Quantum Field Theory II

Notes of lecture course by A. Hebecker

Our primary reference will continute to be the book by Peskin and Schröder [1].

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1 The path integral or functional integral

1.1 Quantum mechanics preliminaries

The very powerful method of path integrals has been developed by Feynman, based on early ideas of Dirac. We explain it using the simplest example, a particle in one dimension with (using $\hbar = 1$)

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) \qquad \text{and} \qquad [\hat{q},\hat{p}] = i\,. \tag{1.1}$$

As is commonly done, we represent the $\hat{p},\hat{q}$-algebra or Heisenberg algebra by operators on the Hilbert space of square integrable functions,

$$\psi:\ q \mapsto \psi(q)\,. \tag{1.2}$$

We will also denote the function ψ, viewed as an element of the Hilbert space, by $|\psi\rangle$. The Heisenberg algebra acts as

$$\hat{p}\ :\ \psi(q) \ \mapsto\ -i\psi'(q) \tag{1.3}$$

$$\hat{q}\ :\ \psi(q) \ \mapsto\ q\,\psi(q)\,, \tag{1.4}$$

where '$\mapsto$' symbolizes a map of functions and the first line is nothing but the familiar rule $\hat{p} \equiv -i\partial/(\partial q)$.

Now, let $|p\rangle$ and $|q\rangle$ be the eigenstates of $\hat{p}$ and $\hat{q}$ with eigenvalues p and q,

$$\hat{p}\,|p\rangle = p\,|p\rangle \qquad \text{and} \qquad \hat{q}\,|q\rangle = q\,|q\rangle\,. \tag{1.5}$$

They explicitly read

$$|p\rangle \ =\ \big(q' \mapsto e^{ipq'}\big) \tag{1.6}$$

$$|q\rangle \ =\ \big(q' \mapsto \delta(q'-q)\big)\,. \tag{1.7}$$

It is easy to check that

$$\langle q|q'\rangle = \delta(q-q') \quad , \quad \langle p|p'\rangle = 2\pi\delta(p-p') \quad , \quad \langle p|q\rangle = e^{-ipq}\,, \tag{1.8}$$

as well as

$$\mathbb{1} = \int dq\, |q\rangle\langle q| = \int \frac{dp}{2\pi}\, |p\rangle\langle p|\,. \tag{1.9}$$

While this should all be very familiar (nevertheless, check all of it!), we will now do a simple calculation in this context in a slightly novel way:

1.2 Path integral for amplitudes

We are interested in the **transition amplitude** from

$$|q_a\rangle \text{ at } t_a = 0 \qquad \text{to} \qquad |q_b\rangle \text{ at } t_b = t\,, \tag{1.10}$$

which by definition reads (with $\hat{H} \to H$ for brevity)

$$\langle q_b|e^{-iHt}|q_a\rangle \;=\; \langle q_b|e^{-iH\Delta}e^{-iH\Delta}\cdots e^{-iH\Delta}|q_a\rangle\,. \tag{1.11}$$

Here, on the r.h. side, we split the time evolution into n time steps with $\Delta \equiv t/n$. Next, we insert $n-1$ identity operators,

$$\langle q_b|e^{-iHt}|q_a\rangle \;=\; \prod_{i=1}^{n-1}\left(\int dq_i\right)\langle q_b|e^{-iH\Delta}|q_{n-1}\rangle\langle q_{n-1}|e^{-iH\Delta}|q_{n-2}\rangle\cdots\langle q_1|e^{-iH\Delta}|q_a\rangle\,. \tag{1.12}$$

Now, we rewrite each of the bra-ket factors on the r.h. side as

$$\langle q_{i+1}|e^{-iH\Delta}|q_i\rangle \;=\; \int\frac{dp}{2\pi}\langle q_{i+1}|p\rangle\,\langle p|e^{-iH\Delta}|q_i\rangle \tag{1.13}$$

and use

$$\langle q_{i+1}|p\rangle \;=\; e^{ipq_{i+1}} \qquad \text{and} \qquad e^{-iH\Delta} \;=\; \mathbb{1} - iH\Delta + \mathcal{O}(\Delta^2)\,. \tag{1.14}$$

Since $\Delta = t/n$ and we will be interested in the limit $n\to\infty$, a sum of n terms $\mathcal{O}(\Delta)$ will contribute, but higher orders in Δ are irrelevant. Next, we apply the relations

$$\langle p|\mathbb{1}|q\rangle \;=\; \langle p|q\rangle \;=\; e^{-ipq} \tag{1.15}$$

and

$$\langle p|H|q\rangle \;=\; \langle p|\left(\frac{\hat{p}^2}{2m}+V(\hat{q})\right)|q\rangle \;=\; \left(\frac{p^2}{2m}+V(q)\right)\langle p|q\rangle \;=\; \left(\frac{p^2}{2m}+V(q)\right)e^{-ipq}\,. \tag{1.16}$$

Collecting all terms we find

$$\langle q_{i+1}|e^{-iH\Delta}|q_i\rangle \;\simeq\; \int\frac{dp}{2\pi}\,e^{ip(q_{i+1}-q_i)}\left[1 - i\left(\frac{p^2}{2m}+V(q_i)\right)\Delta\right] \tag{1.17}$$

$$\simeq\; \int\frac{dp}{2\pi}\,\exp i\left[p(q_{i+1}-q_i) - \left(\frac{p^2}{2m}+V(q_i)\right)\Delta\right]\,. \tag{1.18}$$

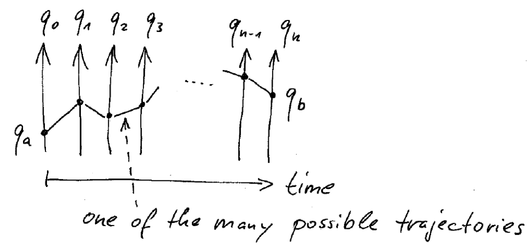


Figure 1: Pictorial representation of the integral over all trajectories.

Changing integration variables according to $p \to p-(q_{i+1}-q_i)m/\Delta$ and using the familiar formula

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}\,, \tag{1.19}$$

we finally arrive at

$$\langle q_{i+1}|e^{-iH\Delta}|q_i\rangle \;\simeq\; \frac{1}{\sqrt{2\pi i\Delta/m}}\exp i\left[\frac{m}{2}\left(\frac{q_{i+1}-q_i}{\Delta}\right)^2 - V(q_i)\right]\Delta \simeq \frac{1}{C(\Delta)}\,e^{iL(q,\dot{q})\Delta}. \tag{1.20}$$

Crucially, in the last step we have 'discretized' the lagrangian L by interpreting q_i and q_{i+1} as the values of a trajectory $q(t)$ at $t = \Delta \cdot i$ and $t = \Delta \cdot (i+1)$ respectively. $C(\Delta) \equiv \sqrt{2\pi\Delta/m}$ is an abbreviation.

We now recall that we have been working on just one of the factors in (1.12). Collecting all such factors gives

$$\langle q_b|e^{-iHt}|q_a\rangle \;=\; \lim_{\Delta\to 0}\frac{1}{C(\Delta)}\prod_{i=1}^{n-1}\left(\int\frac{dq_i}{C(\Delta)}\right)\exp i\sum_{i=0}^{i=n-1}\left[\frac{m}{2}\left(\frac{q_{i+1}-q_i}{\Delta}\right)^2 - V(q_i)\right]\Delta\,, \tag{1.21}$$

where we identified $q_a \equiv q_0$ and $q_b = q_n$. Here the sum in the exponent is a discretized version of the action,

$$S[q] = \int_{t_a}^{t_b} dt\, L(q(t),\dot{q}(t))\,. \tag{1.22}$$

The total of all q_i-integrations can be viewed as the integral over all trajectories (all paths) starting at q_a and ending at q_b, see Fig. 1. Naturally, the object to be integrated in such a 'functional integral' is a functional, in our case $\exp(iS)$. Thus, a less precise but more intuitive version of our last formula is

$$\langle q_b|e^{-iHt}|q_a\rangle \;=\; \int Dq\, e^{iS[q]}\,. \tag{1.23}$$

Here Dq (sometimes also written as $[Dq]$) symbolizes the integral over all smooth (in an appropriate definition not to be discussed here) functions $q : [t_a, t_b] \mapsto \mathbb{R}$ with fixed boundary values $q(t_a) = q_a$ and $q(t_b) = q_b$.

Equation (1.23) is (the simplest version of) the famous path integral formula for amplitudes in quantum mechanics. It is clear on dimensional grounds and easy to check explicitly that S has to be replaced by $S/\hbar$ if we do not set $\hbar = 1$. The classical limit of quantum mechanics becomes very intuitve in this approach: The contribution of trajectories whose action differs from the classical (extremal) action $S_{ext.}$ by more than $\mathcal{O}(\hbar)$ is suppressed due to fast oscillations of $e^{iS/\hbar}$. Hence, the classical trajectory dominates in systems with a large characteristic action (change).

In fact, the path integral is frequently taken to *define* a quantum system, given an action S. Hence one also speaks of 'path integral quantization'.

1.3 Path integral for correlation functions

Let us rewrite our result in the Heisenberg picture,

$$\langle q_b | e^{-iH(t_b - t_a)} | q_a \rangle = \langle q_b | e^{-iHt_b} \, e^{iHt_a} | q_a \rangle \equiv \langle q_b, t_b | q_a, t_a \rangle = \int Dq \, e^{iS[q]} \,, \tag{1.24}$$

where as before the boundary conditions $q(t_{a/b}) = q_{a/b}$ are implicit in the Dq integral.[1] An obvious generalization of the next-to-last expression is

$$\langle q_b, t_b | \hat{q}_{t_m} \cdots \hat{q}_{t_1} | q_a, t_a \rangle \,. \tag{1.25}$$

This is called a **correlation function** (between Heisenberg operators $\hat{q}_t$ at different times t) and is analogous to the Greens functions or correlation functions of fields $\langle 0 | \hat{\phi}(x_m) \cdots \hat{\phi}(x_1) | 0 \rangle$ needed in QFT.

To save writing, let $m = 1$ and return to the Schrödinger picture:

$$\langle q_b, t_b | \hat{q}_t | q_a, t_a \rangle = \langle q_b | e^{-iH(t_b - t)} \hat{q} e^{-iH(t - t_a)} | q_a \rangle \,. \tag{1.26}$$

Furthermore, rewrite

$$\hat{q} = \hat{q} \cdot \mathbb{1} = \hat{q} \int dq \, |q\rangle\langle q| = \int dq \, |q\rangle q \langle q| \,, \tag{1.27}$$

which gives

$$\langle q_b, t_b | \hat{q}_t | q_a, t_a \rangle = \int dq \langle q_b, t_b | q, t \rangle \, q \, \langle q, t | q_a, t_a \rangle \,. \tag{1.28}$$

Under the assumption $t_b > t > t_a$, we can now apply twice our path integral formula for amplitudes (in the discrete form with intermediate q_i-integrations). With $n_1 = (t_b - t)/\Delta$ and $n_2 = (t - t_a)/\Delta$, we have the two products

$$\prod_{i=1}^{n_1 - 1} \int dq_i \qquad \text{and} \qquad \prod_{i=1}^{n_2 - 1} \int dq_i \,, \tag{1.29}$$

[1] Recall that the (by definition time-independent) state $|q_a, t_a\rangle$ characterizes a particle which, at time t_a, is perfectly localized at the position q_a. It is simply the state $|q_a\rangle$ evolved in time from t_a to $t = 0$. Hence the argument t_a is not time in the sense of dynamical evolution but simply, together with q_a, a parameter characterizing the state.

together with the extra dq-integral explicitly appearing in (1.28). After relabelling these $n_1 + n_2 - 1$ integrations can be combined in a single path integral, the only difference to the formula for amplitudes being the extra factor of q at definite intermediate time t appearing in the integrand:

$$\langle q_b, t_b | \hat{q}_t | q_a, t_a \rangle = \int Dq \, q(t) \, e^{iS[q]} \, . \tag{1.30}$$

For $t_b > t_m > \cdots > t_1 > t_a$ this generalizes to

$$\langle q_b, t_b | \hat{q}_{t_m} \cdots \hat{q}_{t_1} | q_a, t_a \rangle = \int Dq \, q(t_m) \cdots q(t_1) \, e^{iS[q]} \, . \tag{1.31}$$

This is nice, but what we really need are **vacuum correlation functions**

$$\langle 0, \, t = \infty | \hat{q}_{t_m} \cdots \hat{q}_{t_1} | 0, \, t = -\infty \rangle \; = \; \langle 0 | \hat{q}_{t_m} \cdots \hat{q}_{t_1} | 0 \rangle \, , \tag{1.32}$$

where we used the fact that the vacuum is by definition an eigenstate of H. To calculate those we consider (letting again $m = 1$ to save writing)

$$\langle q, T | \hat{q}_t | q, -T \rangle = \langle q | e^{-iH(T-t)} \hat{q} \, e^{-iH(t-(-T))} | q \rangle \, , \tag{1.33}$$

with $T > t > -T$. Let's also look at the analogous expression in a QM system where H has been replaced according to

$$H \;\to\; H' \equiv (1 - i\epsilon) \, H \, . \tag{1.34}$$

Here we do not worry about unitarity violation since we will take the limit $\epsilon \to 0$ at the end. We also assume that $|q\rangle$ has some overlap with the vacuum, $|q\rangle = \alpha |0\rangle + \cdots$, where the ellipsis stands for higher energy eigenstates. Now, since in

$$e^{-\epsilon T H} |q\rangle = e^{-\epsilon T H} (\alpha |0\rangle + \cdots) = (\alpha e^{-\epsilon T E_0} |0\rangle + \cdots) \tag{1.35}$$

the vacuum is the **least-suppressed** term on the r.h. side, we have

$$\langle q | e^{-iH'(T-t)} \hat{q} \, e^{-iH'(t+T)} | q \rangle \sim \langle 0 | e^{-iH'(T-t)} \hat{q} \, e^{-iH'(t+T)} | 0 \rangle \, , \tag{1.36}$$

at large T. Note that we are unable to keep track of the normalization since we don't know α. We now replace the l.h. side by our familiar path integral expression, finding

$$\int Dq \, q(t) \, e^{iS'[q]} \sim \langle 0 | \hat{q}_t | 0 \rangle' \, . \tag{1.37}$$

The prime indicates that everything is defined using our modified Hamiltonian H'. Furthermore, the boundary conditions on the l.h. side are defined by demanding $q(-T) = q(T) = q$, with the limit $T \to \infty$ taken at the end. We take this to mean that there are no boundary conditions, i.e. the integral is over *all* functions. That is also consistent with the fact that the q-dependence has dropped out on the r.h. side.

Finally, we generalize to $m > 1$ and take the limit $\epsilon \to 0$:

$$\langle 0|\, T\, \hat{q}_{t_m} \cdots \hat{q}_{t_1} |0\rangle \sim \lim_{\epsilon\to 0} \int Dq\, q(t_1)\cdots q(t_m)\, e^{iS'[q]}\,. \tag{1.38}$$

Here the time-ordering symbol T on the l.h. side replaces our previous assumption that $t_m > \cdots > t_2 > t_1$. Note that the 'natural' (from the path-integral point-of-view) expression on the r.h. side *automatically* gives the time-ordered correlation function which, as we have seen in QFT I, is so central in applications.

Note that one might think that (1.37) is meaningless since it does not depend on t by time-translation invariance and thus it appears to contain no information beyond the (unknown!) normalization. The point is that our derivation of (1.37) was just a shorthand for the analogous derivation of (1.38), with its non-trivial content encoded in the (relative) time dependence.

Before closing this section, we still need to understand how S' differs from the action S used before. This is easy if we recall that H originally always appeared in exponential factors $\exp(-iH\Delta)$. Thus, replacing H by $(1-i\epsilon)H$ is equivalent to the replacement

$$\Delta \to (1-i\epsilon)\Delta \qquad \text{or} \qquad t \to (1-i\epsilon)t\,, \tag{1.39}$$

after returning to continuum notation. Hence

$$\begin{aligned} \exp(iS') &= \exp i \int dt(1-i\epsilon)\left[\frac{m}{2}\left(\frac{dq}{dt}\right)^2\left(\frac{1}{1-i\epsilon}\right)^2 - V(q)\right] &(1.40)\\ &\simeq \exp i \int dt \left[\frac{m}{2}\left(\frac{dq}{dt}\right)^2 (1+i\epsilon) - V(q)(1-i\epsilon)\right]\,. &(1.41)\end{aligned}$$

We see that both large values of $\dot{q}$ and of $V(q)$ are exponentially suppressed. This turns the fast oscillations of e^{iS} for quickly changing S into exponential suppression. The '$i\epsilon$' hence has the intuigtive meaning of a **convergence factor**, which in particular suppresses all non-vacuum contributions at very early and late times.

In fact, for $V(q) = (k/2)q^2 + \cdots$, it is sufficient to keep only the $i\epsilon$ multiplying the force constant k to achive this convergence effect. Jumping ahead, we note that this $(k/2)q^2$-term corresponds to $(m_{qft}^2/2)\phi^2$ in the analogy of a harmonic-oscillator-system with a scalar field theory.[2] Hence, we record for the future that S' simply means for us that we will always use

$$m_{qft}^2 \quad \to \quad m_{qft}^2 - i\epsilon \tag{1.42}$$

whenever this small imaginary part affects the result. In other words, the convergence factor $i\epsilon$ which we introduced above can be identified with the one from the $i\epsilon$-prescription generating the Feynman propagator in QFT.

It is clear from our discussion that one can also characterize the $i\epsilon$-effect by saying that the time integration contour is rotated according to Fig. 2. In fact, this logic can

[2]Do not confuse m_{qft} with the QM particle mass m.

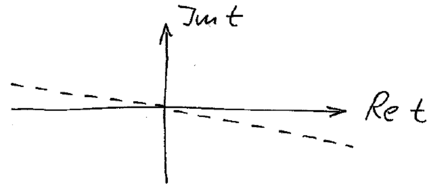


Figure 2: Time integration contour rotated by $(1 - i\epsilon)$.

be developed by letting the rotation angle grow to 90°. This is called **Wick rotation**. The result is a so-called **euclidean theory** with (euclidean) time evolution defined by the operator

$$\exp(-H\tau) \tag{1.43}$$

and the path integral defined with an exponential factor involving the euclidean action S_E,

$$e^{iS} \to e^{-S_E} \qquad \text{with} \qquad S_E \equiv \int d\tau \left[\frac{m}{2} \left(\frac{dq}{d\tau} \right)^2 + V(q) \right] . \tag{1.44}$$

The derivation of the path integral formula for amplitudes and correlation functions proceeds exactly as in this and the previous section, just with $\Delta \to -i\Delta$ and all corresponding modifications which follow from this.

We close this section by giving a version of the path integral formula where the normalization issues associated with the unknown constant α and with the $1/C(\Delta)$ prefactors drop out:

$$\frac{\langle 0 | \, T \, \hat{q}_{t_1} \cdots \hat{q}_{t_m} | 0 \rangle}{\langle 0 | 0 \rangle} = \frac{\int Dq \, q(t_1) \cdots q(t_m) \, e^{iS[q]}}{\int Dq \, e^{iS[q]}} . \tag{1.45}$$

Here and below we write S instead of S', with the $i\epsilon$ and the limiting procedure $\epsilon \to 0$ implicit.

Many modern QM textbooks (e.g. those by Sakurai [2] and Münster [3]) have chapters on the path integral approach. For an in-depth treatment cf. Schulman's book [4] (see also [5–7]).

1.4 Functional integral for the scalar field

Recall that

$$S = \int dt \, L \qquad \text{with} \qquad L = \int d^3x \left(\frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi - V(\varphi) \right) . \tag{1.46}$$

We write L explicitly in '$T - V$' form,

$$L = \int d^3x \left(\frac{1}{2} \dot{\varphi}^2 - \frac{1}{2} (\vec{\nabla} \varphi)^2 - V(\varphi) \right) , \tag{1.47}$$

and legendre-transform to obtain

$$H = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + V(\varphi) \right) , \tag{1.48}$$

where $\pi = \dot{\varphi}$. This can be put on a spatial lattice with lattice spacing Δ, i.e., we replace the continuous variable $\vec{x}$ by $\vec{x} = \vec{n} \cdot \Delta$ with $\vec{n} \in \mathbb{Z}^3$. Now we have

$$H = \sum_{\vec{x}} \left(\frac{1}{2}\pi^2_{\vec{x}} + \frac{1}{2}(\vec{\nabla}\varphi)^2_{\vec{x}} + V(\varphi(\vec{x})) \right) , \tag{1.49}$$

where $(\vec{\nabla}\varphi)_{\vec{x}}$ is a 3-vector with components

$$(\vec{\nabla}\varphi)_{\vec{x},\,i} = \frac{1}{\Delta}\Big(\varphi(\vec{x} + \hat{e}_{(i)} \cdot \Delta) - \varphi(\vec{x}) \Big) . \tag{1.50}$$

This is the Hamiltonian of a many-particle system with coordinates $\varphi_{\vec{x}} \equiv \varphi(\vec{x})$ and conjugate momenta $\pi_{\vec{x}} \equiv \pi(\vec{x})$. Here $\vec{x}$ is simply a discrete label, but we will find it more convenient notationally to keep writing it as an argument. The Hamiltonian can also be written as

$$H = \sum_{\vec{x}} \frac{1}{2}\pi(\vec{x})^2 + \tilde{V}(\, \{\varphi(\vec{x}),\, \vec{x} \in \text{Lattice}\}\,) . \tag{1.51}$$

Here $\tilde{V}$ is the potential of this many-particle system. It is defined by comparing with (1.49) and includes the gradient energy $(1/2)\sum(\vec{\nabla}\varphi)^2_{\vec{x}}$ of the field-theory.

Now we have made explicit that we are dealing simply with a set of many QM particles with canonical kinetic terms (with masses $m = 1$). We can hence apply our previous derivation of the path-integral formula, the only change being the transition to many variables. Explicitly,

$$|q\rangle \quad \to \quad |q_1 \cdots q_n\rangle \quad \to \quad |\, \{\varphi(\vec{x}),\, \vec{x} \in \text{Lattice}\}\, \rangle \tag{1.52}$$

$$|p\rangle \quad \to \quad |p_1 \cdots p_n\rangle \quad \to \quad |\, \{\pi(\vec{x}),\, \vec{x} \in \text{Lattice}\}\, \rangle , \tag{1.53}$$

where the first arrow corresponds to the transition to an n-particle system in standard QM notation and the second arrow to the transition to discretized field theory. (Note that we can always think of a finite volume and hence finite number of lattice points, taking the infinite volume limit at the very end.) To make the analogy even more clear, let's for example give the action of a field operator at some specific lattice point $\vec{y}$:

$$\hat{\varphi}(\vec{y})\,|\,\{\varphi(\vec{x}),\, \vec{x} \in \text{Lattice}\}\,\rangle \;=\; \varphi(\vec{y})\,|\,\{\varphi(\vec{x}),\, \vec{x} \in \text{Lattice}\}\,\rangle . \tag{1.54}$$

Similarly, one can write down expressions analogous to all the formulae of Sect. 1.1. Let us give one more example concerning the normalization of position eigenstates (which now become field eigenstates):

$$\langle q'|q\rangle = \delta(q' - q) \quad \to \quad \cdots \quad \to \quad \langle \varphi'|\varphi\rangle \;=\; \prod_{\vec{x}} \delta\big(\varphi'(\vec{x}) - \varphi(\vec{x})\big) . \tag{1.55}$$

Here we also introduced the abbreviation

$$|\varphi\rangle \equiv |\{\varphi(\vec{x}),\, \vec{x} \in \text{Lattice}\}\rangle\,. \tag{1.56}$$

With these preliminaries, it is mostly an excercise in carefully keeping track of notation to repeat the derivation of Sect. 1.2. The result, which the reader should now be able to derive in all detail, is

$$\begin{aligned}\langle\varphi_b|e^{-iH(t_b-t_a)}|\varphi_a\rangle \;&=\; \prod_{\vec{x}}\left(\int D\varphi(\vec{x})\right)\,\exp i\sum_i\left\{\sum_{\vec{x}}\left[\frac{1}{2}\left(\frac{\varphi_{i+1}(\vec{x})-\varphi_i(\vec{x})}{\Delta_t}\right)^2\right.\right. \\ &\qquad\left.\left. -\tilde{V}(\{\varphi_i(\vec{x}),\vec{x}\in\text{Lattice}\})\right]\right\}\Delta_t\,.\end{aligned} \tag{1.57}$$

Here the time interval length is Δ_t (not to be confused with the lattice spacing Δ) and, as before,

$$\int D\varphi(\vec{x}) \equiv \frac{1}{C(\Delta_t)}\prod_i\left(\int\frac{d\varphi_i(\vec{x})}{C(\Delta_t)}\right)\,. \tag{1.58}$$

In analogy to the discrete version q_i of the function $q(t)$, we now view $\varphi_i(\vec{x})$ as the discrete version of the function $\varphi(t,\vec{x})$ (for each $\vec{x}$). With this, it is clear how to return to continuum notation for both time and space (taking the limits Δ_t, $\Delta\to 0$):

$$\langle\varphi_b|e^{-iH(t_b-t_a)}|\varphi_a\rangle \;=\; \int D\varphi\, e^{iS[\varphi]}\,, \tag{1.59}$$

where

$$S[\varphi] \;=\; \int_{t_a}^{t_b} dt\, L[\varphi(t),\dot{\varphi}(t)] \;=\; \int_{t_a}^{t_b} dt\int d^3x\,\mathcal{L}(\varphi,\partial\varphi)\,. \tag{1.60}$$

Here the argument $\varphi(t)$ of L is the function $\vec{x}\mapsto\varphi(t,\vec{x})$ (and similarly for $\dot{\varphi}(t)$). The symbol $\int D\varphi$ means 'summation' over all smooth functions $\varphi(x)\equiv\varphi(t,\vec{x})$ satisfying $\varphi(t_a,\vec{x}) = \varphi_a(\vec{x})$ and $\varphi(t_b,\vec{x}) = \varphi_b(\vec{x})$. In other words, one integrates over all time-dependent fields or 'fields histories' which interpolate between two given field configurations φ_a and φ_b at some initial and final time. A pictorial representation of this is attempted in Fig. 3

We can also repeat the discussion of Sect. 1.3, which in the field theory case leads to the important formula (assuming $\langle 0|0\rangle = 1$)

$$\langle 0|\,T\,\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_m)|0\rangle \;=\; \frac{\int D\varphi\,\varphi(x_1)\cdots\varphi(x_m)\,e^{iS[\varphi]}}{\int D\varphi\,e^{iS[\varphi]}}\,. \tag{1.61}$$

Here

$$S = \int d^4x\left[\frac{1}{2}(\partial\varphi)^2 - V(\varphi)\right] \qquad\text{and}\qquad V(\varphi) = \frac{1}{2}(m_{qft}^2 - i\epsilon)\varphi^2 + \mathcal{O}(\varphi^3)\,. \tag{1.62}$$

The field operators are Heisenberg picture fields of the fully interacting theory such that (1.61) captures the full dynamics. As explained earlier, the $i\epsilon$ term suppresses the

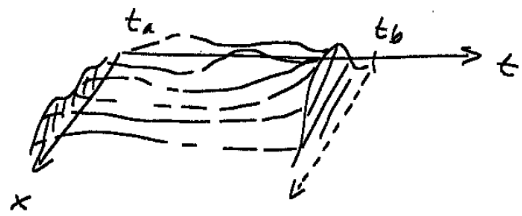


Figure 3: One of the many functions $\varphi(t,x)$ interpolating between $\varphi_a(x) = \varphi(t_a, x)$ and $\varphi_b(x) = \varphi(t_b, x)$.

contribution of non-vacuum states in the limit $t_{b/a} \to \pm\infty$. Technically, this happens because the resulting factor $\exp(-\int \epsilon\varphi^2/2)$ suppresses states with larger φ more strongly.

The crucial conceptual progress we have achieved can be summarized as follows: Let's accept, one the basis of QM experience, that the path integral is a valid approach to quantizing a theory. Then (1.61) is an immediate and natural generalization (without the notationally painful derivation we sketched above) of the corresponding QM formula. It is automatically fully covariant and can be taken to *define* quantum field theory (given of course that we know how to evaluate functional integrals – which we will learn). The only reason we need to refer back to QFT I is the LSZ formula, which we still require to relate correlation functions to scattering amplitudes.

