$$x^k = \frac{z^k}{z^i} = \frac{z^k}{z^j} \cdot \frac{z^j}{z^i} = y^{k+1} \cdot \frac{1}{y^{i+1}} . \tag{26.4}$$

Then comes a special case: For $k = j$,

$$x^j = \frac{z^j}{z^i} = \frac{1}{y^{i+1}} . \tag{26.5}$$

Finally, for $j < k$:

$$x^k = \frac{z^k}{z^i} = \frac{z^k}{z^j} \cdot \frac{z^j}{z^i} = y^k \cdot \frac{1}{y^{i+1}} . \tag{26.6}$$

We may summarize all of this in the compact expression

$$(x^1(y), \cdots, x^n(y)) \;=\; \frac{1}{y^{i+1}}\,(y^1, \cdots, y^i, y^{i+2}, \cdots, y^j, 1, y^{j+1}, \cdots, y^n) . \tag{26.7}$$

Obtaining the explicit form of the Fubini-Study metric is easy: Consecutive differentiation w.r.t. x^i and $\overline{x}^{\bar{\jmath}}$ gives

$$2K_i^{(k)} = \frac{\overline{x}^{\bar{\imath}}}{1 + x^l \overline{x}^{\bar{l}} \delta_{l\bar{l}}} = \frac{\overline{x}^{\bar{\imath}}}{\sigma} \qquad \text{with} \qquad \sigma \equiv 1 + x^l \overline{x}^{\bar{l}} \delta_{l\bar{l}} \tag{26.8}$$

and

$$2g_{i\bar{j}} \equiv 2K^{(k)}_{i\bar{j}} = \frac{\delta_{i\bar{j}}}{\sigma} - \frac{\overline{x}^{\bar{i}} x^j}{\sigma^2} \,. \tag{26.9}$$

Here summation over l and $\bar{l}$ is implicit in the first line.

To see invariance under coordinate change, recall that the two metrics in U_i and U_j are defined as

$$\frac{\partial}{\partial x^k} \frac{\overline{\partial}}{\partial \overline{x}^l} K^{(i)}(x, \overline{x}) \qquad \text{and} \qquad \frac{\partial}{\partial y^k} \frac{\overline{\partial}}{\partial \overline{y}^l} K^{(j)}(y, \overline{y}) \,. \tag{26.10}$$

This obviously defines two tensors which will be definition agree if

$$\frac{\partial}{\partial y^k} \frac{\overline{\partial}}{\partial \overline{y}^{\bar{l}}} K^{(j)}(y, \overline{y}) \; = \; \frac{\partial}{\partial y^k} \frac{\overline{\partial}}{\partial \overline{y}^{\bar{l}}} K^{(i)}(x(y), \overline{x}(\overline{y})) \,. \tag{26.11}$$

Note that here we also have to use the fact that the coordinate change is holomorphic, such that holomorphic and antiholomorphic indices do not mix under reparameterization.

Now, Eq. (26.11) will clearly hold if

$$K^{(i)} = K^{(j)}(y, \overline{y}) + f(y) + \overline{f}(\overline{y}) \,. \tag{26.12}$$

This is, in fact, known as a Kahler transformation – the natural way in which a Kahler potential changes between patches on a Kahler manifold.

Showing that this holds is easy if one notes that

$$2K^{(i)}(x, \overline{x}) = \ln \sigma(x, \overline{x}) \qquad \text{and} \qquad 2K^{(j)}(y, \overline{y}) = \ln \sigma(y, \overline{y}) \tag{26.13}$$

with σ as defined above. Moreover, let us think of a different way of labelling our coordinates as follows:

$$x^k \equiv z^k/z^i \qquad \text{and} \qquad y^k \equiv z^k/z^j \,, \tag{26.14}$$

such that $k = 0, \cdots, n$, but with the caveat that $x^i = 1$ and $y^j = 1$ and hence these two do not count as coordinates. In this notation, one has

$$\sigma(x, \overline{x}) = \sum_{k=0}^{n} |x^k|^2 \qquad \text{and} \qquad \sigma(y, \overline{y}) = \sum_{k=0}^{n} |y^k|^2 \tag{26.15}$$

and

$$\sigma(x(y), \overline{x}(\overline{y})) = \sigma(y, \overline{y}) \, |z^j/z^i|^2 = \sigma(y, \overline{y}) \, |y^j/y^i|^2 \,. \tag{26.16}$$

Hence, we obtain the above form of a general Kahler transformation with $f(y) = \ln(y^j/y^i)$. This completes the demonstration that the metric is well-defined.