Finally, we note that most of what we will do in the next chapters is **perturbation theory** in the path integral approach. However, our previous discretized derivation of QFT path integral formulas can be taken as a non-perturbative definition. This requires that one can (in many cases numerically) perform very high-dimensional integrals and approximate the limits of large volume (or equivalently $\Delta, \Delta_t \to 0$) sufficiently well. That is arguably our best way to actually mathematically define QFT. The whole area of research is called **lattice field theory** (or **lattice gauge theory**, since the main interest is in gauge field theories), see e.g. [10].

1.5 Schrödinger wave functional and other comments

So far, we only considered $\hat{q}$-eigenstates as initial and final states in our QM path integral formula for amplitudes,

$$\langle q_b | e^{-iH(t_b - t_a)} | q_a \rangle \; = \; \int_{q_a, q_b} Dq \, e^{iS[q]} \, . \tag{1.63}$$

Here the indices of the integration sign remind us of the boundary conditions $q(t_{a/b}) = q_{a/b}$. This is easy to generalize to arbitrary states using

$$|\psi\rangle \; = \; \int dq \, \psi(q) \, |q\rangle \, . \tag{1.64}$$

Note that this relation is obvious since $|q\rangle$ corresponds to the wave function $q' \mapsto \delta(q'-q)$. Now we can multiply (1.63) by arbitrary wave functions $\psi_a(q_a)$ and $\psi_b^*(q_b)$ and integrate to obtain

$$\int dq_a \int dq_b\ \psi_b^*(q_b)\langle q_b|e^{-iH(t_b-t_a)}|q_a\rangle\psi_a(q_a) \;=\; \int dq_a \int dq_b\ \psi_b^*(q_b)\psi_a(q_a)\int_{q_a,q_b} Dq\, e^{iS[q]}\,. \tag{1.65}$$

In other words, we have the general path integral formula for amplitudes

$$\langle\psi_b|e^{-iH(t_b-t_a)}|\psi_a\rangle \;=\; \int dq_b\ \psi_b^*(q_b)\int dq_b\ \psi_a(q_a)\int_{q_a,q_b} Dq\, e^{iS[q]}\,. \tag{1.66}$$

It is intuitively clear that, on the r.h. side, we now sum over *all* paths $q(t)$, weighted with the e^{iS} and with the Schrödinger wave functions ψ_a and ψ_b^* for initial and final state respectively.

Next, we recall the QFT analogue of (1.63),

$$\langle\varphi_b|e^{-iH(t_b-t_a)}|\varphi_a\rangle \;=\; \int_{\varphi_a,\varphi_b} D\varphi\, e^{iS[\varphi]}\,, \tag{1.67}$$

where the indices of the integration sign again remind us of the boundary conditions. The initial state $|\varphi_a\rangle$ is a state in which the field value at every position is exactly known. Let's drop the index a for now and write this state, which can be viewed as a product of 'position' (actually, field-value) eigenstates, explicitly as

$$|\varphi\rangle \;=\; \prod_{\vec{x}}|\varphi(\vec{x})\rangle\,. \tag{1.68}$$

Here the many labels $\varphi(\vec{x})$ are nothing but many q_i's, generalizing the single lable q to many-particle QM. If we want to describe a general initial state, we must as before multiply by a wave function and integrate, i.e. our state will be

$$|\Psi\rangle \;=\; \prod_{\vec{x}}\left(\int d\varphi(\vec{x})\right)\ \Psi(\{\varphi(\vec{x}),\,\vec{x}\in\text{Lattice}\})\,|\varphi\rangle\,. \tag{1.69}$$

Returning to the continuum point of view, we see that the Schrödinger wave function Ψ on the r.h. side, with its huge number of arguments, is actually a functional: It takes a field configuration $\varphi(\vec{x})$ as its argument and returns a complex number. We will call this a **Schrödinger wave functional**:

$$|\Psi\rangle \qquad \text{is caracterized by} \qquad \Psi\,:\,\varphi\mapsto\Psi[\varphi]\,, \tag{1.70}$$

where φ depends only on space, not on time. Ψ characterizes a general state in QFT, just as ψ does in QM. Thus, a more general general QFT path integral formula reads

$$\langle\Psi_b|e^{-iH(t_b-t_a)}|\Psi_a\rangle \;=\; \int D\varphi_b\,\Psi_b^*[\varphi_b]\int D\varphi_a\,\Psi_a[\varphi_a]\int_{\varphi_a,\varphi_b} D\varphi\, e^{iS[\varphi]}\,. \tag{1.71}$$

We sum over all field 4-dimensional field configurations $\varphi(t,\vec{x})$ with $t \in [t_a, t_b]$, weighted by e^{iS} and two wave functionals for initial and final state.

To play a bit more with this new notation, let's return to our previous formula be assuming that each of our wave functionals Ψ is non-zero only for one particular classical field configuration. In other words, consider δ-functionals of the form

$$\Psi_{a,b}[\varphi] = \delta[\varphi - \varphi_{a,b}] \equiv \prod_{\vec{x}} \delta\big(\varphi(\vec{x}) - \varphi_{a,b}(\vec{x})\big)\,. \tag{1.72}$$

It is clear that, inserting this in (1.71) gives us (1.67).

Finally, as an independent remark, it may be useful to note that the path integral can be defined not by discretiziation but using **Fourier coefficients** or some analogue thereof. For example, return to QM and decompose $q(t)$ on the interval $[t_a, t_b]$ into a basis of orthonormal functions $q_i(t)$ consistent with the boundary conditions,

$$q(t) = \sum_i \lambda_i q_i(t)\,. \tag{1.73}$$

Then

$$\int Dq\, e^{iS[q]} \quad \to \quad \prod_i \left(\int d\lambda_i\right)\, e^{iS(\lambda_1,\lambda_2,\cdots)}\,. \tag{1.74}$$

An analogous expansion can of course be used in field theory, the only difference being that φ has more arguments and hence i might be a 'multi-index',

$$\int D\varphi\, e^{iS[\varphi]} \quad \to \quad \prod_i \left(\int d\lambda_i\right)\, e^{iS(\lambda_1,\lambda_2,\cdots)}\,, \tag{1.75}$$

where λ_i might correspond to the Fourier modes $\tilde{\phi}(k)$ and the 4-vector k to the index i.

Also, in this context a δ-functional takes the explicit form

$$\delta[\varphi - \varphi^0] = \prod_i \delta(\lambda_i - \lambda_i^0)\,, \tag{1.76}$$

where λ_i^0 are the (Fourier) expansion coefficients of the function φ^0. Clearly, much more could and should be said about path integrals in QM and QFT, but we have to move on.

2 Feynman rules in the path integral approach

2.1 The generating functional

Recall first some basic facts about functional differentiation. Let F be a functional, i.e. a map $F:\, j \mapsto F[j]$, with functions $j: x \mapsto j(x)$. Here $F[j]$ is a number in $\mathbb{R}$ or $\mathbb{C}$.

The functional derivative

$$\frac{\delta F[j]}{\delta j(x)} \tag{2.1}$$

is a function of x (i.e. a number for any fixed x) defined by

$$F[j+\delta j]-F[j]=\int d^4x\,\frac{\delta F[j]}{\delta j(x)}\cdot\delta j(x)+\mathcal{O}\big((\delta j)^2\big)\,. \tag{2.2}$$

Obviously, for fixed x we can also think of $\delta F[j]/(\delta j(x))$ as of a functional depending on j. Two simple examples are

$$a)\qquad F[j]=j(y)\quad;\qquad \frac{\delta}{\delta j(x)}j(y)=\delta^4(x-y)\,, \tag{2.3}$$

$$b)\qquad F[j]=\int d^4y\,j(y)\,\varphi(y)\quad;\qquad \frac{\delta}{\delta j(x)}\int d^4y\,j(y)\,\varphi(y)=\varphi(x)\,, \tag{2.4}$$

where $b)$ is an application of $a)$, using linearity. It will be helpful to always keep in mind the analogy with the simpler, discrete case,

$$\text{continuous } x\qquad\leftrightarrow\qquad\text{discrete set }\{x^i\}\,, \tag{2.5}$$

in which case j is characterized by a discrete set of values j^i and hence the above examples become

$$a)\qquad \frac{\partial}{\partial j^i}\,j^k=\delta_i{}^k \tag{2.6}$$

$$b)\qquad \frac{\partial}{\partial j^i}\sum_k j^k\varphi^k=\varphi^i\,. \tag{2.7}$$

After these preliminaries, let us define the generating functional Z of a scalar field theory with lagrangian $\mathcal{L}$ as

$$Z[j]=\int D\varphi\exp i\int d^4x\,\big(\mathcal{L}(\varphi,\partial\varphi)+j(x)\varphi(x)\big)\,. \tag{2.8}$$

The **crucial feature**, explaining the name of this object (it generates Green's functions), is

$$\langle 0|T\,\varphi(x_1)\cdots\varphi(x_n)|0\rangle=\frac{1}{Z[0]}\left(\frac{\delta}{i\delta j(x_1)}\right)\cdots\left(\frac{\delta}{i\delta j(x_n)}\right)Z[j]\bigg|_{j=0}\,, \tag{2.9}$$

where we assumed $\langle 0|0\rangle=1$.

The **demonstration** of this feature is straightforward. Note first that

$$\frac{\delta}{i\delta j(x_1)}\int D\varphi\exp i\int d^4x(\mathcal{L}+j(x)\varphi(x))=\int D\varphi\,\varphi(x_1)\exp i\int d^4x(\mathcal{L}+j(x)\varphi(x))\,. \tag{2.10}$$

This can be derived from the definition using $\exp(a+\Delta a)-\exp(a)\simeq\Delta a\,\exp(a)$. It can also be viewed as an application of the chain rule for functional differentiation. Next, apply further differentiations with $\delta/(i\delta j(x_2))$ up to $\delta/(i\delta j(x_n))$. This brings down a factor $\varphi(x_1)\cdots\varphi(x_n)$ under the path integral. Noting also that, by definition,

$$Z[0] = \int D\varphi e^{iS}\,, \tag{2.11}$$

the result follows from our previously derived path integral formula for Green's functions. $\Box$

2.2 Generating functional for the free field

We start by rewriting the free action $S_0 = \int d^4x\mathcal{L}_0$ as

$$\begin{aligned} S_0 &= \frac{1}{2}\int d^4x\,\varphi(x)(-\Box - m^2 + i\epsilon)\varphi(x) && (2.12)\\ &= \frac{1}{2}\int d^4x \int d^4y\,\varphi(x)(-\Box_x - m^2 + i\epsilon)\delta^4(x-y)\varphi(y) && (2.13)\\ &= \frac{1}{2}\int d^4x \int d^4y\,\varphi(x)\,iD_F^{-1}(x-y)\varphi(y)\,, && (2.14)\end{aligned}$$

where

$$iD_F^{-1}(x-y) = (-\Box_x - m^2 + i\epsilon)\delta^4(x-y)\,. \tag{2.15}$$

The last equality can be understood as a definiton of the function D_F^{-1}, but it can also be easily checked that this is indeed the inverse operator ('inverse matrix') of the Feynman propagator D_F encountered earlier:

$$\int d^4y\,D_F^{-1}(x-y)\,D_F(y-z) = \delta^4(x-z)\,. \tag{2.16}$$

Here it is useful to visualize the close analogy to the matrix formula $(M^{-1})^i{}_j M^j{}_k = \delta^i{}_k$.

This is a good place for a small mathematical detour: We can think of D_F and D_F^{-1} as of functions (more generally, distributions) of two arguments, x and y. As such, they are 'integral kernels' of linear operators acting on functions or 'kernel representations' of such operators. (Do not confuse with the 'kernel of an operator' in the usual sense.) We describe this using a more general notation: $D_F/D_F^{-1} \to A$. Indeed, let a linear operator A be defined by

$$A : f(x) \qquad \mapsto \qquad g(x) = \int d^4y\,A(x,y)f(y)\,. \tag{2.17}$$

Clearly, $A(x,y)$ is analogous to a (continuous-index) matrix representing an operator. We can also look at the corresponding map for the Fourier-transforms:

$$\tilde{f}(p) = \int d^4x\,e^{ipx}f(x) \qquad \mapsto \qquad \tilde{g}(p) = \int d^4q\,\tilde{A}(p,q)\tilde{f}(q)\,. \tag{2.18}$$

For this to hold we need

$$\tilde{A}(p,q) = \int \frac{d^4x\, d^4y}{(2\pi)^4} e^{i(px-qy)} A(x,y)\,, \tag{2.19}$$

as you should check explicitly. Applying this to D_F^{-1} we find

$$\tilde{D}_F^{-1}(p,q) = -i(p^2 - m^2 + i\epsilon)\delta^4(p-q) \tag{2.20}$$

and hence

$$\tilde{D}_F(p,q) = \frac{i}{p^2 - m^2 + i\epsilon}\delta^4(p-q)\,, \tag{2.21}$$

which you should again check.

While talking about D_F^{-1} and its inverse D_F, a further set of comments might be appropriate: Note that D_F^{-1} is, of course, a differential operator, although in the above we emphasized it's kernel representation $D_F^{-1}(x,y)$. Let's consider, more generally, some differential operator L and it's inverse G, acting on an appropriate function space, e.g. the square-integrable functions on $\mathbb{R}^4$. Then

$$L \cdot G = \mathbb{1} \qquad \text{and} \qquad L_x G(x,y) = \delta^4(x-y), \tag{2.22}$$

where the first expression is a general statement about operators and the second is adopted to the case that L_x acts by differentiation in x. Clearly, knowledge of G allows one to find a function g which, for a given function f, solves

$$L \cdot g = f\,. \tag{2.23}$$

Indeed $g = G \cdot f$ does the job. Hence the 'Green's function' G is such an important object. Now, in many cases, L annihilates a certain subspace of the relevant function space. If L were a finite dimensional matrix, this would simply mean that the inverse, in this case G, does not exist. However, in the infinite-dimensional case, the left-inverse will indeed cease to exist ($H \cdot L = \mathbb{1}$ has no solution H), but the right-inverse G exists. Roughly speaking, G makes use of the infinite-dimensionality of the space to 'free' the zero-eigenvalue subspace of L of any information, such that the combination $L \cdot G$ does not destroy any 'information' and can be equal to the unity operator. However, because of this zero-subspace, the inverse or Green's function G is not unique. Specifically, if $L = i(\Box + m^2)$, without the $i\epsilon$, this inverse $G = L^{-1}$ involves the choice of the pole-prescription, leading to advanced, retarded and Feynman propagators. Here, we adopt the perspective (appropriate for our path-integral approach) that $L = D_F^{-1}$ already comes with an $i\epsilon$, hence it has no zero-subspace and an unambiguous (left and right) inverse $G = D_F^{-1}$, also with the apprpriate $i\epsilon$.

Now we return to our main line of development and introduce a set of convenient shorthand notations for our last expression for S_0:

$$iS_0 = i\int d^4x\, \mathcal{L}_0 = -\frac{1}{2}\varphi_x \left(D_F^{-1}\right)_{xy} \varphi_y = -\frac{1}{2}\varphi^T D_F^{-1}\varphi = -\frac{1}{2}\varphi D_F^{-1}\varphi\,. \tag{2.24}$$

This is borrowed from the standard matrix notation

$$\sum_{ij} a_i M_{ij} b_j = a^T M b \,, \tag{2.25}$$

where in the last expression in (2.24) we even dropped the T for 'transpose' since it is really obvious how to contract the 'indices' x and y.

To further simplify notation, we let $D_F \to D$, such that the generating functional of the free theory now reads

$$Z_0[j] = \int D\varphi \, \exp\left(-\frac{1}{2}\varphi^T D^{-1}\varphi + i j^T \varphi\right) . \tag{2.26}$$

The change of integration variable,

$$\varphi \quad \to \quad \varphi + iDj \,, \tag{2.27}$$

(in more detail,

$$\varphi(x) \quad \to \quad \varphi(x) + i\int d^4y \, D(x-y) j(y) \,, \tag{2.28}$$

but fortunately we don't need this bulky notation any more) gives

$$\begin{aligned} Z_0[j] \;&=\; \int D\varphi \, \exp\Big(-\frac{1}{2}\varphi^T D^{-1}\varphi - \frac{i}{2} j^T D^T D^{-1}\varphi - \frac{i}{2}\varphi^T D^{-1} D j & (2.29) \\ &\qquad + \frac{1}{2} j^T D^T D^{-1} D j + i j^T \varphi - j^T D j \Big) . & (2.30) \end{aligned}$$

Making use of $D^T = D$ (check this explicitly!), we see that we have actually completed the square, such that

$$Z_0[j] \;=\; \int D\varphi \, \exp\left(-\frac{1}{2}\varphi^T D^{-1}\varphi - \frac{1}{2} j^T D j\right) \;=\; Z_0[0] \, \exp\left(-\frac{1}{2} j^T D j\right) . \tag{2.31}$$

We can now show that D is the Feynman propagator without reference to its explicit form (introducing some further shorthand notation as we go along):

$$\begin{aligned} \langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle \;&=\; \langle T\varphi_1\varphi_2\rangle = \frac{1}{Z_0}\left(\frac{\delta}{i\delta j_1}\right)\left(\frac{\delta}{i\delta j_2}\right) Z_0 \exp^{-\frac{1}{2}jDj}\Bigg|_{j=0} & (2.32) \\ &=\; -\frac{\delta}{\delta j_1}\left(-\frac{1}{2} j_x D_{xx_2} - \frac{1}{2} D_{x_2 x} j_x\right) \exp^{-\frac{1}{2}jDj}\Bigg|_{j=0} & (2.33) \\ &=\; \frac{\delta}{\delta j_1}\Big(j_x D_{xx_2}\Big) \;=\; D_{x_1 x_2} \;=\; D_{12} \;=\; D(x_1 - x_2) \,. & (2.34) \end{aligned}$$

Note in particular that we write $\delta/(\delta j_1)$ for $\delta/(\delta j(x_1))$ etc. Also, recall the corresponding Feynman diagram in Fig. 4.

Figure 4: Propagator.

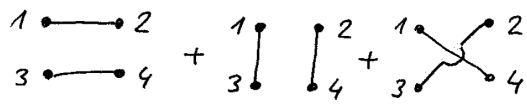


Figure 5: Free-theory 4-point function.

The next-simplest application of the free-theory generating functional is

$$\langle T\varphi_1\varphi_2\varphi_3\varphi_4\rangle \;=\; \frac{\delta}{i\delta j_1}\cdots\frac{\delta}{i\delta_2}\,e^{-\frac{1}{2}jDj}\bigg|_{j=0} \;=\; D_{12}D_{34}+D_{13}D_{24}+D_{14}D_{23}\,, \tag{2.35}$$

with the diagram representing the last expression in a self-evident way given in Fig. 5. Please check explicitly how this expression arises from (functionally) differentiating the exponential.

The underlying general rules (Feynman rules) should now have become clear: Draw the relevant points (in this case $x_1\cdots x_4$). Connext them pairwise in all possible ways by lines. Write $D_{ij}=D(x_i-x_j)$ for a line connecting x_i with x_j. Add all such contributions. It should also be clear that these rules reflect precisely what happens in deriving the analytical expression by differentiation. The key is that each $\delta/(\delta j)$ corresponds to a point and each j to an end of a line, and that all must be paired up for a non-zero result since we set $j=0$ at the end. Thus, one can Taylor-expand the exponent and focus only on the term with the right humber of j's. The factorial and the (1/2)-prefactors take care of multiple possiblities leading to the same final result. Hence, no prefactors are left in the end.

2.3 Generating functional for interacting theories

We will focus on a single scalar field, more specifically, on $\lambda\varphi^4$-theory, but the method generalizes straightforwardly. Consider

$$Z[j]=\int D\varphi\,\exp\left(iS_0[\varphi]+iS_{int.}[\varphi]+ij^T\varphi\right)\,, \tag{2.36}$$

where $iS_0=-(1/2)\varphi^TD^{-1}\varphi$ and, for example,

$$iS_{int.}=-i\int d^4x\frac{\lambda}{4!}\varphi(x)^4\,. \tag{2.37}$$

Rewrite the exponent as

$$e^{iS_{int.}[\varphi]}\;e^{iS_0[\varphi]+ij^T\varphi} \tag{2.38}$$

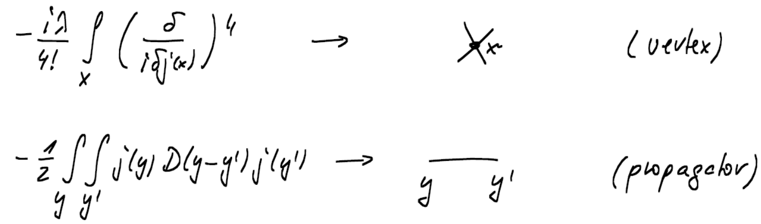


Figure 6: Illustration of how vertex and propagator Feynman rules arise.

and view $\exp(iS_{int.})$ as a power series in φ, i.e.

$$e^{iS_{int.}} = 1 + (-i)\int d^4x \frac{\lambda}{4!}\varphi(x)^4 + (-i)\int d^4x \frac{\lambda}{4!}\varphi(x)^4 \,(-i)\int d^4y \frac{\lambda}{4!}\varphi(y)^4 + \cdots\,. \tag{2.39}$$

The **key idea** is that one can replace each $\varphi(x)$ by a $\delta/(i\delta j(x))$, acting on $\exp(ij\varphi)$. Thus,

$$Z[j] = \int D\varphi \, \exp\left(iS_{int.}\left[\frac{\delta}{i\delta j}\right]\right) \exp\left(iS_0[\varphi] + ij\varphi\right) \tag{2.40}$$

$$= \exp\left(iS_{int.}\left[\frac{\delta}{i\delta j}\right]\right) Z_0[j] \;=\; Z_0[0]\,\exp\left(iS_{int.}\left[\frac{\delta}{i\delta j}\right]\right) e^{-\frac{1}{2}jDj}\,. \tag{2.41}$$

In actual calculations, it will be useful to think of Taylor expansions of the exponentials, i.e.

$$Z[j] = Z_0[j]\left(1 - \frac{i\lambda}{4!}\int_x \left(\frac{\delta}{i\delta j(x)}\right)^4 + \cdots\right)\left(1 - \frac{1}{2}jDj + \frac{1}{2}\left(-\frac{1}{2}jDj\right)^2 + \cdots\right). \tag{2.42}$$

To begin, let's evaluate $Z[0]$, i.e. simply set $j = 0$ after all j-differentiations have been performed. Clearly, there will be infinitely many terms in our perturbative approach. Thus, we set the prefactor $Z_0[0]$ aside for now and start with the term linear in λ:

$$-\frac{i\lambda}{4!}\int_x \left(\frac{\delta}{i\delta j(x)}\right)^4 \frac{1}{2}\left(-\frac{1}{2}\int_y\int_{y'} j(y)D(y-y')j(y')\right)\left(-\frac{1}{2}\int_z\int_{z'} j(z)D(z-z')j(z')\right). \tag{2.43}$$

As an excercise, this should be worked out explicitly. However, the whole point of Feynman rules is of course to systematize such calculations by drawing pictures. Indeed, a preliminary identification of vertex and propagator is given in Fig. 6. Clearly, each differentiation attaches an end of one of the lines to the vertex associated with this differentiation. The corresponding two space-time points are then identified (cf. Fig. 7).

The resulting diagram corresponds to the analytical expression

$$I_8 = -i\lambda\int d^4x D(x-x)D(x-x) \;=\; -i\lambda\int d^4x D(0)^2\,. \tag{2.44}$$

Figure 7: Differentiation corresponds at attaching a line-end to a vertex. The resulting figure-of-eight diagram is shown on the right.

The divergence of $D(x)$ at $x \to 0$ is a standard UV divergence, to be dealt with, e.g., in momentum space by dimensional regularization as already discussed. The divergent x-integration signals (correctly) the proportionality of Z to the space-time volume. Crucially, we have derived that the vertex corresponds precisely (up to combinatorial factors) to $-i\lambda \int d^4x$ and the line attached to vertices at x and y corresponds to $D(x-y)$.

As in our Wick-theorem-based derivation of Feynman rules, we naively expect that the no combinatorial factors arise: The factor $(1/2)$ takes care of exchanging the two ends of one line. The factor $(1/4!)$ takes care of permuting the 4 ends of a vertex. The factors $(1/n!)$ from the Taylor series of the exponential take care of permuting several vertices and lines appearing in the same diagram.

However, this expectations fails in diagrams with symmetries. Indeed, the figure-of-eight diagram comes with a symmetry factor $(1/8)$ such that

$$Z[0] = Z_0[0] \left(1 + \frac{1}{8} I_8 + \mathcal{O}(\lambda^2)\right) . \tag{2.45}$$

The explicit underlying calculation is illustrated in Fig. 8: At each step, one picks one of the vertex-lines (i.e. one of the $\delta/(\delta j)$'s) and lets it act on the propagator line-ends (the j's). The prefactors come from the different choices leading to the same picture. In the end, one multiplies with the standard prefactor,

$$(4 \cdot 2 + 4 \cdot 2 \cdot 2) \cdot \frac{1}{2 \cdot 2 \cdot 4! \cdot 2!} = \frac{1}{8} . \tag{2.46}$$

At order λ^2, the diagrams of Fig. 9 arise, one of them disconnected (are there more?). It is clear that, at any order, only diagrams without external lines (vacuum diagrams) contribute. Hence

$$Z[0] = Z_0[0] \Big(1 \, + \, \{\text{Sum of all vac. diagrams, with appropr. symm. factors}\}\Big) . \tag{2.47}$$

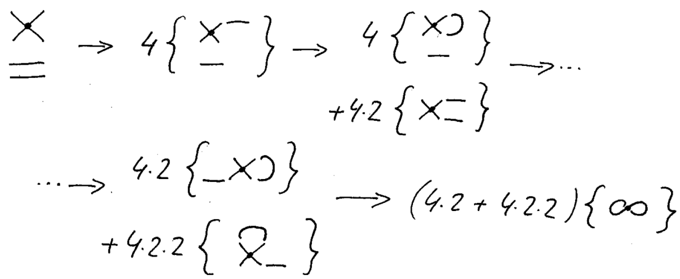


Figure 8: Illustration of symmetry-factor calculation as explained in the text.

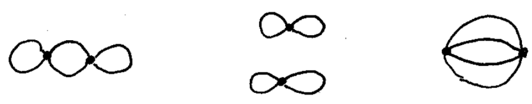


Figure 9: Vacuum diagrams at order λ^2.

2.4 Green's functions

The generalization to derivatives of $Z[j]$ and hence to Green's functions is straightforward:

$$\langle 0|T\varphi_1 \cdots \varphi_n|0\rangle \;=\; \frac{1}{Z[0]}\left(\frac{\delta}{i\delta j_1}\right)\cdots\left(\frac{\delta}{i\delta j_n}\right) Z[j]\Bigg|_{j=0} \tag{2.48}$$

$$= \frac{1}{Z_0[0]\,(1+\text{vac. diagrs.})}\left(\frac{\delta}{i\delta j_1}\right)\cdots\left(\frac{\delta}{i\delta j_n}\right)\exp\left(iS_{int.}\left[\frac{\delta}{i\delta j}\right]\right) Z_0[0]\,e^{-\frac{1}{2}jDj}\Bigg|_{j=0}\,.$$

First, the factor $Z_0[0]$ cancels. Next, we see that now, due to the factors $\delta/(i\delta j_i)$, lines can end not only at vertices but also at the external points x_i. Thus, the above expression generates all diagrams with n external lines ending at the points x_i. As we showed in an excercise in QFT I, from 'all diagrams' a term '(1+vac. diagrs.)' can be factored out. This term is cancelled by the prefactor, giving

$$\langle T\varphi_1\cdots\varphi_n\rangle \;=\; \Big(\text{Sum of all diagrs., without vacuum diagrs.}\Big)\,. \tag{2.49}$$

Here the diagrams are built from lines and vertices with the corresponding analytical expressions summerized in Fig. 10.

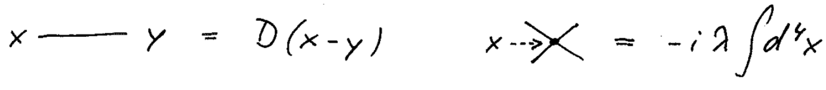


Figure 10: Feynman rules of $\lambda\phi^4$ theory.

To practice, you may want to calculate the two-point function up to order λ^2, cf. Fig. 11, including symmetry factors and the demonstration that no further diagrams contribute at this order. As a reminder, the second diagram (and the only at order λ) gives

$$\int d^4x\, D(x_1 - x)\, D(x - x)\, D(x - x_2)\,. \tag{2.50}$$

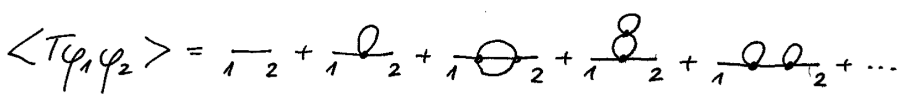


Figure 11: Two-point function in perturbation theory.

There are many immediate generalizations, including other interaction terms, e.g.

$$S_{int.} = -\int \frac{\lambda}{n!}\varphi^n\,, \tag{2.51}$$

and theories with several fields, e.g.

$$iS_0 = -\frac{1}{2}\varphi D_\varphi^{-1}\varphi - \frac{1}{2}\chi D_\chi^{-1}\chi \qquad \text{with} \qquad S_{int.} = -\int \frac{\lambda}{2}\varphi^2\chi\,. \tag{2.52}$$

In the latter case, one needs two source terms, $j_\varphi\varphi$ and $j_\chi\chi$, otherwise everything goes through as before. As an excercise, you may derive the Feynman rules, cf. Fig. 12. Note that the propagators must really be distinguished since the two scalars may have different mass.

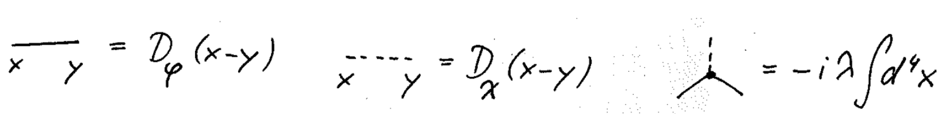


Figure 12: Feynman rules for the theory defined in (2.52).

As a further excercise, you may derive all contributions relevant to $\varphi\varphi \to \varphi\varphi$ scattering at order λ^4, cf. Fig. 13.

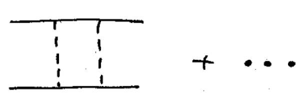


Figure 13: One of the diagrams for 2-to-2 scattering in theory of (2.52).

Very importantly, we can now deal with derivatives in $S_{int.}$ **without** the complications encountered in the canonical approach.[3] Indeed, consider e.g.

$$S_{int.} = \frac{\lambda}{2}\int d^4x\varphi^2(x)\,\partial^\mu\partial_\mu\chi(x)\,. \tag{2.53}$$

[3] At this point we actually *define* the quantized theory using the path integral, without proving that the canonical approach gives the same result.

The picture in the resulting Feynman rule is still the same as in Fig. 12, but the analytical expression is different. To see this, note that

$$S_{int.}\left[\frac{\delta}{i\delta j_\varphi},\frac{\delta}{i\delta j_\chi}\right]=\frac{\lambda}{2}\int d^4x\left(\frac{\delta}{i\delta j_\varphi}\right)^2\partial^\mu\partial_\mu\left(\frac{\delta}{i\delta j_\chi}\right) \tag{2.54}$$

and hence, e.g.,

$$\langle T\varphi_1\varphi_2\chi_3\rangle=\int d^4x D_\varphi(x_1-x)D_\varphi(x_2-x)\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu}D_\chi(x-x_3)\,. \tag{2.55}$$

Thus, the Feynman rule receives a factor ∂^2 acting on the D_χ in real space and a factor $-p_\chi^2$ in momentum space.

Another very important example is that of the complex scalar field, which requires the introduction of a complex source, with j and $\bar{\jmath}$ treated as independent (analogously to the field):

$$Z_0[j,\bar{\jmath}]=\int D\phi\exp i\int d^4x\left[\bar\phi(-\partial^2-m^2+i\epsilon)\phi+\bar\phi j+\bar{\jmath}\phi\right]\,. \tag{2.56}$$

It is a good excercise to evaluate this, then to introduce an interaction term $\sim(\bar\phi\phi)^2$, and to derive the Feynman rules. Note that, while D is the same as for the real scalar, it will appear in the form $\bar{\jmath}^T Dj$. Thus, to keep track of which side belongs to j and which to $\bar{\jmath}$, arrows on the lines will be required.

3 Path integral for fermions

3.1 Bosonic harmonic oscillator in the holomorphic representation

Bosonic (fermionic) fields can be viewed as collections of bosonic (fermionic) harmonic oscillators. Thus, our basic building block will be the path integral treatment of the fermionic harmonic oscillator. This is very similar to the path integral treatment of the bosonic oscillator in the holomorphic representation. While the latter is a standard QM topic in principle, it is not part of every course and we introduce it from scratch. Other common names are the **coherent-state** or **Bargman-Fock** representation and relevant books include those by Zinn-Justin, Fadeev/Slavnov and Itzykson/Zuber [9, 11, 12].

Consider the harmonic oscillator with

$$H=\omega\left(a^\dagger a+\frac{1}{2}\right)\qquad\text{and}\qquad[a,a^\dagger]=1\,. \tag{3.1}$$

We represent

$$a^\dagger\,,\;a\qquad\text{by}\qquad\bar z\,,\;\frac{\partial}{\partial\bar z}\,, \tag{3.2}$$

acting on anti-holomorphic functions $f(\bar{z})$. On this function space, we define the scalar product

$$\langle f_1|f_2\rangle = \int \frac{d^2z}{\pi}\,\overline{f_1(\bar{z})}\,f_2(\bar{z})e^{-z\bar{z}}\,, \tag{3.3}$$

where $d^2z \equiv r\,dr\,d\varphi$ is the standard measure on the (complex) plane and $1/\pi$ is a convenient normalization. Of course,

$$z = re^{i\varphi} \quad \text{and} \quad \bar{z} = re^{-i\varphi}\,. \tag{3.4}$$

The fact that the $a, a^\dagger$-commutation relation is properly represented is obvious. We check that $a^\dagger$ is really the hermitian conjugate of a w.r.t. to this scalar product:

$$\begin{aligned}\langle f_1|a^\dagger f_2\rangle &= \int \frac{d^2z}{\pi}\,\overline{f_1(\bar{z})}\,\bar{z}\,f_2(\bar{z})e^{-z\bar{z}} = \int \frac{d^2z}{\pi}\,\overline{f_1(\bar{z})}\,(-\partial_z)\;f_2(\bar{z})e^{-z\bar{z}} \\ &= \int \frac{d^2z}{\pi}\left(\partial_z\overline{f_1(\bar{z})}\right)\,f_2(\bar{z})e^{-z\bar{z}} = \int \frac{d^2z}{\pi}\,\overline{(\partial_{\bar{z}}f_1(\bar{z}))}\,f_2(\bar{z})e^{-z\bar{z}} = \langle af_1|f_2\rangle\,.\end{aligned} \tag{3.5}$$

It should be immediately clear that the standard (orthonormal) Hilbert-space basis of energy eigenstates is identified as follows:

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle > \qquad \leftrightarrow \qquad \psi_n(\bar{z}) \equiv \frac{\bar{z}^n}{\sqrt{n!}} \qquad \leftrightarrow \qquad |\psi_n\rangle\,. \tag{3.6}$$

While the orthonormality of the ψ_n basis indirectly follows from the above, you should also explicitly (replacing $z, \bar{z}$ by polar coordinates and integrating) check that

$$\langle\psi_m|\psi_n\rangle = \delta_{mn}\,. \tag{3.7}$$

The above are old facts in a different language. A new feature is that one can very easily write down eigenstates $|z\rangle$ of a with eigenvalue z:

$$a|z\rangle = z|z\rangle\,, \qquad \text{explicitly:} \qquad |z\rangle \quad \leftrightarrow \quad f_z(\bar{y}) = e^{z\bar{y}}\,. \tag{3.8}$$

We can think of $f_z(\bar{y})$ as of a power-series in $\bar{y}$ acting on the constant function, i.e. on the vacuum: $f_z(\bar{y}) = f_z(\bar{y}) \cdot 1$. Since our anti-holomorphic variable is identified with $a^\dagger$, we have the identification

$$|z\rangle \quad \leftrightarrow \quad f_z(\bar{y}) = e^{z\bar{y}} \quad \leftrightarrow \quad e^{za^\dagger}|0\rangle\,. \tag{3.9}$$

The states $|z\rangle$ are called **coherent states** and deserve some more detailed comments: Among many other important applications, they are crucial for understanding the classical limit of the harmonic oscillator. Indeed, consider a particle with $E \gg \omega$ oscillating in a quadratic potential. It will have predictible, non-zero values of position and velocity at certain times. However, the expectation values of $a, a^\dagger$ are always zero in energy-eigenstates,

$$\langle n|a|n\rangle = \langle n|a^\dagger|n\rangle = 0\,. \tag{3.10}$$

Since

$$\hat{x},\, \hat{p} \;\sim\; a \pm a^\dagger\,, \tag{3.11}$$

this implies that expectation values of x and p always vanish. By contrast, we have for example

$$\langle \bar{z}|(a+a^\dagger)|z\rangle = \langle \bar{z}|z\rangle(z+\bar{z}) \neq 0\,. \tag{3.12}$$

Thus, coherent states can model the classical situation better.[4] Note also that the 'bar' in $\langle \bar{z}|$ is pure notation. $\langle \bar{z}|$ is just the dual state belonging to $|z\rangle$.

More importantly for us, in QFT the field-operators are also linear superpositions of a's and $a^\dagger$'s. Thus, coherent states are needed to describe situations in which a field has non-zero expectation value. This is of course crucial to understand how ED follows from QED. The name refers to the **coherent** superposition of infinitely many Fock states.

Finally, we calculate the overlap between $|z\rangle$ and $|\psi_n\rangle$,

$$\langle \psi_n|z\rangle = \langle 0|\frac{a^n}{\sqrt{n!}}\, e^{za^\dagger}|0\rangle = \frac{z^n}{\sqrt{n!}}\,, \tag{3.13}$$

and between two coherent states

$$\langle \bar{z}|y\rangle = \sum_n \langle \bar{z}|\psi_n\rangle\langle\psi_n|y\rangle = \sum_n \frac{(\bar{z}y)^n}{n!} = e^{\bar{z}y}\,. \tag{3.14}$$

A further important relation is

$$\mathbb{1} = \int \frac{d^2z}{\pi} e^{-\bar{z}z}|z\rangle\langle\bar{z}|\,. \tag{3.15}$$

It follows from the one-line calculation,

$$\int \frac{d^2z}{\pi} e^{-\bar{z}z}\langle\psi_m|z\rangle\langle\bar{z}|\psi_n\rangle = \delta_{mn} = \langle\psi_m|\mathbb{1}|\psi_n\rangle\,, \tag{3.16}$$

where we used standard $drd\varphi$-integration to get the Kronecker delta. We note that the states $|z\rangle$ are referred to as an 'overcomplete basis' since they can be used to represent unity as above but are not mutually orthogonal. This is no surprise since a is not hermitian.

3.2 Path integral with coherent states

As before, we start with transition amplitudes. Using the completeness relation, we have

$$\langle\bar{z}|e^{-iHt}|y\rangle = \prod_{i=1}^{n-1}\left(\int \frac{d^2z_i}{\pi}e^{-\bar{z}_i z_i}\right)\langle\bar{z}|e^{-iH\epsilon}|z_{n-1}\rangle\langle\bar{z}_{n-1}|e^{-iH\epsilon}|z_{n-2}\rangle\cdots\langle\bar{z}_1|e^{-iH\epsilon}|y\rangle\,, \tag{3.17}$$

[4] Of course, there are many other states with non-zero expectation values of x and p. Coherent states are special in that they saturate the uncertainty inequality $\Delta x\,\Delta p \geq 1/2$ ('minimize uncertainty').

with $t = n\epsilon$. Without loss of generality, we can assume $H = H(a^\dagger, a)$ to be a normal-ordered expression in terms of a and $a^\dagger$. (If it's originally not normal-ordered, we can always re-order it at the expense of adding extra terms. Also, it can clearly involve higher powers of a and $a^\dagger$, beyond the harmonic-oscillator case.) We then find

$$\begin{aligned} \langle \bar{z}_i | e^{-iH(a^\dagger,a)\epsilon} | z_{i-1} \rangle &\simeq \langle \bar{z}_i | (1 - iH(a^\dagger, a)\epsilon) | z_{i-1} \rangle \\ &= e^{\bar{z}_i z_{i-1}} (1 - iH(\bar{z}_i, z_{i-1})\epsilon) \simeq e^{\bar{z}_i z_{i-1} - iH(\bar{z}_i, z_{i-1})\epsilon} \,. \end{aligned} \tag{3.18}$$

Collecting all such terms and renaming $\bar{z} = \bar{z_n}$ and $y = z_0$, we obtain

$$\langle \bar{z}_n | e^{-iHt} | z_0 \rangle = \prod_{i=1}^{n-1} \left(\int \frac{d^2 z_i}{\pi} \right) \exp \left[\bar{z}_0 z_0 + \sum_{i=1}^{n} \left(\bar{z}_i z_{i-1} - \bar{z}_{i-1} z_{i-1} - i\epsilon H(\bar{z}_i, z_{i-1}) \right) \right] . \tag{3.19}$$

We now rewrite

$$\sum_{i=1}^{n} (\bar{z}_i z_{i-1} - \bar{z}_{i-1} z_{i-1}) = \epsilon \sum_{i=1}^{n} \frac{\bar{z}_i - \bar{z}_{i-1}}{\epsilon} z_{i-1} \simeq \int_0^t dt' \dot{\bar{z}} z \,. \tag{3.20}$$

Taking the continuum limit also for the full expression, we can finally write

$$\langle \bar{z}_b | e^{-iHt} | z_a \rangle = \int Dz D\bar{z} \, \exp \left[\bar{z}(0) z(0) + \int_0^t dt' (\dot{\bar{z}} z - iH(\bar{z}, z)) \right] , \tag{3.21}$$

Several important comments have to be made: First, it is crucial to always remember that the functional form of $H(\bar{z}, z)$ is really different from the familiar $H(p, q)$. One must first express H in terms of a and $a^\dagger$ and then switch to classical variables z, $\bar{z}$. Also, please **do not confuse** the identification of $a, a^\dagger$ with the classical variables $z, \bar{z}$ in the path integral with the original representation by $\partial/(\partial \bar{z})$ and $\bar{z}$.

Second, the path integral is over two independent complex functions $z(t')$ and $\bar{z}(t')$ with (partial!) boundary conditions $z(0) = z_a$ and $\bar{z}(t) = \bar{z}_b$. (We switched notation from z_0 and $\bar{z}_n$ to achieve more similarity with the earlier Dx path integral.)

The independence of z and $\bar{z}$ in the path integration can be argued in more detail as follows: First, by definition, we have at each of the n intermediate time slices

$$\int d^2 z = \int r \, dr \, d\varphi = \int dx \, dy \,, \tag{3.22}$$

where we used $z = x + iy$ and $\bar{z} = x - iy$ with real x and y in the last step. However, we can think of x and y as of complex variables which are to be integrated along the contours $\mathrm{Im}(x) = 0$ and $\mathrm{Im}(y) = 0$, i.e.

$$\int d^2 z = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{\infty} dy \,. \tag{3.23}$$

Since the integrand is an analytic function of x and y, we can deform the contours,

$$\int d^2 z = \int_{C_x} dx \int_{C_y} dy \,, \tag{3.24}$$

to the extent allowed by the behaviour at infinity. Now, changing variables back to z, $\bar{z}$, the latter are by definition two independent complex variables and $\bar{z} \neq z^*$ for generic $C_{x,y}$. All of this is not a pure formality: There are, for example, situations where the path integral is well-approximated 'semi-classically' by a stationary point (or saddle-point) of the exponent (which is $S(z,\bar{z})$, as we will see shortly). Such a saddle point may indeed occur for a path with $\bar{z} \neq z^*$.

Third, it may be perceived as unpleasant to have a boundary term only at one of two boundaries. This is easily fixed, a more symmetric form being obtained after integration by parts:

$$\exp\left[\frac{1}{2}\{\bar{z}(t)z(t) + \bar{z}(0)z(0)\} + \int_0^t dt' \left(\frac{1}{2}\{\dot{\bar{z}}z - \bar{z}\dot{z}\} - iH(\bar{z},z)\right)\right] . \tag{3.25}$$

Fourth, one can easily see that up to boundary terms the exponent is the classical action. Indeed, we invert

$$q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger) \quad , \qquad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger) \tag{3.26}$$

to

$$z = a = \frac{1}{2}\left(\sqrt{2\omega}\, q + i\sqrt{\frac{\omega}{2}}p\right) \quad , \qquad \bar{z} = a^\dagger = \frac{1}{2}\left(\sqrt{2\omega}q - i\sqrt{\frac{\omega}{2}}p\right) . \tag{3.27}$$

Thus,

$$\dot{\bar{z}}z \;=\; \frac{\omega}{2}q\dot{q} - \frac{i}{2}\dot{p}q + \frac{i}{2}p\dot{q} + \frac{1}{2\omega}p\dot{p} \;=\; ip\dot{q} \;+\; \text{total derivative} \tag{3.28}$$

and

$$\int_0^t dt'(\dot{\bar{z}}z - iH(\bar{z},z)) \;=\; i\int_0^t dt'(p\dot{q} - H(p,q)) \;+\; \text{boundary terms} \tag{3.29}$$

$$= \; iS[p,q] \;+\; \text{boundary terms}\,. \tag{3.30}$$

Here $S[p,q]$ is the action of the hamiltonian formulation of classical mechanics. Its variation w.r.t. $q(t)$ and $p(t)$ as independent functions gives the Hamilton equations. Given our holomorphic approach, we will need to return to the variables z and $\bar{z}$, writing $S[z,\bar{z}]$ for the classical action in the exponent.

As a side-remark, we note that a (non-holomorphic) Hamiltonian path integral formula

$$\langle q_b|e^{-iHt}|q_a\rangle = \int Dp\, Dq \exp(iS(p,q) + \text{ bnd. terms}) \tag{3.31}$$

also exists. Here the Dp integration is not restricted at the boundaries. You may want to prove this along the lines of our Dq path integral derivation and fix the boundary terms.

Finally, we can as before include Heisenberg-picture operators $a(t)$ and $a^\dagger(t)$ in our amplitude and replace the initial and final states by the vacuum. The latter is, as before, accounted for by introducing the $i\epsilon$ convergence factor in the path integral. Using also the normalization $\langle 0|0\rangle = 1$ we finally obtain

$$\langle 0|Ta(t_1)a^\dagger(t_2)\cdots|0\rangle = \frac{\int Dz\, D\bar{z}[z(t_1)\bar{z}(t_2)+\cdots]\,\exp iS[z,\bar{z}]}{\int Dz\, D\bar{z}\,\exp iS[z,\bar{z}]}\,. \tag{3.32}$$

Boundary terms drop out together with all other prefactors in the ratio on the r.h. side. It is crucial to remember that $S[z,\bar{z}]$ is obtained by expressing the action in terms of the (classical variables) a and $a^\dagger$ and replacing the latter by z and $\bar{z}$.

3.3 Fermionic harmonic oscillator and Grassmann variables

Next, we turn to the so-called fermionic harmonic oscillator, defined by

$$H = \omega\left(a^\dagger a - \frac{1}{2}\right) \qquad \text{with} \qquad \{a,a^\dagger\} = 1 \quad \text{and} \quad \{a,a\} = \{a^\dagger,a^\dagger\} = 0\,. \tag{3.33}$$

The system has just two states, with energies $\pm\omega/2$, the lower one annihilated by a.[5] Let us try to find a representation of the fermionic $a, a^\dagger$ algebra using variables $\theta, \bar{\theta}$, in analogy to the last section. This will turn out to work if $\theta, \bar{\theta}$ are Grassmann variables.

To make the analogy most apparent, we would like to think of our previous function spaces in z and/or $\bar{z}$ as of formal power series in $z, \bar{z}$. Now, instead, we look at formal power series and $\theta, \bar{\theta}$. In both cases we basically work with the algebra generated by unity together with $z, \bar{z}$ or $\theta, \bar{\theta}$.

Before, $z, \bar{z}$ were commuting variables. Now, instead, we have

$$\{\theta,\bar{\theta}\} = \{\theta,\theta\} = \{\bar{\theta},\bar{\theta}\} = 0\,, \tag{3.34}$$

i.e., $\theta\bar{\theta} + \bar{\theta}\theta = 0$, $\theta^2 = \bar{\theta}^2 = 0$. In complete analogy to the bosonic case, we represent

$$a^\dagger\,,\; a \qquad \text{by} \qquad \bar{\theta}\,,\; \frac{\partial}{\partial\bar{\theta}}\,, \tag{3.35}$$

acting on functions of $\bar{\theta}$. This function space is clearly just 2-dimensional since $\bar{\theta}^2 = 0$:

$$f(\bar{\theta}) = f_0 + f_1\bar{\theta}\,. \tag{3.36}$$

With the natural definition

$$\frac{\partial}{\partial\bar{\theta}}(f_0 + f_1\bar{\theta}) = f_1 \tag{3.37}$$

we find

$$\left(\frac{\partial}{\partial\bar{\theta}}\right)^2 = 0\,, \tag{3.38}$$

[5]The negative vacuum energy, i.e. the explicit $(-1/2)$-term in H, is somewhat ad hoc at this stage. One can argue that it follows naturally from the generalization $H_{bosonic} = (\omega/2)(a^\dagger a + aa^\dagger) \rightarrow H_{fermionic} = (\omega/2)(a^\dagger a - aa^\dagger)$. It is also what one finds in QFT coupled to gravity, where the fermion contributes negatively to the vacuum energy.

and

$$\left\{\frac{\partial}{\partial\bar\theta},\bar\theta\right\} f(\bar\theta) = \frac{\partial}{\partial\bar\theta}\bar\theta f(\bar\theta) + \bar\theta\frac{\partial}{\partial\bar\theta}f(\bar\theta) = f_0 + \bar\theta f_1 = f(\bar\theta)\,, \tag{3.39}$$

i.e.

$$\left\{\frac{\partial}{\partial\bar\theta},\bar\theta\right\} = 1\,, \tag{3.40}$$

as desired.

Thinking of the original physical 2-state system, it is natural to write

$$|f\rangle = |0\rangle f_0 + |1\rangle f_1 \tag{3.41}$$

and hence to define the scalar product on the function space by

$$\langle g|f\rangle = \bar g_0 f_0 + \bar g_1 f_1\,. \tag{3.42}$$

To realize this as an integral, we define **Grassmann variable integration** (the **Berezin integral**) by

$$\int d\theta\cdot 1 = \int d\bar\theta\cdot 1 = 0\,, \qquad \int d\theta\,\theta = \int d\bar\theta\,\bar\theta = 1\,. \tag{3.43}$$

Up to normalization, this definition is forced upon us if we want the rule

$$\int d\theta\frac{\partial}{\partial\theta}(\cdots) = 0 \tag{3.44}$$

to hold, which is technically very important. Note also that the operations of integrating and differentiating coincide,

$$\int d\theta\,(\cdots) = \frac{\partial}{\partial\theta}(\cdots)\,, \tag{3.45}$$

and that one can show

$$\int d\theta\,d\bar\theta = -\int d\bar\theta\,d\theta\,, \tag{3.46}$$

using $\theta\bar\theta = -\bar\theta\theta$.

Next, it is easy to check that the definition

$$\langle g|f\rangle = \int d\bar\theta\,d\theta\,\overline{g(\bar\theta)}\,f(\bar\theta)\,e^{-\bar\theta\theta} \tag{3.47}$$

is consistent with the earlier definition using $f_{0,1}$ and $g_{0,1}$. One simply needs to note that $e^{-\bar\theta\theta} = 1-\bar\theta\theta$. With this scalar product, $(a)^\dagger = a^\dagger$, as you should again check explicitly.

We are now ready to define **coherent states**, in this case eigenstates of $a = \partial/(\partial\bar\theta)$ with eigenvalue η:

$$|\eta\rangle \quad\leftrightarrow\quad f_\eta(\bar\theta) = e^{\bar\theta\eta} = 1+\bar\theta\eta = |0\rangle + |1\rangle\eta\,. \tag{3.48}$$

Here η is another Grassmann variable, just like θ. We assume anti-commutation relations between any two of the variables η, $\bar\eta$, θ, $\bar\theta$.

We note that, to write the above relations, we need to be able to multiply Hilbert-space vectors with Grassmann variables. In other words, we work in the tensor product of the Hilbert space and the Grassmann algebra (or several Grassmann algebras, if we need more than one extra variable η). Identifying the Hilbert space with the θ-algebra and calling the extra Grassmann variable η, we can also think of this tensor product as the Grassmann algebra spanned by 1, θ and η. It is then important to choose an ordering convention by identifying $|1\rangle\eta$ with $\theta\eta$ (as opposed to $\eta\theta$).

We also extend the usual anti-linearity of the Hilbert-space scalar product to the Grassmann algebra coefficients,

$$\big(|f_1\rangle\eta\big)\cdot\big(|f_2\rangle\big) = (\eta)^*\langle f_1|f_2\rangle\,. \tag{3.49}$$

For this, we need to introduce a so-called $*$-operation on our Grassmann algebra, which we do be defining

$$(\eta)^* = \bar\eta\;, \quad (\eta\bar\eta)^* = (\bar\eta)^*\eta^* = \eta\bar\eta \tag{3.50}$$

etc.

Now we demonstrate that $|\eta\rangle$ really has the desired property. First, we do so explicitly utilizing the energy eigenstate basis,

$$a|\eta\rangle = a\,(|0\rangle + |1\rangle\eta) = |0\rangle\eta = \eta\,(|0\rangle + |1\rangle\eta) = \eta|\eta\rangle\,, \tag{3.51}$$

where we used $\eta^2 = 0$. Next, we repeat the argument using our differential-operator representation:

$$\frac{\partial}{\partial\bar\theta}(1+\bar\theta\eta) = \eta = \eta\,(1+\bar\theta\eta)\,. \tag{3.52}$$

Finally, we generalize our completeness relation according to

$$\mathbb{1} = \int\frac{d^2z}{\pi}e^{-\bar zz}|z\rangle\langle\bar z| \qquad\to\qquad \mathbb{1} = \int d\bar\theta\,d\theta e^{-\bar\theta\theta}|\theta\rangle\langle\bar\theta|\,. \tag{3.53}$$

This follows from

$$\int d\bar\theta\,d\theta e^{-\bar\theta\theta}|\theta\rangle\langle\bar\theta| = \int d\bar\theta\,d\theta\,(1-\bar\theta\theta)\,(|0\rangle+|1\rangle\theta)\,(\langle 0|+\bar\theta\langle 1|) = |0\rangle\langle 0| + |1\rangle\langle 1|\,, \tag{3.54}$$

where the last step uses our previously defined integration rules.

The other crucial relation we needed in the bosonic case was

$$\langle\bar\theta|\eta\rangle = e^{\bar\theta\eta}\,, \tag{3.55}$$

which is again easy to check.

With these tools, we can now repeat our coherent-state path integral derivation of the bosonic system line by line. Two comments should possibly be made: First, one must be careful with signs which can arise due to $\bar\theta\eta = -\eta\bar\theta$. (We assumed that distinct

Grassmann variables also anti-commute.) Second, we have to 'insert unity' at many different times, such that expressions like

$$\int d\bar{\theta}(t_i)\, d\theta(t_i) \cdots \tag{3.56}$$

for many different times t_i arise together. For us, nothing new happens in the corresponding manipulations since we treat the various θs at different times simply as independent Grassmann variables, just like e.g. θ and η above.