Finally, let us consider the specific case of $\mathbb{C}P^1$. In the patch U_0, we have

$$2g_{x\,\overline{x}} = \frac{1}{1+|x|^2} - \frac{|x|^2}{(1+|x|^2)^2} = \frac{1}{(1+|x|^2)^2} \tag{26.17}$$

and, with $x = r\,\exp(i\phi)$,

$$ds_x^2 = g_{x\overline{x}} dx\, d\overline{x} + g_{\overline{x}x} d\overline{x}\, dx = 2g_{x\overline{x}} dx\, d\overline{x} = \frac{dr^2 + r^2 d\phi^2}{(1+r^2)^2} \,. \tag{26.18}$$

This has to be compared with the round metric on the unit sphere,

$$ds_1^2 = d\theta^2 + \sin^2\theta\, d\phi^2\,. \tag{26.19}$$

Now imagine that this sphere is centered at the origin in $\mathbb{R}^3$ and map it to the x-y-plane using rays originating in the north pole and intersecting the plane and the sphere (stereographic projection).[2] Elementary geometry proves that the ray which intersects the sphere at (θ, ϕ) will enclose an angle $\theta/2$ with the negative vertical axis. Hence, parameterizing the plane by the complex variable $x = r\,\exp(i\phi)$ as above, we have $r = \tan(\theta/2)$. Thus, $2dr/d\theta = 1+\tan^2(\theta/2) = 1 + r^2$, which gives

$$ds_x^2 = \frac{d\theta^2}{4} + \frac{\tan^2(\theta/2)}{(1+\tan^2(\theta/2))^2}d\phi^2 = \frac{d\theta^2}{4} + \tan^2(\theta/2)\cos^4(\theta/2)d\phi^2 = \frac{1}{4}(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{26.20}$$

and $ds_x^2 = ds_1^2/4$, as proposed. Our complex x-coordinate covers the sphere without the north pole. The coordinate change $x \to 1/x$ takes us to the second coordinate patch, which covers the sphere without the south pole.

27 No-scale Kahler potentials and KKLT

Task Using the general supergravity formulae given earlier in the course, calculate the scalar potential of a one-field supergravity model with

$$K(T,\overline{T}) = -\ln\left[(T+\overline{T})^n\right] \qquad \text{and} \qquad W = W_0 = \text{ const.} \tag{27.1}$$

Observe the special feature of the case $n = 3$. Try to generalize to the case of m variables, with e^{-K} being a general homogeneous function of the variables $(T^i + \overline{T}^{\bar{\imath}})$ of degree n.

Returning to the single-modulus case, analyse the so-called 'KKLT potential' arising from the superpotential $W = W_0 + Ae^{-aT}$ for $n = 3$. Use the notation $T = \tau + ic$, set $A = a = 1$ for simplicity and assume $|W_0| \ll 1$. To draw a qualitative plot of $V(\tau)$, after minimizing in c, it is sufficient to understand the qualitative behaviour of V in the two regimes $|e^{-T}| \gg |W_0|$ and $|e^{-T}| \ll |W_0|$. Throughout, assume $\tau \gg 1$.

Hints: The first part is completely straightforward. For the general case, it is useful to prove the relation $(T^i + \overline{T}^{\bar{\imath}})K_i = -n$ and to consider its derivatives.

The discussion of the KKLT potential is a straightforward exercise in parametrically analysing a given function. Note that, in the second regime, you also need to assume that the axionic variable $\operatorname{Im} T = c$ takes the value minimizing the scalar potential. The result is shown in Fig. 28 of the old lecture notes.