After going through the Grassmann version of the derivation (which you should do!), one arrives at

$$\langle 0|Ta(t_1)a^\dagger(t_2)\cdots|0\rangle = \frac{\int D\bar{\theta}\, D\theta[\theta(t_1)\bar{\theta}(t_2) + \cdots]\, \exp iS[\theta,\bar{\theta}]}{\int D\bar{\theta}\, D\bar{\theta}\, \exp iS[\theta,\bar{\theta}]}\,, \tag{3.57}$$

in complete analogy to the bosonic case. Crucially, the fermionic time-ordering needs to be defined with the usual signs for each necessary exchange of operators. (Prove this!) Furthermore, the action is now *by definition*

$$S[\theta,\bar{\theta}] = \int dt(i\bar{\theta}\dot{\theta} - H(\theta,\bar{\theta}))\,. \tag{3.58}$$

3.4 Path integral for fermions in QFT

As before, we start from our canonically quantized system and derive path integral formulae for the canonically defined amplitudes and correlation functions. To this end, we recall that we described the Dirac fermion,

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \qquad \text{with} \qquad \psi = \{\psi_a\} = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_4 \end{pmatrix}\,, \tag{3.59}$$

using the canonical momenta

$$\pi^a = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_a} = i\psi^{\dagger a} \qquad \text{or} \qquad \pi = i\psi^\dagger\,, \tag{3.60}$$

interpreting π as a row-vector. Thus,

$$\begin{aligned} H &= \int d^3x\mathcal{H} = \int d^3x(\pi\dot{\psi} - \mathcal{L}) = \int d^3x(i\psi^\dagger\dot{\psi} - \mathcal{L}) = \int d^3x\, \psi^\dagger\gamma^0(-i\gamma^i\partial_i + m)\psi \\ &= \int d^3x\, d^3y \sum_{a,b=1}^{4} \psi^{\dagger a}(\vec{x}) \left[-i(\gamma^0\gamma^i)_a{}^b\frac{\partial}{\partial x^i} + m\,(\gamma^0)_a{}^b\right] \delta^3(\vec{x}-\vec{y})\psi_b(\vec{y})\,. \end{aligned} \tag{3.61}$$

The last line can be read as an infinite-dimensional row-vector $\psi^{\dagger a}(\vec{x})$ with index $\{a,\vec{x}\}$ multiplying a matrix, which then multiplies a column-vector $\psi_b(\vec{y})$ with index $\{b,\vec{y}\}$.

We quantized the system postulating the anti-communtation relations

$$\{\psi(\vec{x}),\pi(\vec{y})\} = i\delta^3(\vec{x}-\vec{y})\,\mathbb{1} \quad \text{or, equivalently,} \quad \{\psi_a(\vec{x}),\psi^{\dagger b}(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\delta_a{}^b\,, \quad (3.62)$$

with all other anti-commutators of ψ and $\psi^\dagger$ vanishing. Hence, we are precisely in a setting with Hamiltonian

$$H = \sum_{ij} a^{\dagger i} M_i{}^j a_j \qquad \text{and} \qquad \{a_i, a^{\dagger j}\} = \delta_i{}^j\,, \tag{3.63}$$

with i standing for the double-index $\{a,\vec{x}\}$ mentioned before and the matrix M defined in (3.61). But this is just the multi-variable generalization of our fermionic harmonic oscillator

$$H = \omega\, a^\dagger a \qquad \text{with} \qquad \{a, a^\dagger\} = 1 \tag{3.64}$$

of the last section. (Note that here a and $a^\dagger$ are identified with ψ and $\psi^\dagger$, *not* with the particle-type creation and annihilating operators defined in momentum space in QFT I.)

Now our path integral derivation goes through without any difficulty, the only change being that one has to sum (or take products over) the index $i \equiv \{a,\vec{x}\}$ whenever one deals with the variables $a_i \equiv \psi_a(\vec{x})$. The fact that all these variables are independent in the sense of *diagonal* anti-commutation relations (cf. (3.62) and (3.63)) is of course essential. The fact that the Hamiltonian (i.e. the matrix M) is *not diagonal* is of no consequence. Indeed, we never referred to the explicit form of H as a function of a and $a^\dagger$ in our derivation.

Thus, generalizing our previous result (3.57), we have

$$\langle 0|T\psi_{a_1}(x_1)\psi^{\dagger b_2}(x_2)\cdots|0\rangle = \frac{\int D\bar{\theta}D\theta[\theta_{a_1}(x_1)\bar{\theta}^{b_2}(x_2)\cdots]\,\exp iS[\theta,\bar{\theta}]}{\int D\bar{\theta}D\theta\,\exp iS[\theta,\bar{\theta}]}\,, \tag{3.65}$$

where

$$S[\theta,\bar{\theta}] = \int dt\left[\int d^3x \sum_{a=1}^{4}\bar{\theta}^a(x)\dot{\theta}_a(x) - H[\theta,\bar{\theta}]\right]\,, \tag{3.66}$$

and

$$\int D\theta \equiv \prod_{a,\vec{x}}\left(\int D\theta_a(\vec{x})\right)\,. \tag{3.67}$$

Now, it is conventional in QFT *not* to change variable-names when going from the operators to the ‘classical’ Grassmann variables. In other words, the path integral is written using ψ and $\psi^\dagger$ as variables. Furthermore, it is common to replace $\psi^\dagger$ by $\bar{\psi} = \psi^\dagger\gamma^0$, which is just a linear change of variables. Finally, the action (3.66) is, after replacing θ by ψ, just our familiar covariant Dirac action, as we saw at the beginning of this section. Thus, we can summarize by writing

$$\langle 0|T\psi_{a_1}(x_1)\bar{\psi}^{b_2}(x_2)\cdots|0\rangle = \frac{\int D\bar{\psi}D\psi[\psi_{a_1}(x_1)\bar{\psi}^{b_2}(x_2)\cdots]\,e^{iS}}{\int D\bar{\psi}D\psi\,e^{iS}}\,, \tag{3.68}$$

with

$$S = \int d^4x \bar{\psi}(i\partial\!\!\!/ - m + i\epsilon)\psi\,. \tag{3.69}$$

We will comment of the $i\epsilon$ in a moment. Before doing so, we note that we have worked very hard to show that the above formula for correlation functions follows from the canonical definition of QFT I. However, we also see that the path integral for the Dirac fermion is, obviously, a very simple and natural generalization of the bosonic path integral. Thus, it is may be more natural to think of the last two formulae as *defining* the quantum theory. The only drawback is that the Hilbert space as a Fock space and its particle interpretation are more transparent on the canonical side.

There are two subtleties left to discuss, which turn out to be related. The first is the vacuum energy. It should be zero in the above definition of the quantum theory since H was normal-ordered in $\psi^\dagger$, ψ and we did not include an explicit constant. By contrast in the canonical 'Fourier-mode' quantization using $a_{\vec{p}}$, $a^\dagger_{\vec{p}}$ etc., we found a (divergent) constant. This difference is explained by noting that the vacua in the two approaches are actually *different*.[6] Indeed, ψ annihilates the present vacuum, but of course not the canonical particle-vacuum since ψ contains $a_{\vec{p}}$ and $b^\dagger_{\vec{p}}$.

The above does not bother us given that we do not include gravity and are free to subtract a constant as a we please. However, we do have to make sure that the $i\epsilon$ prescription really suppresses higher-energy states. We can not argue the sign of $i\epsilon$ simply as a convergence factor since

$$\int d\bar{\theta} d\theta e^{a\bar{\theta}\theta} \tag{3.70}$$

is well-defined for any a.

Based on what was said above, it is best to discuss this issue in the particle-picture. To do so, recall that

$$\psi(\vec{x}) = \int d\tilde{p}\left(a^s_{\vec{p}} u_s(p) e^{i\vec{p}\vec{x}} + b^{s\dagger}_{\vec{p}} v_s(p) e^{-i\vec{p}\vec{x}}\right) \tag{3.71}$$

and

$$\bar{u}_r(p)u_s(p) = 2m\delta_{rs}\ , \qquad \bar{v}_r(p)v_s(p) = -2m\delta_{rs}\,. \tag{3.72}$$

Now, if we replace m by $m - i\epsilon$, the hamiltonian acquires an extra piece

$$-i\epsilon\int d^3x\bar{\psi}\psi = -i\epsilon\int d\tilde{p}\frac{m}{p_0}\left(a^{s\dagger}_{\vec{p}}a^s_{\vec{p}} - b^s_{\vec{p}}b^{s\dagger}_{\vec{p}}\right) = -i\epsilon\int d\tilde{p}\frac{m}{p_0}\left(a^{s\dagger}_{\vec{p}}a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}}b^s_{\vec{p}}\right)\ +\ \text{const.} \tag{3.73}$$

The coefficient of $(-i\epsilon)$ is clearly larger on excited states than on the vacuum. Hence, the factor

$$e^{-iH\Delta t} \supset e^{-i(-i\epsilon)\Delta t\cdot(\text{positive})} = e^{-\epsilon\,\Delta t\cdot(\text{positive})} \tag{3.74}$$

[6] Of course, we already know that the particle vacuum is the true vacuum in the sense of being the lowest-energy state. This is not in conflict with the above since, in fact, we did not check the positivity properties of our matrix $M_i{}^j$ defining the hamiltonian.

suppresses non-vacuum states more strongly, as required. This justifies the $i\epsilon$ in our path integral formula above, which is in agreement with the pole-prescription-($i\epsilon$) of the canonical approach.

3.5 Feynman rules for fermions

As in the bosonic case, we introduce source fields $\eta(x)$ and $\bar{\eta}(x)$ (which are spinors and Grassmann variables, just like $\psi(x)$ and $\bar{\psi}(x)$) and define

$$Z[\bar{\eta},\eta] \equiv \int D\bar{\psi}\, D\psi\, e^{iS+i\bar{\eta}\psi+i\bar{\psi}\eta}\,. \tag{3.75}$$

We remind the reader of our shorthand notation

$$\text{“ } \bar{\eta}\psi \text{ ”} \;\;\equiv\;\; \int d^4x\, \bar{\eta}(x)\, \psi(x)\,. \tag{3.76}$$

It is straightforward to see that

$$\langle T\psi_1\bar{\psi}_2\cdots\rangle = \frac{1}{Z}\left(\frac{\delta}{i\delta\bar{\eta}_1}\right)\left(\frac{\delta}{-i\delta\eta_2}\right)\cdots Z[\bar{\eta},\eta]\Bigg|_{\bar{\eta},\eta=0}\,. \tag{3.77}$$

Note that the spinor indices of the ψs and ηs are *not* contracted in this expression – we have only suppressed them for brevity. Note also that the minus-sign coming with the η-derivative is needed to compensate for the minus-sign coming from the anti-commutation relations:

$$\frac{\delta}{-i\delta(\eta_2)_b}(i\bar{\psi}\eta) = \int d^4x\, \bar{\psi}^a(x)\frac{\delta}{\delta\eta_b(x_2)}\,\eta_a(x) = \int d^4x\, \bar{\psi}^a(x)\, \delta_a{}^b\, \delta^4(x_2-x) = \bar{\psi}^b(x_2)\,. \tag{3.78}$$

As before, the free case can be treated completely explicitly (note that we suppress the $i\epsilon$ for brevity):

$$iS_0 = i\int d^4x\, \bar{\psi}(i\not{\partial}-m)\psi \equiv -\bar{\psi}S^{-1}\psi\,, \tag{3.79}$$

with the fermionic or spinor propagator S (do not confuse with the action, which is unfortunately also called S) explicitly given by

$$S^{-1}(x,y) = -i(i\not{\partial}_x - m)\,\delta^4(x-y) \qquad \text{and} \qquad S(x,y) = \int d^4k\frac{i}{\not{k}-m}\,e^{ik(x-y)}\,. \tag{3.80}$$

With this, one finds

$$Z_0[\bar{\eta},\eta] = \int D\bar{\psi}\, D\psi\, \exp\left[-\bar{\psi}S^{-1}\psi + i\bar{\eta}\psi + i\bar{\psi}\eta\right] \tag{3.81}$$

and completes the square by the substitutions

$$\psi \quad\to\quad \psi + iS\eta \qquad\quad \text{and} \qquad\quad \bar{\psi} \quad\to\quad \bar{\psi} + i\bar{\eta}S\,. \tag{3.82}$$

Since we are working with the formally defined Grassmann variable integration, the underlying operation of shifting the integration variable requires justification. Indeed, consider the generic function $f(\theta) = f_0 + f_1\theta$. Then, on the one hand,

$$\int d\theta\, f(\theta) = f_1\,, \tag{3.83}$$

but also

$$\int d\theta\, f(\theta - \eta) = \int d\theta\,(f_0 + f_1\theta - f_1\eta) = f_1\,. \tag{3.84}$$

Thus, our final formula reads

$$Z_0[\bar\eta,\eta] = \int D\bar\psi\, D\psi\, e^{-\bar\psi S^{-1}\psi}\, e^{-\bar\eta S\eta} = Z_0[0,0]\, e^{-\bar\eta S\eta}\,. \tag{3.85}$$

The free-theory 2-point-function is thus obtained as

$$\langle T\psi(x)\bar\psi(y)\rangle = \frac{\delta}{i\delta\bar\eta(x)}\left(\frac{\delta}{-i\delta\eta(y)}\right) e^{-\bar\eta S\eta} = S(x-y)\,, \tag{3.86}$$

with the corresponding Feynman rule given in Fig. 14.

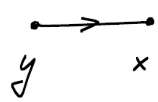


Figure 14: Fermion propagator (the arrow corresponds to going from the right to the left index of the matrix S).

To conclude, interactions are incorporated as in the bosonic case: One rewrites $S_{int.}$ in the definition of Z according to

$$S_{int.}[\bar\psi,\psi,\varphi] \qquad \to \qquad S_{int.}\left[\frac{\delta}{-i\delta\eta}, \frac{\delta}{i\delta\bar\eta}, \frac{\delta}{i\delta j}\right]\,. \tag{3.87}$$

Then, $\exp[iS_{int.}]$ is taken outside the path integral in the definition of Z.

Here we included also the bosonic field φ and its source to allow for the simplest interaction a fermion can have – the Yukawa interaction. Indeed, it is a good exercise to consider the theory defined by

$$iS = -\bar\psi S^{-1}\psi - \frac{1}{2}\varphi D^{-1}\varphi - i\int_x \lambda\bar\psi\psi\,\varphi = -\bar\psi S^{-1}\psi - \frac{1}{2}\varphi D^{-1}\varphi + iS_{int.} \tag{3.88}$$

and to work out the Feynman rules by calculating the simplest n-point functions. Use the 'trick with the sources' as described in the bosonic case (and as recalled just above).

3.6 The 'bosonic' and 'fermionic determinant'

It is instructive and will be useful in what follows to calculate certain simple path integrals even more explicitly. We start with the conventional (bosonic) Gaussian integral

$$\int \frac{d^2 z}{\pi}\, e^{-a\bar{z}z} = \frac{1}{\pi}\int r\, dr\, d\varphi\, e^{-ar^2} = \frac{1}{a} \tag{3.89}$$

and its fermionic analogue

$$\int d\bar{\theta}\, d\theta\, e^{-a\bar{\theta}\theta} = \int d\bar{\theta}\, d\theta\, (1 - a\bar{\theta}\theta) = a\,. \tag{3.90}$$

The very important 'inversion' of the result when going from a commuting to an anti-commuting variable continues to hold in the many-variable case.

Indeed, consider the integral

$$I_B = \prod_i \left(\int \frac{d^2 z_i}{\pi}\right) \exp(-\bar{z}^j A_j{}^k z_k)\,, \tag{3.91}$$

with A a positive hermitian matrix (to ensure manifest convergence). Now, let us interpret, for each i,

$$\int d^2 z_i \qquad \text{as} \qquad \frac{1}{2i}\int d\bar{z}^i \wedge dz_i = \frac{1}{2i}\int (dx_i - idy_i) \wedge (dx_i + idy_i)\,, \tag{3.92}$$

where on the r.h. side we think of integrating a 2-form over the real subspace of $\mathbb{C}^2$ parametrized by x_i and y_i.[7] With the substitution $z_i = U_i{}^j z'_k$ and making use of

$$\prod_i dz_i = \det(U) \prod_i dz'_i \qquad \text{and} \qquad \prod_i d\bar{z}_i = \det(U)^* \prod_i d\bar{z}'_i \tag{3.93}$$

we find

$$I_B = \prod_i \left(\int \frac{d^2 z'_i}{\pi}\right) |\det U|^2 \, \exp(-\bar{z}'^j (U^\dagger)_j{}^k A_k{}^l U_l{}^m z'_m)\,. \tag{3.94}$$

Now we choose U unitary (implying $|\det U| = 1$) such that $U^\dagger A U$ is diagonal. The result then becomes the product of all inverse eigenvalues of A, hence

$$I_B = \prod_i \left(\int \frac{d\bar{z}^i\, dz_i}{2\pi i}\right) e^{z^\dagger A z} = \frac{1}{\det A}\,. \tag{3.95}$$

Next, we repeat this for Grassmann variables:

$$\begin{aligned} I_F &= \prod_i \left(\int d\bar{\theta}^i\, d\theta_i\right) \exp(-\bar{\theta}^j A_j{}^k \theta_k) & (3.96)\\ &= \prod_i \left(\int d\bar{\theta}^i\, d\theta_i\right) \frac{1}{n!} (-\bar{\theta}^{j_1} A_{j_1}{}^{k_1} \theta_{k_1}) \cdots (-\bar{\theta}^{j_n} A_{j_n}{}^{k_n} \theta_{k_n})\,. & (3.97) \end{aligned}$$

[7]For those not familiar with differential forms, consider z to be real and A symmetric. Then the analogy to the anticommuting case is not quite as perfect, but the idea should still be clear. Altrnatively, just accept that one can integrate formally of $d\bar{z}\, dz$ rather than over $dx\, dy$.

Due to the anticommutation relations we have

$$(\bar{\theta}^{j_1}\theta_{k_1})\cdots(\bar{\theta}^{j_n}\theta_{k_n}) = (\bar{\theta}^1\theta_1)\cdots(\bar{\theta}^n\theta_n)\,\epsilon^{j_1\cdots j_n}\epsilon_{k_1\cdots k_n}\,. \tag{3.98}$$

Here we also used the fact that, when exchanging two θs, we don't get signs from the $\bar{\theta}$s since we 'pass over' each $\bar{\theta}$ twice. Since

$$\det\, A = \frac{1}{n!}\epsilon^{j_1\cdots j_n}A_{j_1}{}^{k_1}\cdots A_{j_n}{}^{k_n}\epsilon_{k_1\cdots k_n}\,, \tag{3.99}$$

and since the explicit minus-signs disappear due to $(\bar{\theta}^j\theta_j) = -(\theta_j\bar{\theta}^j)$, we eventually have

$$I_F = \det\, A\,, \tag{3.100}$$

i.e. precisely the inverse of the bosonic result.

If we are now prepared to generalize this formally to infinite-dimensional matrices A, both our bosonic and fermionic results immediately apply to QFT. Ignoring normalization factors, we have

$$Z_0[\bar{\jmath}, j]\Big|_{\bar{\jmath}=j=0} = \int D\bar{\varphi}\, D\varphi\, e^{-\bar{\varphi}D^{-1}\varphi} \sim \frac{1}{\det\, D^{-1}} \tag{3.101}$$

for a complex scalar and

$$Z_0[\bar{\eta}, \eta]\Big|_{\bar{\eta}=\eta=0} = \int D\bar{\psi}\, D\psi\, e^{-\bar{\psi}S^{-1}\psi} \sim \det\, S^{-1} \tag{3.102}$$

for a Dirac fermion. The expressions on the r.h. side contain infinite products of ever-growing eigenvalues of differential operators, but this is not as meaningless as it might seem. Indeed, in many cases one is interested in the logarithm i.e. in (for example in the bosonic case)

$$\ln\, Z_0[0] = -\ln\,\det\, D^{-1} = -\mathrm{tr}\,\ln\, D^{-1} = -\int \frac{d^4k}{(2\pi)^4}\,\ln(k^2-m^2) + \cdots\,, \tag{3.103}$$

where the k-integral replaces the trace and the relevant basis is that of exponentials $\exp(ikx)$. The last expression can then be regularized and calculated. Even without regularization, it is clear that sufficiently high derivatives w.r.t. m^2 are finite.

4 Path integral quantization of gauge theories

4.1 Reminder of the basic structure and preliminary remarks

In QFT I, we discussed the quantization of abelian gauge theories in some detail but introduced non-abelian theories only at the classical level. The reason is that the canonical quantization of non-abelian gauge theories is, in fact, rather complicated and the path integral approach is preferred. Thus, this is the right moment to pick up this topic. Naturally, all that follows includes the abelian or $U(1)$ theory as a special case.

We recall that our starting point was a 'matter' field ψ transforming in some representation R of a gauge group G. For simplicity, we will always think of $SU(N)$ and, when it comes to matter, of its fundamental representation, but you should be able to generalize this if appropriate.

Now, to write down an action invariant under[8]

$$\psi(x) \to U(x)\psi(x)\,, \qquad \text{with} \qquad U(x) \in SU(N)\,, \tag{4.1}$$

we want to ensure that

$$D_\mu\psi \to U D_\mu\psi \tag{4.2}$$

with D_μ an appropriate generalization of the partial derivative ∂_μ. This generalization is provided by

$$D_\mu = \partial_\mu + iA_\mu \qquad \text{with} \qquad A_\mu \in su(N) \tag{4.3}$$

(or $A_\mu \in Lie(G)$ more generally), where A_μ transforms as

$$iA_\mu \to U(\partial_\mu U^{-1}) + U\, iA_\mu\, U^{-1}\,. \tag{4.4}$$

With this, the matter field lagrangian can be made gauge-invariant (consider e.g. $\bar\psi(i\not{D} - m)\psi$).

To write down an action for A_μ, we introduced the field strength

$$F_{\mu\nu} \equiv \frac{1}{i}[D_\mu, D_\nu]\,, \tag{4.5}$$

which transforms as

$$F_{\mu\nu} \to U F_{\mu\nu} U^{-1}\,. \tag{4.6}$$

A gauge invariant A_μ lagrangian is then provided by

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})\,. \tag{4.7}$$

Using a normalized basis T^a of the Lie algebra,

$$\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}\,, \tag{4.8}$$

we can go over to components and write

$$A_\mu = A_\mu^a T^a\,; \qquad F_{\mu\nu} = F_{\mu\nu}^a T^a\,. \tag{4.9}$$

After the rescaling $A_\mu \to gA_\mu$, one finds

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{a\,\mu\nu}\,; \qquad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c\,, \tag{4.10}$$

[8]Replace this with $\psi \to R(U)\psi$ and $U \in G$ and make all the relevant changes in what follows to make sure you understand the general case.

where we used the structure constants of the Lie algebra defined by

$$[T^a, T^b] = i f^{abc} T^c \,. \tag{4.11}$$

The lagrangian naturally splits into free and interacting part, $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int.}$, with

$$\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\nu A^{\mu\, a} - \partial^\mu A^{a\, \nu})\,, \tag{4.12}$$

which is just as in QED but with N vector fields labelled by a. The interactions,

$$\mathcal{L}_{int.} \supset g A^3 \,,\; g^2 A^4 \,, \tag{4.13}$$

obviously induce 3- and 4-point vertices making, in contrast to QED, already the pure gauge theory interacting.

Now, one is tempted to interpret A^a_μ as a set of $4N$ bosonic fields, to define the correlation functions of the corresponding operators by

$$\langle T \hat{A}^a_\mu(x_1) \hat{A}^b_\nu(x_2) \cdots \rangle = \frac{\int DA A^a_\mu(x_1) A^b_\nu(x_2) \cdots e^{iS}}{\int DA e^{iS}} \,, \tag{4.14}$$

and to evaluate the r.h. side with the methods we developed for the real scalar. From this, we would then get scattering amplitudes for photons (or, for $G = SU(3)$, gluons) after Fourier transforming, amputating external lines and going on-shell. (The last of course requires LSZ to go through also for non-abelian gauge theories – which it does, but this requires more thought.)

However, there are two obstacles, both related to gauge invariance. The first is technical: In our perturbative approach to the evaluation of path integrals we need the inverse of the differental operator defining $\mathcal{L}_0$. But, and we already encountered this problem in QFT I, this inverse doesn't exist for gauge theories. We will learn how to deal with this in a moment.

Second, we have a conceptual problem since we are trying to evaluate a non-gauge invariant quantity. Really, we should not use (4.14) but rather

$$\langle 0 | \, T \, O[\hat{A}] \, | 0 \rangle = \frac{\int DA \, O[A] \, e^{iS}}{\int DA \, e^{iS}} \,, \tag{4.15}$$

where $O[\hat{A}]$ is a gauge-invariant expression depending on the fields A^a_μ (an 'observable'). We now briefly argue that specific versions of (4.14) fall into this category.

In QED, we had a proof (based on Ward-Takahashi identities) that with all the external fermions on-shell the photon lines (external and internal [9]) are insensitive to gauge change ($k^\mu \mathcal{M}_\mu = 0$). Thus, amplitudes evaluated on the basis of (4.14) and even correlation functions for physical photons (on-shell and contracted with a physical polarization vector), i.e. after

$$A_\mu(x) \quad \rightarrow \quad \epsilon^\mu(k) \tilde{A}_\mu(k) \,, \tag{4.16}$$

[9] This holds since an internal photon line always connects to either an external fermion line or a fermion loop.

are gauge invariant. Hence, if we use (4.14) to calculate only such gauge-invariant quantities, we are basically calculating expectation values of gauge-invariant operators, as in (4.15). This is sensible.

For non-abelian theories, such as QCD, the gauge invariance issue is much more subtle due to the direct coupling between gauge bosons, cf. Fig. 15. Nevertheless, on-shell amplitudes with all gluons physically polarized are gauge invariant. Thus, we may again say that

$$T\,O[\hat{A}] = \text{Fourier-trf. / on-shell / phys.-pol. version of } \Big\{ T\,A_\mu^a(x_1)A_\nu^b(x_2)\cdots \Big\}\,. \tag{4.17}$$

In this interpretation, (4.15) again makes sense.

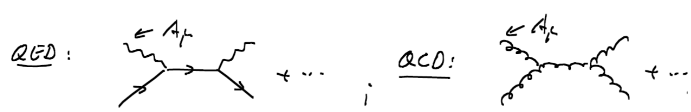


Figure 15: Illustration of the more complicated gauge boson interaction in non-abelian theories.

Finally, there are of course important physical situations when (4.15) can be applied as it stands. For example, the lightest particles in pure non-abelian gauge theory (pure **'Yang-Mills theory'**) are glueballs. There are operators G, built from the fundamental fields, such that

$$\langle 0|\, G[A](x)\, |\text{glueball}\rangle \neq 0\,. \tag{4.18}$$

Then we can set

$$O[A] = TG(x_1)G(x_2)\cdots \tag{4.19}$$

and, after Fourier-transforming etc., obtain scattering amplitudes and decay widths of glueballs. Very roughly speaking, this is what's done on the lattice, where the path integral formula (4.15) is used as it stands, after S is discretized appropriately. It's not possible for us, being bound to analytical methods and hence mostly perturbation theory.

Furthermore, we can couple our pure Yang-Mills theory to matter and we will have plenty of examples of practically relevant, manifestly gauge-invariant operators to be placed under the path integral. The two-photon correlation function is one such example, as should be clear from Fig. 16. (Of course, the path integral must now include also quarks and photons.)

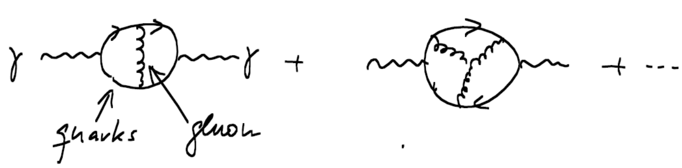


Figure 16: Gluons contributing to the photon self-energy at higher order.

Anyway, we have now provided enough motivation for learning how to evaluate path integrals of the type

$$\int DA\, O[A]\, e^{iS} \tag{4.20}$$

with O a gauge invariant expression in terms of the gauge fields A_μ^a. Let's now actually do it.

4.2 Fadeev-Popov method

We immediately anticipate a technical problem related to gauge invariance. Indeed, recall that under a gauge rotation by

$$U = e^{-ig\chi}\,; \qquad \chi(x) \in su(N) \tag{4.21}$$

we have

$$iA_\mu \quad \to \quad iA_y^\chi \equiv U(\partial_\mu U^{-1}) + iUA_\mu U^{-1}\,. \tag{4.22}$$

If we now fix some set of field configurations $A^{(0)}$ which are not related to each other by gauge transformation, the path integral splits as

$$\int DA = \int DA^{(0)}\, D\chi\,. \tag{4.23}$$

The χ integration looks divergent since the integrand is by definition independent of χ. However, this is only an artifact of perturbation theory. Indeed, the divergence is only real if we think of A_μ in terms of a Taylor expansion around $A_\mu = 0$. In this case, the $D\chi$-integral is just a product of unbounded real integrals, each being divergent.

However, since the relevant gauge groups are always compact (e.g. $U(1) \sim S^1$, $SU(2) \sim S^3$), it is clear that χ should actually be integrated only over a finite range. For example, in the $U(1)$ case $e^{2\pi i} = 1$ and larger χ brings one back to field configurations which one already encountered.

In numerical appoaches, on the lattice, one can define the DA-integration such that the gauge redundancy is explicitly finite (apart, of course, from infinities related to the infinite-volume or zero-lattice-spacing limit). One can then proceed without the gauge-fixing that we are going to develop next.

We note that, if we write (recall that the x-integration is implicit in this notation)

$$iS[A] = -\frac{1}{2}A_\mu (D^{-1})^{\mu\nu} A_\nu\,, \tag{4.24}$$

the operator $(D^{-1})^{\mu\nu}$ has eigenfunctions with zero eigenvalue, for exactly the reason just discussed. This was already mentioned in Sect. 4.1 and it was solved in QFT I by gauge fixing. The same idea works here:

We fix the gauge by demanding $G(A) = 0$ for some non-gauge-invariant function G of A. For example,

$$G(A) = \partial_\mu A^\mu - \omega\,, \tag{4.25}$$

with an arbitrary function $\omega(x)$. Both ω and G take their values in the Lie algebra.

Next, generalizing the familiar relation

$$1 = \int dx\, \delta(f(x))\, |f'(x)| \tag{4.26}$$

to the case of functionals, we have

$$1 = \int D\chi\, \delta[G(A^\chi)] \left| \det \left(\frac{\delta G(A^\chi)}{\delta \chi} \right) \right| . \tag{4.27}$$

Here it may be useful to recall that, by definition,

$$\delta G(A^\chi)(x) = \int d^4y \frac{\delta G(A^\chi)(x)}{\delta \chi(y)} \cdot \delta\chi(y) + \cdots . \tag{4.28}$$

Hence $\delta G(A^\chi)/\delta\chi$ is a linear operator on the space of Lie-algebra-valued functions, and $\delta G(A^\chi)(x)/\delta\chi(y)$ is its operator kernel. The determinant of such objects can be defined after UV/IR regularization, as mentioned earlier.

Using (4.27), we have

$$\langle T O[\hat{A}] \rangle = \frac{\int D\chi \int DA\, \delta[G(A^\chi)] \left| \det \left(\frac{\delta G(A^\chi)}{\delta\chi} \right) \right| O[A]\, e^{iS[A]}}{\int D\chi \int DA\, \delta[G(A^\chi)] \left| \det \left(\frac{\delta G(A^\chi)}{\delta\chi} \right) \right| e^{iS[A]}} , \tag{4.29}$$

which can be interpreted as splitting the integration over all A according to Fig. 17. Since numerator and denominator differ only by the insertion of $O[A]$, we can focus on the numerator in what follows.

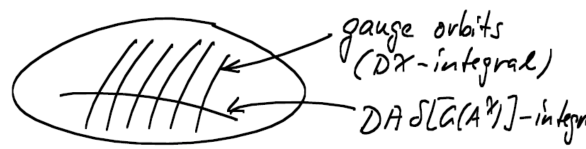


Figure 17: One first integrates over fields A_μ subject to the gauge-fixing condition with a specific χ, and subsequently over all gauge orbits, i.e. over χ.

First, we have

$$O[A]\, \exp(iS[A]) = O[A^\chi]\, \exp(iS[A^\chi]) \tag{4.30}$$

due to gauge invariance. Next, we can change the integration variable from A to A^χ,

$$\int DA = \int DA^\chi \left| \det \left(\frac{\delta A^\chi}{\delta A} \right) \right|^{-1} . \tag{4.31}$$

To evaluate the determinant, focus first on infinitesimal χ, evaluating δA^χ in linear order in χ only. Referring back to (4.22), we have

$$\delta A^\chi = \delta A - i[g\chi, \delta A] . \tag{4.32}$$

The functional derivative $\delta(A^\chi)^a(x)/\delta A^b(y)$ is clearly proportional to the unit matrix w.r.t. its space-time indices, $\sim \delta^4(x-y)$. It is non-trivial with respect to its gauge indices. However, one can easily show (using the total antisymmetry of f^{abc}, which you should also prove), that

$$\frac{\delta A^\chi}{\delta A} \sim \mathbb{1} + M\,, \tag{4.33}$$

with M an antisymmetric matrix (cf. problems). Hence $\det(\mathbb{1}+M) = 1 + O(M^2)$ and our full determinant is unity in linear order in χ. However, finite gauge transformations can be built as an infinite product of infinitesimal ones, so that we have in full generality

$$\det\left(\frac{\delta A^\chi}{\delta A}\right) = 1\,. \tag{4.34}$$

We can thus simply replace DA by DA^χ. Finally, we can also write

$$\left|\det\left(\frac{\delta G(A^\chi)}{\delta\chi}\right)\right| = \left|\det\left(\frac{\delta G(A^{\chi+\chi'})}{\delta\chi'}\right)\right|_{\chi'=0}\,. \tag{4.35}$$

Putting everything together, we get

$$\int D\chi \int DA^\chi\,\delta[G(A^\chi)]\,\left|\det\left(\frac{\delta G(A^{\chi+\chi'})}{\delta\chi'}\right)\right|_{\chi'=0} O[A^\chi]\,e^{iS[A^\chi]}\,. \tag{4.36}$$

Now we rename $A^\chi \to A$, making it apparent that the $D\chi$-integral gives just an overall, constant prefactor, which we are allowed to drop. After that, we rename $\chi' \to \chi$ and have

$$\langle TO[\hat{A}]\rangle = \frac{\int DA\,\delta[G(A)]\,\left|\det\left(\frac{\delta G(A^\chi)}{\delta\chi}\right)\right|_{\chi=0} O[A]\,e^{iS[A]}}{\int DA\,\delta[G(A)]\,\left|\det\left(\frac{\delta G(A^\chi)}{\delta\chi}\right)\right|_{\chi=0} e^{iS[A]}}\,. \tag{4.37}$$

At the conceptual level we are now finished: The freedom is fixed and our path integral is well-defined. Nevertheless, a few more steps are needed before the expression becomes practically useful.

First, it is often convenient to replace the gauge choice $G = \partial A - \omega = 0$ with an arbitrary but fixed ω by a so-called 'averaged' gauge. Indeed, while ω explicitly appears under the path integral, we know that the result actually is gauge invariant and can *not* really depend on it. Thus, we may simply introduce a further path integral

$$\int D\omega\,\exp\left(-i\int\frac{\lambda}{2}\omega(x)^2\right) \tag{4.38}$$

in numerator and denominator. (We use the shorthand ω^2 for $\omega^a\omega^a$ here and below.) The ω integration can then be performed making use of

$$\delta[G(A)] = \delta[\partial A - \omega]\,. \tag{4.39}$$

Focussing just on the numerator, we now have

$$\int DA \left|\det\left(\frac{\delta G(A^\chi)}{\delta\chi}\right)\right|_{\chi=0} O[A]\, \exp i\left(S[A]-\frac{\lambda}{2}\int(\partial A)^2\right)\,, \tag{4.40}$$

where we recognize the gauge-fixing term familiar from the Gupta-Bleuler approach.

Second, we use the previously derived formula

$$\det M = \int D\bar\theta\, D\theta\, e^{-\bar\theta M\theta} \tag{4.41}$$

to evaluate the **Fadeed-Popov determinant** $\det(\delta G/\delta\chi)$. Explicitly, we recall that

$$(A^\chi_\mu)^a = A^a_\mu + \partial_\mu\chi^a + gf^{abc}\chi^b A^c_\mu \tag{4.42}$$

and hence

$$\begin{aligned}\frac{\delta G(A^\chi)^a(x)}{\delta\chi^d(y)} &= \frac{\delta}{\delta\chi^d(y)}\big(\partial A^a(x)+\partial^2\chi^a(x)+gf^{abc}\partial^\mu(\chi^b(x)A^c_\mu(x))\big) &(4.43)\\ &= \delta^{ad}\partial^2\delta^4(x-y)+gf^{adc}\partial^\mu_x(\delta^4(x-y)A^c_\mu(x))\,. &(4.44)\end{aligned}$$

The unphysical fermionic fields introduced to evaluate the determinant of this last matrix are called **ghosts** and are frequently denoted by $\bar c$ and c, such that

$$\det\left(\frac{\delta G}{\delta\chi}\right) = \tag{4.45}$$
$$\int D\bar c\, Dc\, \exp\left[-i\int d^4x\, d^4y\, \bar c^a(x)\left\{\delta^{ab}\partial^2\delta^4(x-y)+gf^{abc}\partial^\mu_x(\delta^4(x-y)A^c_\mu(x))\right\}c^b(y)\right].$$

The overall prefactor and hence the "i" are irrelevant.

The expression under the exponent is equivalent to the introduction of a ghost action

$$iS_{ghost} = i\int d^4x\,\mathcal{L}_{ghost} = i\int d^4x\,\bar c^a[-\delta^{ab}\partial^2 - gf^{abc}\partial^\mu A^c_\mu]c^b \tag{4.46}$$

into our theory. Note that here ∂^μ acts both on A and on c.

The ghosts carry gauge indices just like A and one can think of them as of fields transforming in the 'adjoint representation' of the group. Indeed, apart from their Grassmann nature, one can think of c as living in the Lie algebra

$$c = c^a T^a \tag{4.47}$$

and transforming correspondingly:

$$c \qquad\to\qquad UcU^{-1} = e^{-ig\chi}c\,e^{ig\chi} = c - ig[\chi, c]+\cdots\,. \tag{4.48}$$

This is (quite generally) the so-called adjoint representation by which a group acts on a vector space isomorphic to its own Lie algebra. In the last expression, you see the infinitesimal version, i.e. the action of the Lie-algebra element χ on another Lie-algebra element, in this case c. Correspondingly, one can define a covariant derivative of a field taking values in the Lie-algebra,

$$D_\mu c = \partial_\mu c + ig[A_\mu, c]\,. \tag{4.49}$$

In components, this reads

$$(D_\mu c)^a = \partial_\mu c^a - g f^{abc} A^b_\mu c^c\,, \tag{4.50}$$

and hence we can also write

$$\mathcal{L}_{ghost} = -\bar{c}\,\partial^\mu\big(\partial_\mu c + ig[A_\mu, c]\big) = -\bar{c}\,\partial^\mu D_\mu c\,. \tag{4.51}$$

To play with the new concept of a field in the adjoint representation, you may want to prove gauge invariance and work out the component form of the lagrangian

$$\mathcal{L} = -\mathrm{tr}\big(D_\mu \Phi D^\mu \Phi\big) \tag{4.52}$$

of a so-called adoint scalar Φ (a field taking values in the Lie algebra or, if you wish, in its complexification). Determine the relevant representation matrices $R_{adj.}(A_\mu)_a{}^b$ and derive the Feynman rules for the coupling of Φ to gauge bosons!

In summary, we have derived the fundamental result

$$\langle T\, O[A]\rangle = \frac{1}{Z[0]}\int DA\, D\bar{c}\, Dc\, O[A]\, e^{iS[A]+iS_{gf}[A]+iS_{ghost}[\bar{c},c,A]} \tag{4.53}$$

where

$$\mathcal{L}_{gf} = -\frac{\lambda}{2}\,(\partial A)^2 \qquad \text{and} \qquad \mathcal{L}_{ghost} = -\bar{c}\,\partial^\mu D_\mu c\,, \tag{4.54}$$

are the so-called gauge-fixing and ghost lagrangians. $Z[0]$ is defined by the path integral given in (4.53), but without the insertion of $O[A]$.

4.3 Feynman rules

Introduce sources j^a_μ, $\bar{\eta}^a$, η^a for the fields A^a_μ, c^a, $\bar{c}^a$, such that

$$S \quad \to \quad S + \int d^4x\, (j^{\mu\, a} A^a_\mu + \bar{\eta}^a c^a + \bar{c}^a \eta^a)\,, \tag{4.55}$$

and define a generating functional Z by

$$Z[j_\mu, \eta, \bar{\eta}] = \int DA\, D\bar{c}\, Dc\, \exp i[S + Aj + \bar{\eta}c + \bar{c}\eta]\,. \tag{4.56}$$

Write the action as $S = S_0 + S_{int.}$, where S_0 contains all terms that are quadratic in $A, c, \bar{c}$, and define Z_0 on the basis of S_0. As usual, the source-dependence of Z_0 can be determined by completing the square:

$$Z_0[j, \eta, \bar{\eta}] = Z_0[0]\, \exp\left[-\frac{1}{2} j D_A j - \bar{\eta}\, D\eta\right]\,. \tag{4.57}$$

Next, $S_{int.} = S_{int.}[A, \bar{c}, c]$ is incorporated by writing its field-arguments as derivatives w.r.t the sources, i.e.

$$Z[j,\eta,\bar{\eta}] = \exp\left[iS_{int.}\left(\frac{\delta}{i\delta j}, \frac{\delta}{-i\delta\eta}, \frac{\delta}{i\delta\bar{\eta}}\right)\right] Z_0[j,\eta,\bar{\eta}]\,. \tag{4.58}$$

This is already our final formula from which the Feynman rules follow by working out 2-,3- and 4-point functions and identifying the analytical terms corresponding to propagators and vertices. The only unusual feature of the above theory are the ghosts, which are scalars and at the same time fermions (Grasssmann variables). This contradicts the spin-statistic theorem and is only permissible since the ghosts are not physical particles. Their sources, η and $\bar{\eta}$ will always be set to zero, such that ghosts appear only in loops.

Let us start with the central element, the gauge boson propagator. It follows from the two quadratic pieces

$$-\frac{1}{2}\mathrm{tr}\,F^2 \supset \frac{1}{2}(-\partial_\mu A_\nu \partial^\mu A^\nu + \partial_\mu A^\mu \partial_\nu A^\nu) \tag{4.59}$$

and

$$-\frac{\lambda}{2}(\partial A)^2 = -\frac{\lambda}{2}\partial_\mu A^\mu \partial_\nu A^\nu\,. \tag{4.60}$$

Here the gauge index a of A^a_μ and its summation is suppressed for brevity. The two pieces above combine into an inverse propagator in momentum space

$$D_A^{-1}(k)_{\mu\nu} = i[k^2\eta_{\mu\nu} - k_\mu k_\nu(1-\lambda)]\,, \tag{4.61}$$

where the prefactor (1/2) has disappeared and a prefactor 'i' has been introduced as a result of our definition of the propagator, of the Fourier transform and the opposite signs of the momenta k belonging to the two A_μ-factors.

The inverse follows by making the ansatz

$$D_A(k)_{\mu\nu} = A\eta_{\mu\nu} + Bk_\mu k_\nu \tag{4.62}$$

and determining A and B. It reads

$$D_A(k)_{\mu\nu} = \frac{-i}{k^2+i\epsilon}\left(\eta_{\mu\nu} + \frac{(1-\lambda)}{\lambda}\cdot\frac{k_\mu k_\nu}{k^2}\right) \tag{4.63}$$

or, with the frequently used parameterization by $\xi = 1/\lambda$,

$$D_A(k)^{ab}_{\mu\nu} = \frac{-i}{k^2+i\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\,\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab}\,. \tag{4.64}$$

In the last expression, we restored the previously suppressed gauge indices. We should have also have followed the '$i\epsilon$' convergence factor throughout the analysis. However, it was easy to restore it in the end recalling that the physical polarizations of A_μ should behave like massless real scalars. This fixes the sign unambiguously since we want an exponential suppression of high-energy states. Hence, k^2 is replaced by $k^2 + i\epsilon$. The $i\epsilon$

associated with the $k_\mu k_\nu$ term could in principle be determined, but it will drop out together with the gauge parameter dependence when evaluating observables.

These are all "general Lorentz" or "covariant" gauges. The specific choice $\xi = 1$ is called Feynman gauge, the choice $\xi = 0$ Landau gauge. The latter gauge is special since it is the only one of the covariant gauges which is "truly fixed" rather than just averaged, as explained before. Indeed, recall that the gauge fixing lagrangian emerged from

$$\int D\omega \, e^{-i(\lambda/2)\int \omega^2} \,, \tag{4.65}$$

with $\omega = \partial_\mu A^\mu$. Thus, only at $\lambda = \infty$ (i.e. $\xi = 0$) is the condition $\partial_\mu A^\mu = 0$ really strictly enforced. Not surprisingly, the Landau-gauge propagator is transverse, i.e. longitudinal gauge bosons do not propagate in Landau gauge:

$$(\eta_{\mu\nu} - k_\mu k_\nu / k^2)\, k^\mu = 0 \,. \tag{4.66}$$

The ghost propagator is a massless, scalar Feynman propagator, $D(k) = i/(k^2 + i\epsilon)$. We leave the remaining Feynman rules, of which especially the 3- and 4-gauge-boson vertices are worth checking in detail, as an excercise.

To derive the latter, it useful to write down explicitly how the terms in $S_{int.}$ act on a set of propagator terms. Symbolically, one has

$$\left(\int d^4x \left(\frac{\delta}{i\delta j(x)}\right)\left(\frac{\delta}{i\delta j(x)}\right)\left(\frac{\delta}{i\delta j(x)}\right)\right) \left(- \int_y \int_z j(y) D_A(y-z) j(z)\right)\left(\ldots\right)\left(\ldots\right). \tag{4.67}$$

Here we have for brevity suppressed all prefactors, indices, space-time derivatives acting $\delta/(\delta j(x))$ etc. Then, one goes over to the Fourier transform, keeping in mind some fixed convention about the momentum flow, e.g. that in

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \, \frac{i}{k^2} e^{-ik(x-y)} \tag{4.68}$$

the momentum flows from y to x. (This is relevant for the sign of the 3-gauge-boson vertex.) The result is summaried in Fig. 18. Note that this also applies to the abelian case, including the ghosts. However the ghosts decouple and the 3/4-gauge-boson vertices vanish since $f^{abc} = 0$ (actually, since there is just one generator, $f^{aaa} = 0$).

Let us include charged fields, in the simplest case Dirac fermions in the fundamental representation of $SU(N)$,

$$\mathcal{L}_{matter} = \bar{\psi}^i [(i\not{D} - m)\psi]_i \,, \tag{4.69}$$

where the fundamental-representation index i runs from 1 to N (cf. the 'color-index' of quarks in QCD running over 'r,g,b'). We chose to write the fundamental index as a subscript, the anti-fundmantal one as a superscript, but this is of course merely conventional. The two extra Feynman rules are easy to derive and the result is given in Fig. 19. Finally, just as reminder, the Feynman rules for incoming/outgoing particles are

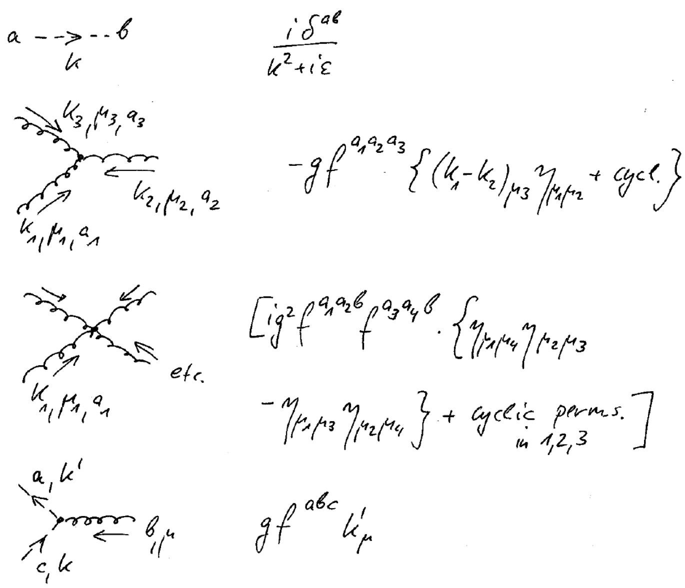


Figure 18: Feynman rules for non-abelian gauge theories.

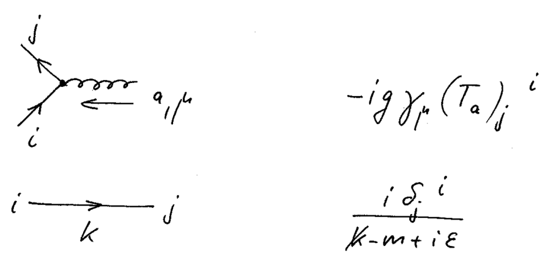


Figure 19: Coupling to Dirac matter in the fundamental representation.

summarized in Fig. 20. Note that, concerning the gauge boson states, we are working at the moment by analogy to QED – understanding the corresponding Hilbert space in non-abelian theories is our task for the next Section.

Among the large number of related books, we recommend in particular those by Weinberg and Pokorski [13, 14].

5 BRST symmetry, Hilbert space and canonical quantization

In the last section, we have learned to calculate correlation functions of gauge invariant operators. Physical states can be constructed by letting such operators act on the vacuum. This gives us the Hilbert space and the ability to calculate transition amplitudes from correlation functions.

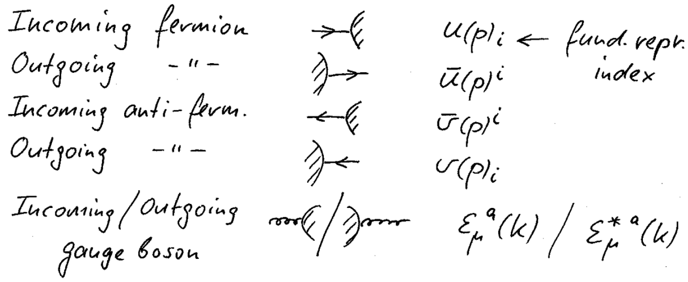


Figure 20: Incoming/outgoing states.

Even more directly, if we recall the concept of the Schrödinger wave functional and restrict these functionals by the requirement of gauge invariance, we have the Hilbert space. The path integral immediately allows for the calculation of transition amplitudes. Clearly, the Hamiltonian is indirectly defined by our ability to follow the time-evolution of states: We just call H the operator generating this evolution.

So why invest more time?

One very good reason is that both experimentally[10] and calculationally, the particle-based / perturbative way of thinking is very useful. However, the wave-functional approach to the Hilbert space is somehow 'orthogonal' to this – cf. our discussion of coherent states and how they differ from energy eigenstates. Thus, we want to be able to construct the Hilbert space also perturbatively, as a Fock space corresponding to the free part of the lagrangian. This is also necessary if we want to repeat the LSZ argument to relate correlation functions to scattering amplitudes.[11] For all of the reasons above, it is crucial to understand the Fock space and, in this context, what the physical gauge boson states are.

Jumping ahead, the outcome will be as expected: Analogously to QED, external gluons (my shorthand for 'non-abelian gauge bosons' from now on) have to be transversally polarized, external ghosts are not allowed.

[10] One important way in which we how know about QCD, an $SU(3)$ gauge theory with fundamental fermions (quarks), is through quark and gluon scattering at machines like the LHC. QCD is in think context also crucial as a stepping stone towards new ('beyond-the-standard-model') physics to be studied at particle colliders. While the LHC really collides protons and produces 'jets' of hadrons, it is by now well-understood how to relate this to the underlying process where quarks/gluons scatter into quarks/gluons. (See also later in this course.) Similarly, electroweak interactions are described by an $SU(2) \times U(1)$ gauge theory and we probe them via scattering processes involving the corresponding gauge-bosons ($W^{\pm}, Z, \gamma$).

[11] Conceptually, this may seem like a step back since gauge invariance will only emerge order-by-order in perturbation theory, as opposed to our 'gauge-invariant-operator' point of view. Nevertheless, technically such amplitudes are central.

5.1 BRST symmetry

Let us write the gauge fixing condition as

$$\delta[(\partial A) - \omega] \sim \int Db \exp\, i \int d^4x\, b^a(\omega - \partial A)^a\,, \tag{5.1}$$

where we ignore normalization issues and appeal to the standard formula for the δ-function, but at every point in space-time. The field $b = b^a T^a$ is the so-called Nakanishi-Lautrup field, transforming in the adjoint representation.

With this, we go back to our derivation of last section, replace the gauge-fixing condition by the above and do *not* carry out the integral over ω. The resulting lagrangian is

$$\mathcal{L} = -\frac{1}{2}\text{tr}\, F^2 + b^a(\omega - \partial A)^a - \frac{\lambda}{2}\omega^a\omega^a - \bar{c}^a\partial^\mu D_\mu c^a + \bar{\psi}(i\not{D} - m)\psi\,, \tag{5.2}$$

where we suppressed the gauge indices of the matter fields. The corresponding path integral is over $A_\mu, \omega, b, c, \bar{c}$ and $\bar{\psi}, \psi$.

Integrating first over b and then over ω corresponds to what we did before, giving the known result with gauge-fixing term $\sim (\partial A)^2$. Here, it will be useful to integrate over ω first. Suppressing gauge indices, we write

$$b(\omega - \partial A) - \frac{\lambda}{2}\omega^2 = -\frac{\lambda}{2}(\omega - b/\lambda)^2 + \frac{1}{2\lambda}b^2 - b(\partial A)\,, \tag{5.3}$$

where we completed the square w.r.t. ω. If we now shift ω by b/λ, the path integral over ω becomes a simple Gaussian. This gives just an irrelevant overall prefactor.

As a side-remark, what we just did is called 'integrating out' the field ω. It was possible in this simple and explicit way since ω did not have a kinetic term, one also says that it was just an 'auxiliary field'.

Thus, we can now work with the lagrangian

$$\mathcal{L} = -\frac{1}{4}F^2 + \frac{1}{2\lambda}b^2 - b(\partial A) - \bar{c}\partial^\mu D_\mu c + \bar{\psi}(i\not{D} - m)\psi\,, \tag{5.4}$$

where we keep suppressing gauge indices, writing in this spirit $\text{tr}F^2 = F^aF^a/2 = F^2/2$. Clearly, we could go on to integrate out the Nakanishi-Lautrup field b as above: One completes the square, shifts the field and drops the trivial Gaussian integration. The result would be our gauge fixed action of last section (Check this!).

As a further side-remark, this way of integrating out an auxiliary field is equivalent to writing down its equation of motion, solving for the auxiliary field and plugging the result back into the action. Indeed, the EOM for b is

$$\frac{b}{\lambda} - \partial A = 0\,, \tag{5.5}$$

hence $b = \lambda(\partial A)$ and

$$\frac{1}{2\lambda}b^2 - b(\partial A) \qquad \to \qquad -\frac{\lambda}{2}(\partial A)^2\,. \tag{5.6}$$

As a small excercise, argue why this second way of integrating-out must be equivalent to the first one! Under which conditions is it appropriate to integrate out a massive, dynamical (i.e. possessing a kinetic term) field in this way?

For us, the most useful form of the action or lagrangian is that of (5.4). The reason is that the **BRST transformation**

$$\delta_\epsilon A_\mu = \epsilon D_\mu c \tag{5.7}$$

$$\delta_\epsilon \psi = -ig\epsilon c\psi \qquad \Big(\text{short for} \quad (\delta_\epsilon\psi)_i = -ig\epsilon c^a (T^a)_i^{\,j}\psi_j\Big) \tag{5.8}$$

$$\delta_\epsilon c = -ig\epsilon c^2 \qquad \Big(\text{short for} \quad (\delta_\epsilon c)^a = \frac{1}{2} g\epsilon f^{abc}c^b c^c\Big) \tag{5.9}$$

$$\delta_\epsilon \bar{c} = -\epsilon b \tag{5.10}$$

$$\delta_\epsilon b = 0 \tag{5.11}$$

is a symmetry of this action (Becchi/Rouet/Stora '76 – Annals of Phys. & Tyutin '75 – Lebedev Institute preprint).