Solution: First, we have

$$K = -n\,\ln(T+\overline{T})\,, \qquad K_T = K_{\overline{T}} = \frac{-n}{T+\overline{T}}\,, \qquad K_{T\overline{T}} = \frac{n}{(T+\overline{T})^2} = (K^{T\overline{T}})^{-1} \tag{27.2}$$

[2] Beware that an alternative form of the stereographic projection uses a unit sphere centered at $(0,0,1) \in \mathbb{R}^3$. This corresponds to scaling distances on the plane up by a factor of two.

and hence

$$V(T,\overline{T}) = e^K(K^{T\overline{T}}|K_T\,W_0|^2 - 3|W_0|^2) = e^K|W_0|^2(n-3)\,. \tag{27.3}$$

We see that for $n = 3$ the potential vanishes identically, implying that T remains a modulus in spite of $W \neq 0$.

Now consider the multi-variable case, with

$$K = -\ln f(T^1 + \overline{T}^1, \cdots, T^k + \overline{T}^k) \tag{27.4}$$

and

$$f(\alpha(T^1 + \overline{T}^1), \cdots, \alpha(T^k + \overline{T}^k)) = \alpha^n f(T^1 + \overline{T}^1, \cdots, T^k + \overline{T}^k)\,, \tag{27.5}$$

as proposed. By Euler's homogeneous function theorem, we have

$$(T^i + \overline{T}^{\bar{\imath}})\partial_i(e^{-K}) = n\,e^{-K} \tag{27.6}$$

and hence

$$(T^i + \overline{T}^{\bar{\imath}})K_i = -n\,. \tag{27.7}$$

Differentiation w.r.t. $\overline{T}^{\bar{\jmath}}$ gives

$$K_{\bar{\jmath}} + (T+\overline{T})^i K_{i\bar{\jmath}} = 0\,, \tag{27.8}$$

where we used $K_j = K_{\bar{\jmath}}$. Multiplying by the inverse metric one obtains

$$K^{i\bar{\jmath}}K_{\bar{\jmath}} + (T+\overline{T})^i = 0\,, \tag{27.9}$$

and after further multiplication by K_i and application of (27.7),

$$K_i K^{i\bar{\jmath}} K_{\bar{\jmath}} = n\,. \tag{27.10}$$

Now one immediately finds the multi-variable result

$$V = e^K(K^{i\bar{\jmath}}(K_iW_0)\,(K_{\bar{\jmath}}\overline{W}_0) - 3|W_0|^2) = e^K|W_0|^2(n-3)\,. \tag{27.11}$$

Finally, we turn to the discussion of the model with $n = 3$ and superpotential $W_0 + e^{-T}$. In the first regime, $\mathrm{Re}\,T \ll \ln(1/|W_0|)$, we may set $W \simeq e^{-T}$. Then the second scalar potential term, $3|W|^2$, is suppressed by two powers of the large quantity τ with respect to the F-term squared. Similarly, $D_TW \simeq \partial_T W$. Hence,

$$V \simeq e^K K^{T\overline{T}}|\partial_T\,e^{-T}|^2 \sim \frac{1}{T+\overline{T}}|e^{-T}|^2 \sim \frac{e^{-2\tau}}{\tau}\,. \tag{27.12}$$

This is positive and monotononically falling.

In the second regime, $\mathrm{Re}\,T \gg \ln(1/|W_0|)$, the naively leading term is obtained by setting $W = W_0$. But this vanishes by the no-scale property. Hence, we need to consider the formally subleading terms, which involve one power of W_0 and one power of e^{-T}. Such terms, $\sim W_0\,e^{-T}$, appear both in the F-term squared and in $-3|W|^2$. But the second contribution suffers a relative suppressed by one power of the large quantity τ. (This is due to the enhancement of the F-term squared by $K^{T\overline{T}}$, which is only partially compensated by K_T.) Thus, we find

$$V \simeq e^K K^{T\overline{T}}[(\partial_T\, e^{-T})\, K_{\overline{T}}\overline{W}_0 + \text{h.c.}] \sim \frac{e^{-\tau}}{\tau^2}|W_0|[e^{i(c+\operatorname{Arg} W_0)} + \text{h.c.}] \sim -\frac{e^{-\tau}}{\tau^2}|W_0|\,. \qquad (27.13)$$

In the last step, we assumed that c takes the value minimizing $\cos(c + \operatorname{Arg} W_0)$ at minus unity. We see that, at large τ, V is negative and approaches zero from below.

Our two results for large and 'small' (still much larger than unity) values of τ guarantee the presence of a local minimum at negative value of V and with $\tau \sim 1/|W_0|$.