Before demonstrating this claim, a few comments are in order. First, the infinitesimal parameter ϵ has to be a Grassmann variable, for reasons which will become apparent later on. This affects us only through the signs – everything you learned about continuous symmetries still holds. Next, we show that c^2 is in fact not zero (as one might naively have suspected from its Grassmann nature):

$$\begin{aligned} c^2 &= (c^aT^a)(c^bT^b) = c^ac^b\,T^aT^b = \frac{1}{2}(c^ac^b - c^bc^a)T^aT^b = \frac{1}{2}(c^ac^b\,T^aT^b - c^ac^bT^bT^a) \\ &= \frac{1}{2}c^ac^b[T^a,T^b] = \frac{i}{2}f^{abc}c^ac^bT^c = \frac{i}{2}f^{abc}c^bc^cT^a \equiv (c^2)^aT^a\,. \end{aligned} \tag{5.12}$$

Here we used the the total antisymmetry of f^{abc}. The above confirms that the two forms of (5.9) coincide. Finally, as we will see, the BRST symmetry is so important since it represents the 'gauge-fixed/quantum version' of the gauge symmetry.

Now we check in detail that the BRST transformation leaves the action invariant. Note first that the usual gauge transformation of the gauge potential reads

$$\delta_\chi A_\mu = \partial_\mu\chi - i[\chi, gA_\mu] = D_\mu\chi\,, \tag{5.13}$$

where in the last expression we treat χ as a conventional adjoint field, with corresponding action of the covariant derivative. We thus discover that the BRST transformation of A_μ is nothing but a gauge transformation with gauge parameter

$$\chi = \epsilon c\,. \tag{5.14}$$

Similary, the BRST transformation of ψ is manifestly a gauge transformation with gauge parameter ϵc. Thus, the BRST invariance of

$$-\frac{1}{4}F^2 + \bar{\psi}(i\not{D} - m)\psi \tag{5.15}$$

is proven.

The b^2 term is trivially invariant.

We are left with the task of establishing the invariance of

$$b^a(\partial A)^a + \bar{c}^a\partial^\mu D_\mu c^a = 2\mathrm{tr}\{b(\partial A) + \bar{c}\partial^\mu D_\mu c\}\,. \tag{5.16}$$

Now, since

$$\delta_\epsilon A_\mu = \epsilon D_\mu c \qquad \text{and} \qquad \delta_\epsilon \bar{c} = -\epsilon b\,, \tag{5.17}$$

the effect of varying A in the first term of (5.16) precisely cancels the effect of varying $\bar{c}$ in the second term. Since b does not vary, all that is left is calculating the effect of the variation of $D_\mu c$:

$$\begin{aligned}\delta_\epsilon(D_\mu c) &= \delta_\epsilon\{\partial_\mu c + ig[A_\mu, c]\} = D_\mu\delta_\epsilon c + ig[(\delta_\epsilon A_\mu), c] \qquad (5.18)\\ &= -ig\epsilon D_\mu c^2 + ig[\epsilon D_\mu c, c] = ig\epsilon\Big(-[(D_\mu c)c + c(D_\mu c)] + \{D_\mu c, c\}\Big) = 0\,.\end{aligned}$$

This ends our proof, but as an excersice you might want to repeat the last part of the calculation using the component form of $\delta_\epsilon c$, such that f^{abc}s appear explicitly. $\square$

5.2 The BRST operator

Let us now consider the classical field theory defined by the gauge-fixed lagrangian, in the form including the auxiliary and ghost fields b and $c, \bar{c}$. This system can be quantized, either by the path integral (in which case the Hilbert space are the wave functionals) or canonically (see below). The (global!) classical symmetry transformation δ_ϵ of the lagrangian system implies a corresponding symmetry of the Hamiltonian system. The latter is generated by via the Poisson bracket by an observable, which we call Q and, after quantization, by the operator $\hat{Q}$. This is the BRST operator or BRST charge.

Equivalently, the Noether theorem implies the existence of a current j^μ associated with δ_ϵ. The charge Q is then defined by $\int d^3x\, j^0$, as usual. (We will drop the 'hat' on Q in what follows – it should always be clear from the context whether the operator or the classical observable are meant.)

Let us try to make the operator Q more explicit. To this end, let us adopt the definition

$$\delta_\epsilon \equiv -\epsilon Q \qquad \text{or, more concretely} \qquad \delta_\epsilon\varphi \equiv -\{\epsilon Q, \varphi\} \tag{5.19}$$

at the classical level, with φ standing for any of the relevant fields and $\{\cdots\}$ the Poisson bracket. By the standard Dirac quantization rule $\{A, B\} \to -i[A, B]$, we find at the quantum level

$$\delta_\epsilon \equiv i\epsilon Q \qquad \text{or, more concretely} \qquad \delta_\epsilon\varphi \equiv [i\epsilon Q, \varphi]\,. \tag{5.20}$$

As we have learned, operators become classical fields under the path integral. Hence, operator relations (like the above definition of a symmetry transformation of an operator) become classical relations under the path integral. Thus, our knowledge about how the BRST symmetry acts classically allows us to write

$$\langle\Psi_b|\left[\,i\epsilon Q, O[\varphi]\,\right]\cdots|\Psi_a\rangle = \int_{\Psi_a,\Psi_b} D\varphi\Big((\delta_\epsilon O[\varphi])\cdots\Big)e^{iS}\,. \tag{5.21}$$

Crucially, our classical defintion of δ_ϵ **defines** the quantum operator Q within the path integral approach. Indeed, we can calculate arbitrary expectation values of a transformed operator $O[\varphi]$. We can even transform states, for example by defining states through some operator acting on the vacuum,

$$|\Psi\rangle \equiv O[\varphi]\,|0\rangle\,, \tag{5.22}$$

and then transforming the operator as above. That allows us to calculate the overlap of a BRST-transformed state with some arbitrary other state.

The punchline of all this that, even without having written down Q explicitly in terms of annihiliation and creation operators, we can work with it rather explicitly. In particular, we can check the crucial claim that Q is **nilpotent**,

$$Q^2 = 0 \tag{5.23}$$

by a classical calculation. Namely, we observe that

$$\delta_{\epsilon'}\delta_\epsilon = \epsilon\epsilon' Q^2 \tag{5.24}$$

vanishes if and only if $Q^2 = 0$. Thus it is sufficient to establish $\delta_\epsilon\delta_{\epsilon'} = 0$. Indeed,

$$\delta_\epsilon b = 0 \quad\Rightarrow\quad \delta_{\epsilon'}\delta_\epsilon b = 0\,. \tag{5.25}$$

Similarly,

$$\delta_\epsilon\bar{c} = -\epsilon b \quad\Rightarrow\quad \delta_{\epsilon'}\delta_\epsilon\bar{c} = \delta_{\epsilon'}(-\epsilon b) = -\epsilon\delta_{\epsilon'}b = 0\,. \tag{5.26}$$

Furthermore,

$$\delta_\epsilon c = -ig\epsilon c^2 \tag{5.27}$$

implies

$$\delta_{\epsilon'}\delta_\epsilon c = -ig\epsilon\delta_{\epsilon'}c^2 = (-ig)^2\epsilon[(\epsilon' c^2)c + c(\epsilon' c^2)] = (-ig)^2\epsilon\epsilon'[c^3 - c^3] = 0\,. \tag{5.28}$$

Here we have extended the action δ_ϵ on products of fields by the Leibniz rule

$$\delta_\epsilon(fg) = (\delta_\epsilon f)g + f(\delta_\epsilon g)\,. \tag{5.29}$$

Here it proves convenient that we have defined ϵ to be Grassmann, such that δ_ϵ is 'bosonic'. Finally, one the application of $\delta_{\epsilon'}\delta_\epsilon$ to ψ and A_μ is left as an excercise. Similarly, you should convince yourself that $Q^2\varphi_i = 0$, with φ_i any one of our many fields, implies that $Q^2\varphi_{i_1}\cdots\varphi_{i_n} = 0$, i.e. Q^2 vanishes on any polynomial function of fields.

To summarize, the BRST symmetry is generated by the nilpotent BRST operator Q. Our knowledge of the BRST transformation on fields allows us to work with Q rather explicitly, given that we understand both operators and states via fields inserted under the path integral.

5.3 The Hilbert space

Recall that we are working here with the theory defined by the gauge-fixed lagrangian, i.e.

$$\mathcal{L} = \mathcal{L}(A_\mu, \psi, \bar{\psi}, c, \bar{c}, b)\,. \tag{5.30}$$

We have not yet quantized canonically, so our understanding of the 'Hilbert space' of this theory relies on the path integral: Field operators acting on the vacuum define states, the overlap of such states is defined via the path integral,

$$\Big(\langle 0|O_{t'}\Big) \cdot \Big(O_t|0\rangle\Big) = \int D\varphi\, O_{t'} O_t e^{iS}\,. \tag{5.31}$$

However, both from the experience with canonical quantization of QED, from the wrong sign of the zero-zero part of the 2-gauge-boson correlator (e.g. in Feynman gauge),

$$\langle T A_\mu(x) A_\nu(y)\rangle = \int \frac{d^4k}{(2\pi)^4}\, \frac{-i\eta_{\mu\nu}}{k^2 + i\epsilon}\, e^{-ik(x-y)}\,, \tag{5.32}$$

and from the violation of the spin-statistics theorem by the ghosts we suspect that this space is not positive-definite. (Hence our quotation marks on 'Hilbert space' above.) This is resolved as follows:

Let us denote the whole linear space (of wave functionals) by $\mathcal{F}$. Let us denote by $\mathcal{F}_{phys}$ its subspace annihiliated by Q,

$$\mathcal{F}_{phys} \equiv \mathrm{Ker}\, Q \subset \mathcal{F}\,. \tag{5.33}$$

The index 'phys' stands for physical subspace and is justified as follows: The BRST symmetry is closely related to gauge symmetry, most obviously on the physical fields A_μ and ψ. As a result, we expect physical and hence gauge invariant states to be invariant under the BRST transformation.

Note furthermore that states in

$$\mathcal{F}_0 \equiv \mathrm{Im}\, Q \subset \mathcal{F} \tag{5.34}$$

have zero norm due to $Q^2 = 0$. Indeed, the overlap of two states $Q|\Psi_1\rangle$ and $Q|\Psi_2\rangle$ from $\mathcal{F}_0$ always vanishes,

$$\langle \Psi_1|Q^\dagger Q|\Psi_2\rangle = \langle \Psi_1|Q^2|\Psi_2\rangle = 0\,, \tag{5.35}$$

since Q is hermitian.[12] More generally, states in $\mathcal{F}_0$ (such as $Q|\Psi_1\rangle$) have zero overlap with physical states, say $|\Psi_{phys}\rangle$, with $Q|\Psi_{phys}\rangle = 0$. Indeed,

$$\langle \Psi_1|Q^\dagger|\Psi_{phys}\rangle = \langle \Psi_1|Q|\Psi_{phys}\rangle = 0\,. \tag{5.36}$$

[12]The latter is certainly expected of a observable, but due to the involvement of ghosts it is not compeletely obvious. With the appropriate definition of the hermitian conjugation on ghost fields (see below) Q is indeed hermitian.

All of this culminates in the mathematically highly natural definiton of our true Hilbert space,

$$\mathcal{H} \equiv \frac{\mathcal{F}_{phys}}{\mathcal{F}_0} \equiv \frac{\operatorname{Ker} Q}{\operatorname{Im} Q}\,, \tag{5.37}$$

as a quotient space or the space of equivalence classes of states within $\mathcal{F}_{phys}$. Linearity and scalar product on this new space are inherited from $\mathcal{F}$ in a natural way. (You should be able to prove that, addition, multiplication by complex numbers and the scalar product can be defined on $\mathcal{H}$ on the basis of representatives since they do not depend on which representative one chooses.)

The above construction is also known as the **cohomology of** Q. It should be familiar in the context of the exterior derivative d, which also satisfies $d^2 = 0$. Indeed

$$d_{(p)}:\ p\text{-forms} \quad \rightarrow \quad (p+1)\text{-forms}\,, \tag{5.38}$$

and

$$H^p(M) \equiv \frac{\text{closed } p\text{-forms}}{\text{exact } p\text{-forms}} \equiv \frac{\operatorname{Ker} d_{(p)}}{\operatorname{Im} d_{(p-1)}}\,. \tag{5.39}$$

For an n-dimensional compact manifold M, this space H^p of 'cohomology classes' of p-forms is dual to the space H_{n-p} of 'homology classes' of $n-p$ cycles. An example with $n = 2$ and $p = 1$ and M a Riemann surface of genus 3 is illustrated in Fig. 21. As the cohomology of d reveals the topology of the manifold, the cohomology of Q reveals the gauge-invariant structure of the space of wave functionals.

Figure 21: Illusration of the (6-dimensional) space of homology classes of 1-cycles of a Riemann surface of genus 3.

The next important point is to demonstrate that physical amplitudes do not depend on the gauge choice. Indeed, let us split the lagrangian as

$$\mathcal{L} = -\frac{1}{4}F^2 + \bar{\psi}(i\not{D} - m)\psi + \mathcal{L}_{gf} + \mathcal{L}_{ghost} \tag{5.40}$$

with

$$\mathcal{L}_{gf} = \frac{1}{2\lambda}b^2 - b(\partial A) \qquad \text{and} \qquad \mathcal{L}_{ghost} = -\bar{c}\partial^\mu D_\mu c\,. \tag{5.41}$$

We now claim that the part in which all gauge-choice-dependence resides, i.e. the combination $\mathcal{L}_{gf} + \mathcal{L}_{ghost}$ is BRST exact, i.e., there exists an operator Φ such that

$$\mathcal{L}_{gf} + \mathcal{L}_{ghost} = [Q, \Phi]\,. \tag{5.42}$$

In terms of classical fields, this means that

$$\epsilon(\mathcal{L}_{gf} + \mathcal{L}_{ghost}) = -i\delta_\epsilon\Phi\,. \tag{5.43}$$

That this is the case follows from the simple calculation

$$\delta_\epsilon \,\bar{c}\left((\partial A)-\frac{1}{2\lambda}b\right)=-\epsilon b\left((\partial A)-\frac{1}{2\lambda}b\right)+\bar{c}\epsilon\partial^\mu D_\mu c=\epsilon\left(-b(\partial A)+\frac{1}{2\lambda}b^2-\bar{c}\partial^\mu D_\mu c\right)\,, \tag{5.44}$$

which also defines which Φ to choose. It is true more generally, i.e. for gauge-fixing functions other than (∂A), that the gauge-fixing/ghost part of $\mathcal{L}$ is BRST exact. Any change of this gauge dependent part of $\mathcal{L}$ (in the simplest case the change of λ) can thus be written as

$$\delta_\lambda S=[Q,\delta_\lambda\Phi]\,, \tag{5.45}$$

where δ_λ symbolizes some change of the gauge-fixed action due to a change of the gauge condition and $\delta_\lambda\Phi$ is an appropriate operator. Now, clearly, a transition amplitude between physical states Ψ_a and Ψ_b changes by

$$\int_{\Psi_a,\,\Psi_b} D\varphi\, e^{i(S+\delta_\lambda S)}\;-\;\int_{\Psi_a,\,\Psi_b} D\varphi\, e^{iS} \tag{5.46}$$

for such a gauge condition change. Infinitesimally, one has

$$\int_{\Psi_a,\,\Psi_b} D\varphi\,[Q,\delta_\lambda\Phi]\,e^{iS}=0\qquad\text{because}\qquad\langle\Psi_b|Q\;=\;Q|\Psi_a\rangle\;=\;0\,. \tag{5.47}$$

This establishes that physics is gauge-choice-independent.

Finally, as a short but very important remark, Q is by definition conserved, i.e. it commutes with H and hence with time evolution. Thus, our definition of the physical subspace commutes with time evolution. As a result, the dynamics (e.g. scattering) will never force us to leave the physical subspace.

5.4 Canonical quantization

Fundamentally, we are now done. Of course, the proofs that the resulting Hilbert space $\mathcal{H}$ is positive definite and that the scattering matrix on this space is unitary are missing. We will not give these proofs but refer to the book by Kugo [15], where this is carried in the canonically quantized theory.

Independently, it desirable to understand the relation of the Hilbert space formally defined in the path integral approach to the Fock space. This is also crucial for the LSZ formalism. Finally, we want to get some intuition about how BRST quantization relates to Gupta-Bleuler quantization defined earlier.

Thus, we have good to reason to try and canonically quantize the theory defined by

$$\mathcal{L}=-\frac{1}{4}F^2-b(\partial A)+\frac{1}{2\lambda}b^2-\bar{c}\partial^\mu D_\mu c\,. \tag{5.48}$$

Here we have dropped the matter part since its treatment is straightforward and it does not contribute to the gauge-theory subtleties we are interested in. Our treatment will be a highly simplified version of the detailed analysis in [15].

As in the abelian case, we have (always suppressing the gauge index)

$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = F^{i0}\,. \tag{5.49}$$

Since $F^{00} = 0$, the corresponding contribution to π^0 vanishes. However, the gauge-fixing part of $\mathcal{L}$ contains $\dot{A}_0$ such that one finds

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = -b\,. \tag{5.50}$$

The Nakanishi-Lautrup field b has no canonical momentum, such that one only finds the four commutation relations (at $t = x^0 = 0$)

$$[A_i(\vec{x}), \pi^k(\vec{y})] = i\delta_i{}^k\, \delta^3(\vec{x}-\vec{y}) \tag{5.51}$$

$$[A_0(\vec{x}),\, -b(\vec{y})] = i\delta^3(\vec{x}-\vec{y})\,. \tag{5.52}$$

For the ghosts, we have (after integrating by parts, such that ∂^μ acts on $\bar{c}$)

$$\pi_c = \frac{\partial \mathcal{L}}{\partial \dot{c}} = \dot{\bar{c}} \tag{5.53}$$

$$\pi_{\bar{c}} = \frac{\partial \mathcal{L}}{\partial \dot{\bar{c}}} = \dot{c} - ig[A_0, c]\,. \tag{5.54}$$

Thus, quantizing with anti-commutators as appropriate for fermions, we have

$$\{c(\vec{x}), \pi_c(\vec{y})\} = \{\bar{c}(\vec{x}), \pi_{\bar{c}}(\vec{y})\} = i\delta^3(\vec{x}-\vec{y})\,. \tag{5.55}$$

We make the familiar ansatz for expressing the *free* fields (corresponding to $g = 0$) in terms of creation and annihilation operators:

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \left(a^\mu_{\vec{k}} e^{-ikx} + a^{\mu\,\dagger}_{\vec{k}} e^{ikx} \right) \tag{5.56}$$

$$b(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \left(b_{\vec{k}} e^{-ikx} + b^\dagger_{\vec{k}} e^{ikx} \right) \tag{5.57}$$

$$c(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \left(c_{\vec{k}} e^{-ikx} + c^\dagger_{\vec{k}} e^{ikx} \right) \tag{5.58}$$

$$\bar{c}(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \left(\bar{c}_{\vec{k}} e^{-ikx} - \bar{c}^\dagger_{\vec{k}} e^{ikx} \right)\,. \tag{5.59}$$

Thus, b is simply a real scalar. It may be surprising that we make this ansatz, motivated by the dynamics of the Klein-Gordon equation for a non-dynamical scalar. However, at $g = 0$ we have from the variation of the action

$$\partial_\mu F^{\mu\nu} = \partial^\nu b \qquad \text{and hence} \qquad \partial^2 b = 0\,. \tag{5.60}$$

So the ansatz makes sense.

In addition, another conceptual issue arises in this context: For the five creation/annihilation operator pairs $a^{\mu\,\dagger}/a^\mu$ and $b^\dagger/b$ we only have the four canonical commutation relations (5.51) and (5.52). To machieve a consistent Fock space construction, we need to reduce the number of independent creation/annihiliation operators. This is achieved by demanding the b-field equation of motion at the oprator level, i.e.

$$b = \lambda\,\partial A\,. \tag{5.61}$$

Concerning the unusual sign in the ansatz for $\bar{c}$ (the minus sign in in front of $\bar{c}^\dagger_{\vec{k}}$), this is a purely notational issue which requires a comment: Our ghost lagrangian is only real if

$$c^\dagger = c \qquad \text{and} \qquad \bar{c}^\dagger = -\bar{c}\,. \tag{5.62}$$

This could be fixed easily be redefining c or $\bar{c}$ by an i, giving *real* ghosts and an i prefactor in the ghost lagrangian. However, then we would clash with much of the literature cited in the last sections. Presumably, this would overall be advantageous and indeed, the very careful text by Kugo uses such conventions. For the record:

$$b_{Kugo} = -b\ , \quad c_{Kugo} = -c\ , \quad \bar{c}_{Kugo} = i\bar{c}\,. \tag{5.63}$$

We will stick with the 'antihermitian-$\bar{c}$' convention in what follows.

We do not derive the creator/annihilator commutation relations, which is standard. As usual, one restricts oneself to $g = 0$ and uses Feynman gauge, where $b = \partial A$, so that b-excitations correspond to unphysical A polarizations. The result is

$$[a^\mu_{\vec{k}}, a^\nu_{\vec{q}}] \;=\; -\eta^{\mu\nu}\, 2k_0\,(2\pi)^3\,\delta^3(\vec{k}-\vec{q})\,, \tag{5.64}$$

$$\{c_{\vec{k}}, \bar{c}^\dagger_{\vec{q}}\} \;=\; \{\bar{c}_{\vec{k}}, c^\dagger_{\vec{q}}\} \;=\; 2k_0\,(2\pi)^3\,\delta^3(\vec{k}-\vec{q})\,. \tag{5.65}$$

Crucially, due to higher-order terms in $\pi^i = F^{i0}$ and in $\pi_{\bar{c}}$, the interaction lagrangian differs from minus the interaction hamiltonian and non-covariant pieces arise in the canonically derived Feynman rules. This is the same complication we encountered before in scalar QED and we will not dwell on this.

The (*free-particle*) Fock space is still defined by starting from a vacuum, annihiliated by all annihiliation operators, and applying creation operators to get the excited states. However, this state is not positive definite and, differently from QED, the operation of making it positive definite mixes states of different particle number. This is where the BRST symmetry and the BRST operator are *really* needed. Indeed, the corresponding Noether current is

$$J_\mu = bD_\mu c - (\partial_\mu b)c + \frac{i}{2}g(\partial_\mu\bar{c})\cdot\{c,c\} - \partial^\nu(F_{\mu\nu}c) \tag{5.66}$$

and the charge explicitly reads

$$Q = \int d^3x\,\left(b^a(D_0c)^a - (\partial_0 B)^a c^a + \frac{i}{2}g(\partial_0\bar{c})^a f^{abc}c^b c^c\right)\,. \tag{5.67}$$

Here we have re-established explicit gauge indices, just to be sure that they are not entirely frogotten.

With this operator Q, we can approach our Fock space, construct the physical subspace and mod out the gauge redundancies. The statement $\mathcal{H} = \mathrm{Im}\, Q/\mathrm{Ker}\, Q$ says it all. Of course, Q has to be given explicitly in terms of creation/annihiliation operators of the fields, but this is now easy. The actual cohomology construction is, however, highly complex due to the cubic piece in Q: Indeed, Q mixes states with different particle number. This makes the Hilbert space very different from the free-particle Fock space we started with.

The non-trivial statements to be made are that $\mathcal{H}$ is positive-definite and the scattering matrix is unitary. We refer to Kugo's book for the relatively lengthy proof.

We can, however, develop some intuition for what's going on by restricting our attention to the abelian case or, equivalently, to the limit $g \to 0$. Moreover, this is also what's relevant for the 1-particle sector of the Fock and Hilbert space. As you recall, the 1-particle sector is sufficient to understand LSZ, where we argue from locality that all incoming and outgoing states are far apart and we only care about the overlap of each such state with the 1-particle-sector of the Fock space.

In this limit, we have

$$Q = \int d^3x \, b \overset{\leftrightarrow}{\partial}_0 c = i \int \frac{d^3k}{(2\pi)^3 2k_0} \, [c^\dagger_{\vec{k}} b_{\vec{k}} - b^\dagger_{\vec{k}} c_{\vec{k}}] \, . \tag{5.68}$$

We also recall that we are in Feynman gauge, such that $b = \partial A$. Furthermore, we use the following polarization vectors for $A_\mu(k)$, with $\vec{k} = \{k^i\}$ pointing in positive z-direction:

$$\epsilon^\mu_\pm = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} \, , \quad \epsilon^\mu_L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \, , \quad \epsilon^\mu_U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \, , \tag{5.69}$$

where L and U stand for 'longitudinal' and 'unphysical'. We can also switch from the creation/annihiliation operators $a^\mu / a^{\mu\,\dagger}$ to operators $\alpha_\pm / \alpha^\dagger_\pm$, $\alpha_L / \alpha^\dagger_L$ and $\alpha_U / \alpha^\dagger_U$, corresponding to the above polarizatios (cf. QFT I). We then have

$$\alpha^\dagger_{\{\pm, L, U\}, \vec{k}} = \epsilon^\mu_{\pm, L, U}(k) \, a_{\vec{k}\,\mu} \tag{5.70}$$

and

$$Q \sim \int d^3k \, [c^\dagger_{\vec{k}} \alpha_{L, \vec{k}} - \alpha^\dagger_{L, \vec{k}} c_{\vec{k}}] \, . \tag{5.71}$$

At the 1-particle level, we encounter (suppressing the momentum index) the two physical gluon polarizations

$$\alpha^\dagger_\pm |0\rangle \tag{5.72}$$

and the so called **BRST quartet**

$$\bar{c}^\dagger |0\rangle \;\to\; \alpha^\dagger_L |0\rangle \;\to\; 0 \qquad , \qquad \alpha^\dagger_U |0\rangle \;\to\; c^\dagger |0\rangle \;\to\; 0 \, , \tag{5.73}$$

where the arrows correspond to the action of Q. To appreciate this structure, it is crucial to also remember that the non-zero scalar products within this quartet are

$$\langle 0|c\bar{c}^\dagger|0\rangle \qquad \text{and} \qquad \langle 0|\alpha_U \alpha_L^\dagger|0\rangle\,. \tag{5.74}$$

We see that, generalizing the Gupta-Bleuler approach, we now have $\alpha_U^\dagger/\bar{c}^\dagger$ as forbidden excitations and $\alpha_L^\dagger/c^\dagger$ as residual-gauge-freedom excitations. Crucially, and showing this would require more work, this quartet structure extends to the full Fock space and allows for a systematic understanding that, in the LSZ approach to scattering amplitudes, only the poles associated with $\alpha_\pm^\dagger$ contribute. In other words, **the physical S-matrix is unitary**. This also requires the use so called **Slavnov-Taylor identities**. The latter are derived using BRST symmetry in analogy to how **Ward-Takahashi identities** are derived using gauge symmetry in the abelian case. We will return to this issue.

To appreciate better what one would need to show, we give a demonstration in the much simpler case of QED. Since the (gauge-fixed) lagrangian is real, H is hermitian and S unitary,

$$SS^\dagger = \mathbb{1}\,. \tag{5.75}$$

It acts on the full (non-positive definite) Fock space $\mathcal{F}$. Let us define $\mathcal{F}_\pm$ as the subspace built only from physically polarized photons ($\mathcal{F}_\pm \subset \mathcal{F}_{phys} \subset \mathcal{F}$). Let P be the projector on $\mathcal{F}_\pm$ (which implies $P^2 = P$ and $P^\dagger = P$). Unitarity of the physical S-matrix then means

$$(PSP)(PSP)^\dagger = P\,. \tag{5.76}$$

This is equivalent to

$$PSPS^\dagger P = P\,, \tag{5.77}$$

which will hold if

$$SPS^\dagger = SS^\dagger\,. \tag{5.78}$$

Thus, we need to show that we can remove a projector P in between two S-matrix elements. Focus on one of the external photon lines of such an S-matrix element, let's say with index μ and momentum k (which we will suppress). We simply need to prove that

$$\sum_\pm \mathcal{M}_\mu \epsilon_\pm^\mu \epsilon_\pm^{\nu\,*} \mathcal{M}_\nu^* = \mathcal{M}_\mu(-\eta^{\mu\nu})\mathcal{M}_\nu^*\,. \tag{5.79}$$

But this immediately follows from the completeness relation

$$\sum_\pm \epsilon_\pm^\mu \epsilon_\pm^{\nu\,*} + \epsilon_L^\mu \epsilon_U^{\nu\,*} + \epsilon_U^\mu \epsilon_L^{*\,\nu} = -\eta^{\mu\nu} \tag{5.80}$$

together with the Ward-Takahashi identity $\mathcal{M}_\mu k^\mu = 0\,.$ $\square$

A 'slick' argument for the non-abelian case can be given as follows (cf. [1]): Let $g \to 0$ at $t \to \pm\infty$. Our initial states are then defined in the well-understood Hilbert spaces of the free theory (cf. Gupta-Bleuler approach 'plus ghosts', es explained above). We are certain that the physical part of the full Fock-space is positive definite in this case. Now, as we very slowly[13] switch on g, this simple physical space of transverse gauge bosons evolves into the space $\mathcal{H}$ formally defined using Q. Since Q commutes with H, we will never leave this space in the dynamical evolution. Furthermore, since the dynamics is unitary by defintion (given hermitian H), the dynamics restricted to $\mathcal{H}$ is also unitary. Thus, after switching off g in the far future, we find an outgoing physical state in the free space of transverse gluons and we are assured that the scattering matrix is unitary.

For a more details, see e.g. [11, 13, 15–20].

6 Running coupling and β-function, in QCD and in general

6.1 QCD

Most of the concrete calculations in this section will be performed in QCD, so let us give a brief summary: We are dealing with an $SU(3)$ gauge theory with 6 flavors of Dirac fermions (quarks) in the fundamental representation:

$$\mathcal{L} = -\frac{1}{2g^2} \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu} + \sum_f \bar{\psi}_f (i\not{D} - m_f)\psi_f \,, \tag{6.1}$$

where $f \in \{u, d, s, c, b, t\}$. This should be familiar as part of the Standard Model, but we are at the moment not interested in the additional $SU(2)\times U(1)$ gauge symmetry and the lepton and Higgs fields. Jumping ahead, we note that this theory has a strong-coupling or confinement scale at $\Lambda_{QCD} \sim 200$ MeV, and from the prespective of that scale one sometimes refers to $\{u, d, s\}$ as 'light' and the rest as 'heavy' quarks.

We have

$$A_\mu = A^a_\mu T^a \qquad \text{with} \qquad T^a = \frac{1}{2}\lambda^a \,, \tag{6.2}$$

with the latter known as Gell-Mann matrices. The first three are explicitly given by placing the Pauli-matrixes in the upper-left corner, corresponding to an $SU(2)$ subgroup,

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,,\ \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,,\ \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,, \tag{6.3}$$

the next four by placing the off-diagonal Pauli-matrices in the 'middle'

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \,,\ \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \,, \tag{6.4}$$

[13] This is the catch: We need to think about Poincare and BRST symmetry in this limit.

and the lower-right blocks,

$$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}. \tag{6.5}$$

The last is the unqiue traceless element orthogonal to λ^3,

$$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{6.6}$$

$SU(3)$ has a maximal commuting subalgebra of dimension 2 (one says the group has rank 2), spanned by $\lambda^{3,8}$:

$$[\lambda^3, \lambda^8] = 0\,. \tag{6.7}$$

Our perturbative methods will allow us to deal with this theory only at high energies, above Λ_{QCD}, where quarks and gluons are useful degrees of freedom. We will learn that going to lower energies the coupling blows up near Λ_{QCD}, making perturbative calculations impossible. Fortunately, by now one has enough control of the path integral numerically (on a lattice) to be sure that the very same fundamental lagrangian is responsible for the wealth of meson/hadron physics observed in that low-energy domain.

6.2 Dimensional regularization and minimal subtraction

We will do perturbation theory, using the Feynman rules derived with the Fadeev-Popov method and working in d space-time dimensions, as in our QED analysis in QFT I. In $d \neq 4$ dimensions, the gauge coupling is dimensionful. Indeed, A_μ has mass-dimension one,

$$D_\mu = \partial_\mu + iA_\mu \qquad \Rightarrow \qquad [A_\mu] = 1\,, \tag{6.8}$$

which implies

$$S = \int d^dx\, \frac{1}{2g^2}\, \mathrm{tr}(F^2) \quad \Rightarrow \quad -d - [g^2] + 2 + 2[A_\mu] = 0 \quad \Rightarrow \quad [g^2] = 4 - d\,. \tag{6.9}$$

Of course, the mass dimension of A_μ will change if we rescale A_μ by g, as is frequently done, but the non-trival mass dimension of g remains.

It will be convenient to write $d \equiv 4 - \epsilon$ and to **redefine** g according to

$$g^2 \quad \to \quad g^2\mu^\epsilon\,, \tag{6.10}$$

where μ is a new mass scale which is necessarily introduced in the process of renormalization. Thus, from now on, g is again **dimensionless**.

Let us now calculate some observable which is not sensitive to the "mess" near Λ_{QCD} which we are not able to control analytically. (This is a very rough description of what is called an **IR-safe observable** and can of course be defined more precisely.) The example we choose is the cross-section for

$$e^+e^- \to \text{hadrons}\,, \tag{6.11}$$

which can also be thought of as the 'decay width' of γ^* to hadrons, with some of the relevant diagrams shown in Fig. 22.

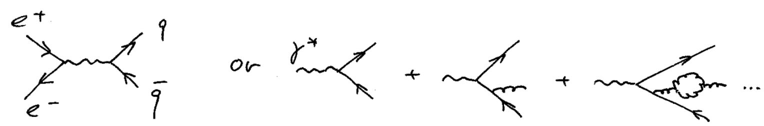


Figure 22: Some diagrams contributing to $e^+e^- \to$ hadrons or, what is only relevant here, $\gamma^* \to$ hadrons.

Let us go to high cms-energies $s = Q^2 = q^2$, with q the momentum of γ^*, and normalize the result according to

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}\,. \tag{6.12}$$

For n_f active flavors ('active' means light from the perspective of $s = Q^2$), we will find for this by definition dimensionless quantity

$$R = R(Q^2/\mu^2, g^2) \;=\; 3n_f\left[1 + g^2\mu^\epsilon Q^{-\epsilon} f_1(\epsilon) + g^4\mu^{2\epsilon}Q^{-2\epsilon} f_2(\epsilon) + \mathcal{O}(g^6)\right]. \tag{6.13}$$

As a side-remark, the prefactor '3' was historically an important piece of evidence for the color-degree-of-freedom.

In Eq. 6.13, the factors μ^ϵ accompany factors of g^2 due to our redefinition of the coupling. Since Q is the only other dimensionful parameter (we take $m/Q \to 0$ for all particle masses involved), the Q-dependence is then fixed. The functions $f_{1,2,\dots}$ follow from explicit loop calculations and the ϵ-dependence can, in particular, involve divergencies (poles) at $\epsilon \to 0$:

$$\int \frac{d^dk}{(k^2+q^2)^2} \sim (q^2)^{d/2-2} \int \frac{d^dx}{(x^2+1)^2} \sim Q^{-\epsilon}\left(\frac{1}{\epsilon} + \cdots\right). \tag{6.14}$$

In the concrete example above, $f_1(0) = 1/4\pi^2$ is finite, which can be understood before any calculation. Indeed, according to what we already learned any possible divergence must be absorbed by a counterterm, in our case by writing

$$g^2 \equiv g_0^2 = g_{phys}^2 Z_g^2 \qquad \text{with} \qquad Z_g^2 \simeq 1 + 2\delta Z_g \equiv 1 + \frac{g^2 \cdot \#}{\epsilon}\,, \tag{6.15}$$

where we used the fact that odd orders in g do not arise in convetional perturbation theory. Thus, the first $1/\epsilon$ from a counterterm arises at $\mathcal{O}(g^4)$. Hence, the first $1/\epsilon$ from a loop can appear in f_2. A divergence found in f_1 would represent an inconsistency.

With these remarks, it is now clear that we expect to find

$$R = 3n_f \left[1 + \frac{g^2}{4\pi^2}\left(\frac{\mu}{Q}\right)^\epsilon \left(1 + g^2\left(\frac{\mu}{Q}\right)^\epsilon f(\epsilon) + \mathcal{O}(g^4)\right)\right] . \tag{6.16}$$

If we now assume, as calculations will indeed show, that

$$f(\epsilon) = \frac{c_1}{\epsilon} + c_2 + c_3\epsilon + \cdots , \tag{6.17}$$

and if we write the second $(\mu/Q)^\epsilon$ term as

$$\left(\frac{\mu}{Q}\right)^\epsilon = e^{\epsilon \ln(\mu/Q)} = 1 + \epsilon \ln(\mu/Q) + \mathcal{O}(\epsilon^2) , \tag{6.18}$$

we arrive at

$$R = 3n_f \left[1 + \frac{g^2}{4\pi^2}\left(\frac{\mu}{Q}\right)^\epsilon \left(1 + g^2\left\{c_1\frac{1}{\epsilon} + c_2 + c_1\ln(\mu/Q) + \mathcal{O}(\epsilon)\right\} + \mathcal{O}(g^4)\right)\right] . \tag{6.19}$$

Now we renormalize by replacing the g with $g_{phys}Z_g$ and choosing

$$Z_g = 1 - \frac{g^2 c_1}{2\epsilon} , \tag{6.20}$$

i.e. such that it **precisely cancels the pole**. This defines the **minimal substraction** or **MS** scheme, where minimal refers to the the fact that just the pole is being cancelled. Thus, we find

$$R = 3n_f \left[1 + \frac{g^2_{phys}}{4\pi^2}\left(\frac{\mu}{Q}\right)^\epsilon \left(1 + g^2_{phys}\left\{c_2 + c_1\ln(\mu/Q) + \mathcal{O}(\epsilon)\right\} + \mathcal{O}(g^4_{phys})\right)\right] . \tag{6.21}$$

Now we can take the limit $\epsilon \to 0$ and, after renaming $g_{phys} \to g$ for notational simplicity, we have

$$R = 3n_f \left[1 + \frac{g^2}{4\pi^2}\left(1 + g^2\left\{c_2 + c_1\ln(\mu/Q)\right\} + \mathcal{O}(g^4)\right)\right] . \tag{6.22}$$

This works to all orders in g, so that we quite generally obtain a finite expression for

$$R = R(Q^2/\mu^2, g^2) \tag{6.23}$$

of the type displayed above, with g now being the physical coupling.

We note that we did not renormalize any fields (e.g. $A_\mu \to A_\mu Z_A^{1/2}$). This would not affect our result since the corresponding Z_A-factors would precisely cancel between vertices and external legs (where they induce a modifications of the Z-factors of LSZ). Of course, we can not expect to get finite Green's functions in this approach, but that's not important for us at the moment. If needed, it is clear how to extend the MS scheme to Green's functions.

Furthermore, we note that the MS scheme we just defined actually represents a whole 1-parameter family of schemes, with the so-far arbitrary parameter being μ. The observable R we just calculated appears to explicitly depend on this parameter, which of course should not be the case.

6.3 The β function

The resolution is simple: Our MS-scheme physical coupling g also depends on μ, in just the right way for R to be μ-independent:

$$g = g(\mu) \qquad \text{such that} \qquad 0 = \mu^2 \frac{d}{d\mu^2} R(Q^2/\mu^2, g^2(\mu))\,. \tag{6.24}$$

Note that it is common and useful to think in terms of the so-called logarithmic μ-derivative,

$$\mu^2 \frac{d}{d\mu^2} = \frac{d}{d\ln(\mu^2)}\,. \tag{6.25}$$

The above implies

$$0 = \mu^2 \left(\frac{\partial}{\partial\mu^2} + \frac{dg^2}{d\mu^2} \cdot \frac{\partial}{\partial g^2} \right) R(Q^2/\mu^2, g^2) \tag{6.26}$$

or

$$\mu^2 \frac{dg^2}{\mu^2} = -\left(\mu^2 \frac{\partial}{\partial\mu^2} R \right) \Big/ \left(\frac{\partial}{\partial g^2} R \right)\,. \tag{6.27}$$

The expression on the r.h. side formally depends on g^2 and (Q^2/μ^2), but in fact the (Q^2/μ^2)-dependence drops out. This must be the case since the l.h. side does not depend on Q. Indeed, while we use a particular observable to determine the μ-dependence of g, the latter is a universal quantity and it should not matter which obersvable we use. Thus, the observable-specific quantity Q must disappear. The parameter μ always appears together with Q and hence also disappears.

The upshot is that we have found a way to calculate (via a specific observable) the universal quantity

$$\mu \frac{dg}{d\mu} \equiv \beta(g)\,, \tag{6.28}$$

the so-called β-function in the minimal subtraction scheme. In our concrete example we have

$$\frac{dg^2}{d\ln\mu^2} = -\frac{3n_f c_1 g^4/(4\pi)^2 + \mathcal{O}(g^6)}{3n_f/(4\pi^2) + \mathcal{O}(g^2)}\,, \tag{6.29}$$

and hence

$$\beta(g) = -\frac{c_1}{2} g^3\,. \tag{6.30}$$

It is important to remember that we must always work consistently to a given order in perturbation theory to make sure that the Q^2-dependence of the underlying observable drops out. It is also worth noting that, once the fundamentals are clear, we do not need to pay much attention to R or any other observable: Indeed, the leading-order β-function coefficient is just the coefficient of the leading ϵ-pole in Z_g.

Jumping ahead, we note that for an $SU(N_c)$ gauge theory with N_f flavors one finds

$$\beta(g) = -\frac{g^3}{16\pi^2} \left(\frac{11}{3} N_c - \frac{2}{3} N_f \right)\,. \tag{6.31}$$

Recall that in QFT I we derived

$$\beta(e) = \frac{e^3}{12\pi^2}\,, \tag{6.32}$$

for QED with just the electron running in the loop. As a simple excercise, try to understand (6.32) as a 'limit' of (6.31) without doing any calculation. We also recall that the leading-order β-function is universal, i.e. the results in other schemes and even that for the bare coupling as a function of the cutoff Λ agree.

In practice, we can only get a fixed-order result for R and, to make the unknown higher orders as small as possible we must avoid large logarithms. Thus, we choose $\mu = Q$ and use

$$R = R(1, g^2(Q^2)) \tag{6.33}$$

to compare with experiment. In fact, for this very reason one uses $g = g(s)$ if one has a leading-order result for some observable (e.g. a cross-section) depending on a single mass scale $\sqrt{s}$.

We also note that the true expansion parameter is actually not g or g^2 but

$$\frac{g^2}{16\pi^2}N_c \equiv \frac{\alpha_s}{4\pi}N_c\,, \tag{6.34}$$

since we get further factors N_c at each higher-loop level as well as further 'loop suppression factors' $1/(16\pi^2)$ with each new loop. To understand the last point, note that

$$\int \frac{d^4k}{(2\pi)^4} = \int \frac{k^3\,dk\,Vol(S^3)}{(2\pi)^4} = \frac{1}{16\pi^2}\int k^2\,d(k^2)\,, \tag{6.35}$$

where we used $Vol(S^3) = 2\pi^2$.

Finally, we can think of determining $\alpha_s(\mu^2)$ by solving the **renormalization group equation** (6.28). The result is

$$\alpha_s(\mu^2) \sim \frac{1}{\ln \mu^2/\Lambda_{QCD}^2}\,, \tag{6.36}$$

with Λ_{QCD} being the integration constant and the normalization changing (in the simplest approximation in steps) with growing energy due to changing N_f. It is known from LEP (the predecessor of LHC at CERN) that $\alpha_s(m_z) \simeq 0.12$, which implies $\Lambda_{QCD} \simeq 200$ MeV. The behavior $\alpha_s \to 0$ at $\mu \to \infty$ is famously known as **asymptotic freedom**.

As an important generalization of this section, we note that the crucial relation

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\right) R(g, Q/\mu) = 0 \tag{6.37}$$

generalizes to the **Callen-Symanzik equation** for Green's function:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + n\gamma(g)\right) G^{(n)}(\{x_i\}, g, \mu) = 0\,. \tag{6.38}$$

In this renormalization group equation, a term depending on the field-renormalization appears:

$$\gamma \equiv -\mu \frac{\partial}{\partial \mu} \, \delta Z_\varphi \, . \tag{6.39}$$

Note that, for simplicity, we have here returned to the simplest case of a theory with just one real scalar field. Also, in the above μ is interpreted as the renormalization scale of some general renormalization procedure. Specifically in dim. reg. with minimal subtraction, a little more work has to be invested since, naively, the δZ's contain just the pole and no μ dependence. However, everything works out OK is one takes into account how the dimension of the fields changes in $d \neq 4$.

6.4 The 1-loop β-function of QCD

6.4.1 General strategy

The explicit calculation of the β-function is unavoidably somewhat technical and we will follow closely the rather detailed presentation of [1]. It is more convenient not to discuss a particular observable but to renormalize the lagrangian. This involves renormalizing fields and the gauge-fixing parameter. We start with A_μ and the gauge parameter ξ,

$$(A^0)^a_\mu = Z_3^{1/2} A^a_\mu \, , \qquad \xi_0 = Z_\xi \xi \, . \tag{6.40}$$

We will demonstrate that we must demand $Z_3 = Z_\xi$ since it is know from QCD Ward identities (or Slavnov-Taylor identities)[14] that the gluon self-energy is transverse. Thus, our counterterm should be transverse as well, which will only be the case if the above relation holds.

Indeed, the counterterm follows from (suppressing the gauge index)

$$\begin{aligned} S \;&\supset\; -\frac{1}{2}\int_k (A^0)_\mu \left(k^2\eta^{\mu\nu} - k^\mu k^\nu \left(1 - \frac{1}{\xi_0}\right)\right) (A^0)_\nu & (6.41)\\ &\supset\; -\frac{1}{2}\int_k Z_3 \, A_\mu \left(k^2\eta^{\mu\nu} - k^\mu k^\nu \left(1 - \frac{1}{Z_\xi \xi}\right)\right) A_\nu \, . & (6.42) \end{aligned}$$

If $Z_\xi = Z_3 = 1 + \delta Z_3$, this gives

$$S \;\supset\; -\frac{1}{2}\int_k \left[A_\mu \left(k^2\eta^{\mu\nu} - k^\mu k^\nu \left(1 - \frac{1}{\xi}\right)\right) A_\nu \;-\; \delta Z_3 \, A_\mu \Big(k^2\eta^{\mu\nu} - k^\mu k^\nu\Big) A_\nu \right] \, , \tag{6.43}$$

such that the term proportional to δZ_3 vanishes if $A_\mu(k) \sim k_\mu$, which is what we call transverse.

In addition, we have

$$\psi_0 = Z_2^{1/2}\psi \, , \qquad c_0 = Z_c^{1/2} c \, , \qquad m_0 = Z_m m \, , \qquad g_0 = Z_g g \, . \tag{6.44}$$

[14] While these are in general more complicated than in QED, the crucial feature of a **transverse self-energy** survives.

Plugging all of this into the lagrangian and separating all the terms involving δZs, we get the **counterterm lagrangian**. Among others, it involves the counterterm for the fermion-fermion-gluon vertex,

$$\sim g\,\bar{\psi}\not{A}\psi\,(Z_2 Z_3^{1/2} Z_g - 1) \equiv g\,\bar{\psi}\not{A}\psi\,(Z_{1,F} - 1)\,, \tag{6.45}$$

which implies that

$$Z_g = Z_{1,F} Z_2^{-1} Z_3^{-1/2}\,. \tag{6.46}$$

Thus, to find Z_g we need to calculate the counterterms $\delta Z_{1,F}$, δZ_2 and δZ_3 which make the fermion-fermion-gluon vertex, the fermion self-energy and the gluon self-energy finite.[15] As in QED, we can then follow the logic (in cutoff notation) that

$$\beta(g) = \frac{d}{d\,\ln\Lambda}\,g_0 = g\frac{d}{d\,\ln\Lambda}(Z_g)\,. \tag{6.47}$$

Recalling that the $\ln\Lambda$-term corresponds precisely to the coefficient of the ϵ-pole, all we need to do is to extract the coefficients of $1/\epsilon$ from $Z_{1,F}$, Z_2^{-1} and $Z_3^{1/2}$, to add them and to multiply by g. The result will be the leading order β-function. This of course precisely corresponds to what we saw in (6.20) and (6.30).

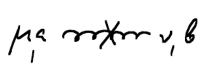
6.4.2 Gluon self energy

Let us start with the **gluon self energy**, i.e. with Z_3. We already saw the corresponding lagrangian term in Fourier space above. It gives rise to the counterterm Feynman rule of Fig. 23. This conterterm has to cancel the divergence in

$$i\Pi^{(1)}_{\mu\nu,\,ab} = i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\,\Pi_{(1)}(q^2)\delta_{ab}\,, \tag{6.48}$$

where on the r.h. side we assumed transversality, as discussed earlier. For the separate diagrams see Fig. 24. The quantity we need for our β-function calculation is

$$\delta Z_3 = \Pi_{(1)}(0)\Big|_{\frac{1}{\epsilon}\text{–term}}\,. \tag{6.49}$$



$= i(-\eta^{\mu\nu}q^2 + q^\mu q^\nu)\,\delta^{ab}\,\delta Z_3$

Figure 23: Couterterm Feynman rule for gluon self energy.

[15]Note that in QED we similarly had $Z_e = Z_1 Z_2^{-1} Z_3^{1/2}$, with $Z_1 \equiv Z_{1,F}$ since we do not need to specify which vertex we are talking about. However, since we also knew that $Z_1 = Z_2$, it was sufficient to calculate the photon self-energy or vacuum palarization to fix Z_3.

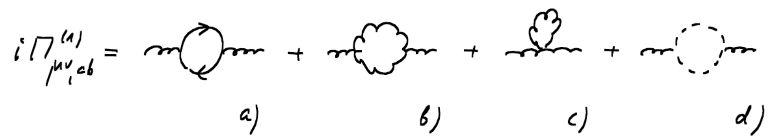


Figure 24: Gluon self energy.

a) Fermion loop

Let us now proceed diagram by diagram. Contribution a) is as in QED, except for the group-theoretic factor 1/2 that comes from

$$T^a_{ij} T^b_{ji} = \mathrm{tr}(T^a T^b) = \frac{1}{2}\delta^{ab} \,. \tag{6.50}$$

For N_f flavors we hence obtain

$$\Pi^{a)}_{(1),\,QED} = -\frac{e^2}{6\pi^2\epsilon} \qquad \Rightarrow \qquad \Pi^{a)}_{(1)} = -\frac{g^2 N_f}{12\pi^2\epsilon} \,. \tag{6.51}$$

Before carrying on, it is useful to think about this a bit more generally: The fermions could transform in any representation, called r, generated by matrices

$$(T^a_r)_{ij} \qquad \text{with} \qquad \mathrm{tr}(T^a_r T^b_r) = C(r)\delta^{ab} \equiv T(r)\delta^{ab} \,, \tag{6.52}$$

where both the notations $C(r)$ and $T(r)$ can be found in the literature. What is crucial is that the matrices T^a_r must be normalized such that they satisfy precisely the same commutation relations as the matrices

$$T^a_F \equiv T^a \equiv \frac{\lambda^a}{2} \tag{6.53}$$

of the fundamental representation. The quantity $C(r)$, which is group-theoretically related to the 'Dynkin index' is then characteristic of the representation r. Our specific factor 1/2 arises as a result of

$$C(F) = \frac{1}{2} \,. \tag{6.54}$$

With this notation, the Dirac fermion loop contribution in its most general form reads

$$I_{a)} = i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\delta^{ab}\left(-\frac{g^2 N_f C(r)}{6\pi^2\epsilon}\right) \,. \tag{6.55}$$

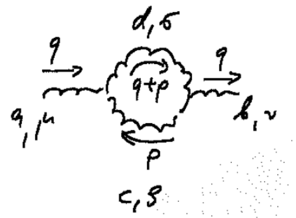
b) Gluon loop with 3-vertices

Next, we evaluate contribution b) is Feynman gauge, $\xi = 1$. First, let us recall the 3-gluon-vertex Feynman rule, cf. Fig 25.

$$= -g f^{a_1 a_2 a_3}\left\{(k_1-k_2)_{\mu_3}\eta_{\mu_1\mu_2} + \text{cyclic}\right\}$$

Figure 25: Feynman rule for 3-gluon vertex.

This rule gives rise to the gluon-loop diagram in Fig. 26 and the corresponding analytical expression. We note that, in contrast to the scalar $\lambda\varphi^3/3!$ term, there is no $1/3!$ factor in the corresponding term of the interaction lagrangian. Hence, one expects $3!$ terms to arise in the analytical expression. This does indeed happen in the form of the two terms coming with k_1 and k_2 and the additional terms from cyclic permutations. Multiplication of the two vertices the gives 36 terms, as above. The prefactor $(1/2)$ is a symmetry factor coming from the symmetry of the diagram under reflection about a horizontal axis. It arises in the very same manner as the corresponding factor $(1/2)$ in the analogous diagram in $\lambda\varphi^3$ theory.



$$= \frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\cdot\frac{-i}{p^2}\cdot\frac{-i}{(p+q)^2}\, g^2 f^{acd} f^{bcd}$$

$$\cdot\left\{(\underset{1}{q}-\underset{2}{p})_{\underset{3}{\sigma}}\eta_{\rho\mu} + (\underset{2}{p}-\underset{3}{(-q-p)})_{\underset{1}{\mu}}\eta_{\rho\sigma} + (\underset{3}{(-q-p)}-\underset{1}{q})_{\underset{2}{\rho}}\eta_{\mu\sigma}\right\}$$

$$\cdot\left\{(\underset{1}{-q}+\underset{2}{p})^{\underset{3}{\sigma}}\eta_\nu{}^\rho + (\underset{2}{-p}-\underset{3}{(q+p)})_{\underset{1}{\nu}}\eta^{\sigma\rho} + (\underset{3}{q+p}-\underset{1}{(-q)})^{\underset{2}{\rho}}\eta_\nu{}^\sigma\right\}$$

Figure 26: Gluon-self-energy diagram based on 3-gluon vertices. The way in which the different terms arise from cyclic permutations is emphasized.

Finally, let us turn to the group theoretic factor. One can show that

$$f^{acd} f^{bcd} \sim \delta^{ab}\,. \tag{6.56}$$

The reason is basically that one expects an 'invariant tensor' with two adjoint indices and δ^{ab} is the only candidate.

To understand this structure in more general terms, think of $(-if^{acd})$ as of an $(N^2-1)\times(N^2-1)$ matrix. This is a generator of the adjoint representation of $SU(N)$:

$$-if^{acd} = (-if^a)^{cd} \equiv (T_A^a)^{cd}\,. \tag{6.57}$$

This interpretation should be self-evident if you recall what we said when discussing how the covariant derivative acts on ghost fields (see also problems). We then have

$$f^{acd}f^{bcd} = (-if^a)^{cd}(-if^b)^{dc} = \mathrm{tr}(T_A^a T_A^b) = C(A)\delta^{ab}\,, \tag{6.58}$$

where we used total antisymmetry of the structure constants and our earlier definition of $C(r)$.

Alternatively, we can rewrite our color factor as

$$f^{acd}f^{bcd} = f^{dac}f^{dbc} = (-if^d)^{ac}(-if^d)^{cb} = (T_A^d T_A^d)^{ab} \equiv C_2(A)\delta^{ab}\,. \tag{6.59}$$

This is a special case of the more general definition of a parameter $C_2(r)$ which exists for any irreducible representation r:

$$(T_r^a T_r^a)_{ij} \equiv C_2(r)\delta_{ij}\,. \tag{6.60}$$

The operator

$$T_r^a T_r^a \equiv C_2(r)\,\mathbb{1} \tag{6.61}$$

is called the 'quadratic Casimir operator' of the representation r. Somewhat confusingly, this name is then often also used for the parameter $C_2(r)$. Note that, as before, the fact that the r.h. side of (6.60) is proportional to δ_{ij} follows because this is the only invariant tensor available.

In the course of our discussion, we have just proved that

$$C_2(A) = C(A)\,. \tag{6.62}$$

This is a special case of the more general relation

$$d(r)\,C_2(r) = d(A)\,C(r)\,, \tag{6.63}$$

where $d(r)$ is the dimension of the representation r. The proof is an easy excercise.

Finally, we state without proof that for $SU(N)$

$$C_2(A) = C(A) = N\,, \tag{6.64}$$

which is hence our color factor associated with diagram b). For more group-theoretic details see [1] or any of the various books on group and representation theory (e.g. [21]).

We now turn to the evaluation of the loop integral using the idea of the 'Feynman parameter' introduced in the context of QED:

$$\frac{1}{p^2}\cdot\frac{1}{(p+q)^2} = \int_0^1 dx\,\frac{1}{[(1-x)p^2 + x(p+q)^2]^2} = \int_0^1 dx\,\frac{1}{(k^2-\Delta)^2}\,, \tag{6.65}$$

where

$$k \equiv p + xq \qquad \text{and} \qquad \Delta \equiv -x(1-x)q^2\,. \tag{6.66}$$

The integration variable is now changed, $\int d^dp \to \int d^dk$.

Let us now turn to the numerator, i.e. the product of the two big curly brackets in Fig. 26 with open indices μ, ν. It is straightforward to rewrite this structure in terms of k, q and x. Since it appears under an $SO(1,3)$ symmetric k-integral,

$$\int d^d k \, \frac{1}{(k^2 - \Delta)^2} \Big[\cdots \Big] \,, \tag{6.67}$$

any term linear in k will vanish. Furthermore, any term quadratic in k will give a contribution $\sim \eta^{\mu\nu}$, and the prefactor is easy to determine. Indeed, by $SO(1,3)$ symmetry

$$\int d^d k \, f(k^2) \, k^\mu k^\nu \;\; = \;\; a \, \eta^{\mu\nu} \,, \tag{6.68}$$

and hence

$$\int d^d k \, f(k^2) \, k^2 \;\; = \;\; a \cdot d \,. \tag{6.69}$$

Thus,

$$a = \int d^d k \, f(k^2) \, \frac{k^2}{d} \,, \tag{6.70}$$

and we can use the substitution

$$k^\mu k^\nu \quad \to \quad \frac{k^2}{d} \, \eta^{\mu\nu} \tag{6.71}$$

inside the square bracket in (6.67). Doing all this, we have (with the curly brackets defined in Fig. 26)

$$\begin{aligned} &\int \frac{d^d p}{(2\pi)^d} \cdot \frac{1}{p^2 (p+q)^2} \cdot \Big\{ \cdots_\mu \Big\} \cdot \Big\{ \cdots_\nu \Big\} = \qquad\qquad (6.72) \\ &\quad = \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \cdot \frac{1}{(k^2 - \Delta)^2} \cdot \Bigg\{ - \eta_{\mu\nu} k^2 6 \left(1 - \frac{1}{d} \right) \\ &\qquad\quad - \eta_{\mu\nu} q^2 \big[(2-x)^2 + (1+x)^2 \big] + q_\mu q_\nu \big[(2-d)(1-2x)^2 + 2(1+x)(2-x) \big] \Bigg\} . \end{aligned}$$

We now Wick-rotate, $k_0 \to i k_0$, and subsequently apply the more generally useful formulae

$$\int \frac{d^d k}{(2\pi)^d} \cdot \frac{1}{(k^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \cdot \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)} \left(\frac{1}{\Delta} \right)^{n - \frac{d}{2}} \tag{6.73}$$

and

$$\int \frac{d^d k}{(2\pi)^d} \cdot \frac{k^2}{(k^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \cdot \frac{d}{2} \cdot \frac{\Gamma\left(n - \frac{d}{2} - 1\right)}{\Gamma(n)} \left(\frac{1}{\Delta} \right)^{n - \frac{d}{2} - 1} \,. \tag{6.74}$$

The reader may recall that we already used such integrals, in the special case $n = 2$, in QED. Here, we will provide some more calculational detail and mathematical background:

First, recall that (in euclidean signature)

$$\int d^d k = \int d\Omega_{d-1} \int_0^\infty k^{d-1} dk = Vol(S^{d-1}) \int_0^\infty k^{d-1} dk \,. \tag{6.75}$$

Next, write

$$\int_0^\infty \frac{k^{d-1}dk}{(k^2+\Delta)^n} = \frac{1}{2}\int_0^\infty d(k^2)\frac{(k^2)^{d/2-1}}{(k^2+\Delta)^n} = \cdots \,, \tag{6.76}$$

which, after the substitutions

$$(k^2+\Delta) \equiv \frac{\Delta}{y}\,, \quad d(k^2) = -\frac{\Delta}{y^2}\,dy\,, \quad k^2 = \Delta\cdot\left(\frac{1}{y}-1\right) = \frac{\Delta\cdot(1-y)}{y}\,, \tag{6.77}$$

becomes

$$\cdots = \frac{1}{2}\left(\frac{1}{\Delta}\right)^{n-d/2}\int_0^1 dy\, y^{n-1-d/2}(1-y)^{d/2-1}\,. \tag{6.78}$$

We now use the definition of the **Beta-function of mathematics** (not to be confused with the β**-function of QFT**),

$$\int_0^1 dy\, y^{a-1}(1-y)^{b-1} \equiv B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\,, \tag{6.79}$$

where the second equality is an important and useful identity. Of course, the expression in terms of Γ functions could also have been derived directly, by evaluating the integral explicitly for integer a, b and analytically continuing.

The integral of (6.74) is now also easily done writing the k^2 in the numerator as $(k^2+\Delta)-\Delta$. Putting all of the above together, diagram b) of Fig. 24 gives

$$\begin{aligned} I_{b)} = \frac{ig^2}{(4\pi)^{d/2}}\,C_2(A)\,\delta^{ab}\int_0^1 dx\,\frac{1}{\Delta^{2-d/2}}\cdot\Bigg\{&\Gamma\left(1-\frac{d}{2}\right)\eta^{\mu\nu}q^2\left[\frac{3}{2}(d-1)x(1-x)\right]+ \\ &+\Gamma\left(2-\frac{d}{2}\right)\eta^{\mu\nu}q^2\left[\frac{1}{2}(2-x)^2+\frac{1}{2}(1+x)^2\right] \\ &-\Gamma\left(2-\frac{d}{2}\right)q^\mu q^\nu\left[\left(1-\frac{d}{2}\right)(1-2x)^2+(1+x)(2-x)\right]\Bigg\}. \end{aligned} \tag{6.80}$$

Let us postpone further evaluation and turn to the other gluon loop:

c) Gluon loop – tadpole

Given what we have done so far, it is straightforward (in fact, much simpler than for diagram b)) to bring this to the form

$$I_{c)} = -g^2 C_2(A)\delta^{ab}\int\frac{d^dp}{(2\pi)^d}\cdot\frac{1}{p^2}\cdot\eta^{\mu\nu}(d-1)\,. \tag{6.81}$$

Figure 27: Self-energy of $\lambda\varphi^4$ theory – tadpole diagram.

The result is consistent with the expectation from Fig. 24 that, in this contribution, the external momentum q does not flow through the loop (with loop momentum p). We immediately recognize the quadratic divergence at $d = 4$, which is completely analogous to that of $\lambda\varphi^4$ theory (cf. Fig. 27).

Here, we do not expect such divergences since we do not have a dimensionful lagrangian parameter (such as the scalar mass-squared term) which could absorb a Λ^2-contribution. Thus, we expect to see that other diagrams cancel the quadratic divergence.

We also observe that, in dimensional regularization, $I_{c)} \equiv 0$ since, more generally,

$$\int \frac{d^d p}{(p^2)^n} \equiv 0\,. \tag{6.82}$$

The latter is obvious since, under the substitution $p \to \alpha p$, this integral acquires an overall factor α^{d-2n}, which is only consistent if the integral evaluates to zero.

The presence of a quadratic divergence and the zero value are not in contradiction. Indeed, a quadratic divergence in $d = 4$ corresponds to a log-divergence in $d = 2$, implying a pole in ϵ. But in $d = 2$ we also have an infrared divergence (a divergence at $p^2 \to 0$), i.e.

$$d \to 2: \qquad \int \frac{d^{2-\epsilon} p}{p^2} \sim \frac{1}{\epsilon} - \frac{1}{\epsilon} = 0\,. \tag{6.83}$$

While it is thus completely consistent to set $I_{c)}$ to zero, it will be instructive to separate IR and UV divergence and see the cancellation of the latter explicitly. This is achieved by rewriting the integrand as

$$\frac{1}{p^2} = \frac{(p+p)^2}{p^2(p+q)^2}\,, \tag{6.84}$$

introducing a Feynman parameter x in complete analogy to diagram b), and performing the p-integration:

$$\begin{aligned} I_{c)} &= \frac{ig^2}{(4\pi)^{d/2}} C_2(A)\delta^{ab} \int_0^1 \frac{dx}{\Delta^{2-d/2}} \cdot \eta^{\mu\nu} \cdot q^2 \times \\ &\times \left\{ -\Gamma\left(1 - \frac{d}{2}\right) \frac{1}{2} d(d-1)x(1-x) - \Gamma\left(2 - \frac{d}{2}\right)(d-1)(1-x)^2 \right\}, \end{aligned} \tag{6.85}$$

where, as before,

$$\Delta = -x(1-x)q^2\,. \tag{6.86}$$

We see that the first and second Γ functions provide the expected poles at $d = 2$ (corresponding to the quadratic divergence at $d = 4$) and the pole at $d = 4$ (the log-divergence).

The IR-divergence now hides in the divergence at $x = 0$. Indeed, the term multiplying $\Gamma(2 - d/2)$ is non-vanishing at $x \to 0$, such that

$$\int_0^1 \frac{dx}{[-q^2(x(1-x)]^{2-d/2}}\bigg|_{d=2} \quad \supset \quad \int_0 \frac{dx}{x}\,. \tag{6.87}$$

d) Ghost loop

Nothing conceptually new happens, although it may be worthwhile recalling the factor (-1) coming with every fermion loop:

$$\begin{aligned} I_{d)} &= (-1)\int \frac{d^dp}{(2\pi)^d}\cdot\frac{i}{p^2}\cdot\frac{i}{(p+q)^2}\cdot g^2 f^{dac}f^{cbd}(p+q)^\mu p^\nu && (6.88)\\ &= \frac{ig^2}{(4\pi)^{d/2}}C_2(A)\delta^{ab}\int_0^1\frac{dx}{\Delta^{2-d/2}}\times && (6.89)\\ &\quad\times\left\{-\Gamma\left(1-\frac{d}{2}\right)\eta^{\mu\nu}q^2\frac{1}{2}x(1-x)+\Gamma\left(2-\frac{d}{2}\right)q^\mu q^\nu x(1-x)\right\}. && (6.90)\end{aligned}$$

Combining the diagrams

Let us first combine the coefficients of $\Gamma(1-d/2)$ in diagrams b), c) and d). All functional dependence except that on the number of dimensions d is the same, such that one arrives at

$$I_{b)} + I_{c)} + I_{d)}\bigg|_{\Gamma(1-d/2)} \sim (3d - 3 - d^2 + d - 1) = -2\left(1-\frac{d}{2}\right)(2-d)\,. \tag{6.91}$$

We find a factor $(1 - d/2)$ which cancels the pole $1/(1 - d/2)$ of the Γ function. Thus, indeed, there is no quadratic divergence in $d = 4$.

Using the freedom to exchange

$$x \quad \leftrightarrow \quad (1-x) \tag{6.92}$$

in any term in the numerator one finds, after some algebra,

$$\begin{aligned} I_{b)} + I_{c)} + I_{d)} &= \frac{ig^2}{(4\pi)^{d/2}}C_2(A)\delta^{ab}\int_0^1 dx\,\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}\cdot\left(\eta^{\mu\nu}q^2 - q^\mu q^\nu\right)\times && (6.93)\\ &\qquad\times\left[\left(1-\frac{d}{2}\right)(1-2x)^2+2\right].\end{aligned}$$

This can be fully evaluated in terms of Γ functions (cf. Itzykson/Zuber, [12]), for generic d. We will limit ourselves to extracting the pole,[16] which is easy since

$$\Delta^{2-d/2} \to 1 \qquad \text{at} \qquad d \to 4\,. \tag{6.94}$$

[16] To get the finite terms right, one expands

$$\Delta^{-\epsilon/2} = 1 - \frac{\epsilon}{2}\ln[-q^2x(1-x)] + \cdots$$

before doing the x-integration.

At this point, we include the fermion loop diagram $I_{a)}$ and recall that, according to Fig. 24, we have now obtained $\Pi^{(1)}_{\mu\nu}(q^2)$. This is related to $\Pi_{(1)}(q^2)$ and eventually to δZ_3. Thus, we find

$$\begin{aligned}\delta Z_3 = \Pi_{(1)}(0)\Big|_{1/\epsilon\text{-part}} &= \left. -\frac{g^2 N_f}{6\pi^2\epsilon}C(r) - \frac{g^2}{(4\pi)^2}\left(-\frac{5}{3}\right)C_2(A)\Gamma\left(2-\frac{d}{2}\right) + \cdots \right|_{1/\epsilon\text{-part}} \\ &= -\frac{2g^2}{16\pi^2\epsilon}\left[\frac{4}{3}N_f C(r) - \frac{5}{3}C_2(A)\right] ,\end{aligned} \tag{6.95}$$

where we used

$$\Gamma\left(2-\frac{d}{2}\right) = \frac{1}{2-d/2} + \text{finite} = \frac{2}{\epsilon} + \text{finite} . \tag{6.96}$$

In the last expression for δZ_3, we clearly see that by replacing $C(r)$ with unity we would recover the vacuum polarization effect of QED. In QED, this is all that contributes to the β function. Also, we see that an opposite-sign effect proportional to $C_2(A)$ arises due to the non-abelian nature of our theory. In fact, its coefficient 5/3 is going to be enhanced to 11/3 by δZ_2 and $\delta Z_{1,F}$.

6.4.3 Fermion self energy

We only need the fermion self energy, cf. Fig. 28, to determine δZ_2. The corresponding analytical expression reads

$$I_F = (ig)^2 \int \frac{d^d p}{(2\pi)^d}\gamma^\mu T^a \frac{i}{\not{p} - \not{k}}\gamma_\mu T^a \frac{(-i)}{p^2} . \tag{6.97}$$

Evaluating this is a straightforward excercise in γ-matrix algebra and loop integrals. One also needs to use

$$(T^a T^a)_{ij} = C_2(r)\delta_{ij} . \tag{6.98}$$

Finally, one needs to write down the corresponding counterterm diagram proportional to δZ_2 and demand cancellation of the divergence. This gives

$$\delta Z_2 = -\frac{g^2}{16\pi^2}\cdot\frac{2}{\epsilon}\cdot C_2(r) . \tag{6.99}$$

Figure 28: Fermion self-energy.

6.4.4 Gluon-fermion-fermion vertex

The relevant diagrams, let us call them A and B, are shown in Fig. 29. Their divergence has to be cancelled by the counterterm

$$\mathcal{L} \supset i(Z_{1,F}-1)\bar{\psi} ig \not{A} \psi = -\delta Z_{1,F} \bar{\psi} g \not{A} \psi \,. \tag{6.100}$$

Straightforwardly applying the Feynman rules gives

$$I_A = g^3 \int \frac{d^d p}{(2\pi)^d} T^b T^a T^b \gamma^\nu \frac{1}{\not{p}+\not{k}'} \gamma^\mu \frac{1}{\not{p}+\not{k}} \gamma_\nu \cdot \frac{1}{p^2} \,. \tag{6.101}$$

There is no quadratic divergence. To extract the log-divergence, it will be sufficient to evaluate the diagram at $k = k' = 0$.

Figure 29: The two diagrams, referred to as A and B in the text, contributing to the vertex.

The most interesting part of the calculation is the group theory factor:

$$T^b T^a T^b = T^b T^b T^a + T^b [T^a, T^b] = C_2(r) T^a + T^b i f^{abc} T^c \,. \tag{6.102}$$

The second term can be further rewritten as

$$i f^{abc} \frac{1}{2} [T^b, T^c] = \frac{1}{2} i f^{abc} i f^{bcd} T^d = \frac{1}{2}(-C(A)) \delta^{ad} T^d = -\frac{1}{2} C_2(A) T^a \,, \tag{6.103}$$

such that one eventually finds

$$T^b T^a T^b = \left[C_2(r) - \frac{1}{2} C_2(A) \right] T^a \,. \tag{6.104}$$

Note that the color structure of the tree-level vertex is, of course, simply T^a. Thus, our result is consistent with multiplicative renormalization.

We also have

$$\int \frac{d^d p}{(2\pi)^d} \cdot \frac{\gamma^\nu \not{p} \gamma^\mu \not{p} \gamma_\nu}{(p^2)^3} = \int \frac{d^d p}{(2\pi)^d d} \cdot \frac{\gamma^\nu \gamma^\rho \gamma^\mu \gamma_\rho \gamma_\nu}{(p^2)^2} \,. \tag{6.105}$$

Here we used the substitution

$$p^\rho p^\sigma \quad \to \quad p^2 \frac{\eta^{\rho\sigma}}{d} \,, \tag{6.106}$$

which is allowed under a Lorentz-invariant p-integral. Doing the γ algebra and extracting the pole is now straightforward. One need not be concerned that the last inegral also appears to have an IR divergence – the latter is just an artifact of setting k and k' to zero.

Diagram B is evaluated in complete analogy - we leave it as an excercise. The most interesting part is again the color factor,

$$f^{abc}T^bT^c = \frac{1}{2}f^{abc}if^{bcd}T^d = \frac{i}{2}C_2(A)T^a\,. \tag{6.107}$$

Combining A and B and requiring the overall $(1/\epsilon)$ term to be cancelled, one obtains

$$\delta Z_{1,F} = -\frac{g^2}{16\pi^2}\cdot\frac{2}{\epsilon}\big(C_2(r)+C_2(A)\big)\,. \tag{6.108}$$

6.4.5 Summary

Finally, recall that

$$Z_g = Z_{1,F}Z_2^{-1}Z_3^{-1/2} \tag{6.109}$$

and hence

$$\begin{aligned}
\beta(g) &= gZ_g\Big|_{\text{coeff. of }(1/\epsilon)} = g\left(-\frac{1}{2}\delta Z_3-\delta Z_2+\delta Z_{1,F}\right)\Big|_{\text{coeff. of }(1/\epsilon)}\\
&= \frac{g^3}{16\pi^2}\left(\ \left[\frac{4}{3}N_fC(r)-\frac{5}{3}C_2(A)\right]\quad+\quad\Big[2C_2(r)\Big]\quad+\quad\Big[2C_2(r)-2C_2(A)\Big]\ \right)\\
&= \frac{g^3}{16\pi^2}\left(\frac{4}{3}N_fC(r)-\frac{11}{3}C_2(A)\right)\,.
\end{aligned} \tag{6.110}$$

This is a famous formula which, in particular, implies that non-abelian gauge theories are asymptotically free as long as there is not too much matter.

It may be the right moment to recall how we arrived at this result at the conceptual level: Quite generally, we have a renormalization scale M and a cutoff scale Λ, such that

$$g_{bare}(\Lambda) = Z_g(\Lambda/M)g_{phys.}(M) \tag{6.111}$$

and

$$\beta(g_{phys.}) = \frac{d}{d\ \ln M}\,g_{phys.}(M)\,. \tag{6.112}$$

This last derivative can be rewritten as

$$\begin{aligned}
\frac{d}{d\ \ln M}\,g_{phys.}(M) &= \frac{d}{d\ \ln M}\,Z_g^{-1}(\Lambda/M)\,g_{bare}(\Lambda) = -\frac{d}{d\ \ln\Lambda}\left(Z_g^{-1}(\Lambda/M)\right)g_{bare}(\Lambda)\\
&= \frac{dZ_g(\Lambda/M)}{d\ \ln\Lambda}\,Z_g^{-2}(\Lambda/M)\,g_{bare}(\Lambda)\simeq\frac{dZ_g(\Lambda/M)}{d\ \ln\Lambda}\,g_{phys.}(M)\,,
\end{aligned} \tag{6.113}$$

where the last simplification is only valid at leading order in g.

Now, specifically in dim. reg., M is identified with μ and

$$\frac{dZ_g(\Lambda/M)}{d\ln\Lambda} = \delta Z_g\Big|_{\text{coeff. of }(1/\epsilon)} \tag{6.114}$$

in the MS scheme at leading order. This can be extended to higher orders, where higher poles in ϵ appear and a more complicated μ-dependence arises from expanding μ^ϵ to higher order in ϵ. We will not pursue this.

As a concluding remark, we note that it is often useful to visualize the β function of theory in diagrams such as that of Fig. 30. The arrows indicate in which direction the coupling 'flows' with growing μ. Zeros of the β function are called 'fixed points' for obvious reasons.

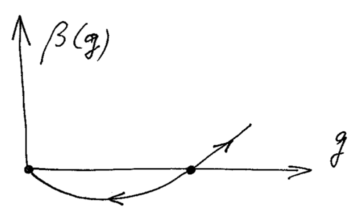


Figure 30: Schematic illustration of the β function of QCD inside the so-called 'conformal window'. The change of sign at larger g can arise dur to the $\sim g^5$ term in $\beta(g)$ for appropriately chosen N_c and N_f. See problems.

7 Operator product expansion and its simplest application in QCD

The operator product expansion (OPE) is a very powerful tool which is useful in many QFTs. It dates back to the work of Wilson in 1969 and treats the limit in which the space-time arguments of two local operators converge ($(x-y) \to 0$):

$$A(x)B(x) \sim \sum_C F_C^{AB}(x-y) \cdot C(y)\,. \tag{7.1}$$

The non-trivial point is that, on the r.h. side, all possible divergences arising in this limit reside in the complex coefficients $F(x-y)$ while the operators C are well-defined and finite in this limit. The OPE has historically been very important in QCD, which is still its most important application in the realm of experimentally accessible particle physics. However, it is also absolutely essential in string theory, where it is applied to the 2d worldsheet QFT (more precisely a CFT) describing the embedding of the string in the so-called target space. Moreover, it is crucial in the study many non-perturbative QFTs, especially CFTs, also outside $d = 2$.

7.1 e^+e^- to hadrons at leading order (LO)

Before we can develop and apply the OPE, we need to gain some basic physical understanding of the process that will serve as our main example application. This is also worthwhile in itself.

Recall our calculation of $e^+e^- \to \mu^+\mu^-$ in QFT I, cf. Fig. 31, which gave the result

$$\frac{d\sigma}{d\Omega} = \frac{\alpha_{em}^2}{4s}(1+\cos^2\theta)\,. \tag{7.2}$$

From this, the total cross section follows by applying

$$\int d\Omega \cdots = \int d\varphi\, \sin\theta\, d\theta \cdots = 2\pi \int_{-1}^{1} d\cos\theta \cdots\,, \tag{7.3}$$

i.e.

$$\sigma = \frac{2\pi\alpha_{em}^2}{4s}\int_{-1}^{1} dx\,(1+x^2) = \frac{4\pi\alpha_{em}^2}{3s}\,. \tag{7.4}$$

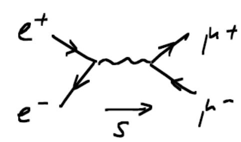


Figure 31: Diagram for $e^+e^- \to \mu^+\mu^-$.

For $e^+e^- \to q\,\bar{q}$ (cf. Fig. 32), the relevant part of the lagrangian is

$$\mathcal{L} \supset \sum_f \bar{q}_f(i\not{D} - m_f)q_f \supset \sum_f \bar{q}_f(-eQ_f \not{A}_\gamma)q_f\,, \tag{7.5}$$

where it is crucial to remember that $\not{D}$ contains the $SU(3)$, $SU(2)_L$ and $U(1)_Y$ gauge connection in the full theory. Here, we may assume that the electroweak symmetry is already broken, $SU(2)_L \times U(1)_Y \to U(1)_{em}$ and we need only $SU(3)$ and $U(1)_{em}$. At tree level, only the latter is relevant. Nevertheless, we need to remember that we have suppressed the color index and, actually, $q_f \to q_{f,i}$ with $i \in \{1,2,3\}$. It follows that

$$\sigma_{e^+e^-\to q\,\bar{q}} = \sigma_{e^+e^-\to\mu^+\mu^-}\cdot 3 \sum_{f\,,\,m_f\ll s} Q_f^2 = 3\cdot\frac{4\pi\alpha_{em}^2}{3s}\cdot\sum_{f\,,\,m_f\ll s} Q_f^2\,. \tag{7.6}$$

We also need to remember that $Q_{u,c,t} = 2/3$ and $Q_{d,s,b} = -1/3$ and that we must stay below 100 GeV unless we want to also include the Z boson contribution.

All of this of course also works if we keep the cross section differential in θ and φ. The corresponding observable is the angular distribution of two **jets**, cf. Fig. 33.

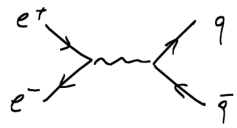


Figure 32: Diagram for $e^+e^- \to q\,\bar{q}$.

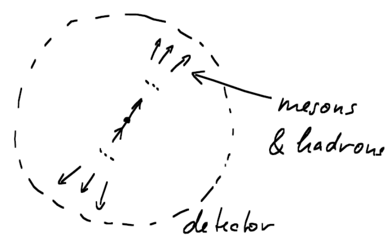


Figure 33: Two hadronic jets from $e^+e^- \to q\,\bar{q}$.

7.2 $e^+e^- \to$ hadrons at next-to-leading order (NLO)

The relevant diagrams, shown in Fig. 34, fall into two categories – **virtual** and **real** corrections. Note that, because $\alpha_{em} \ll \alpha_s$, we ignore QED corrections. Hence, there are no loops or real corrections (i.e. corrections associated with the radiation of extra **real** particles) attached to the incoming electron and positron. The correction to the **total** cross section arising from the sum of all QCD corrections given symbolically in Fig. 34 is finite. This is non-trivial and deserves some discussion.

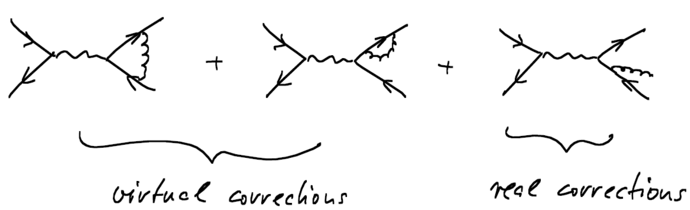


Figure 34: NLO corrections (virtual + real) to $e^+e^- \to q\,\bar{q}$.

First, we recall that we already explained the **UV** finiteness in Sect. 6.2. As a reminder, the argument was that the LO 'R-ratio' (of $e^+e^- \to hadrons$ and $e^+e^- \to \mu^+\mu^-$) was just a number and hence unable to absorb a divergence. Without referring to R, we can also simply argue that the LO result depends on only on α_{em}. But the LO renormalization of α_{em} is due exclusively to the vacuum polarization diagram, Fig. 35. The latter knows about α_{em} and the electromagnetic charges of all light particles, but not about α_s. Hence, we can't find a UV divergence in the above cross section. Technically, the cancellation occurs between the two first diagrams in Fig. 34.

Next, we turn to the infrared or, in more detail, **soft** and **collinear** divergences. Soft divergences are those associated with gluon momentum near zero. Collinear divergences

Figure 35: Vacuum polarization. All sufficiently light charged particles run in the loop.

come from the region in the gluon-momentum integral when the latter is nearly parallel to one of the outgoing quark momenta.[17] Divergences of these types come from all three diagrams in Fig. 34. In particular, divergences of the real correction come from the relevant region of the phase space integral. They may be treated by evaluating the phase space in d dimensions. This makes it possible to see explicitly how divergences cancel between virtual and real corrections. Note also that divergences up to $1/\epsilon^2$ arise since the gluon can be soft and collinear at the same time.

It is worth remembering that the second diagram in Fig. 34 actually needs to be resummed, i.e., many such fermion self energies are allowed to appear on the same outgoing line. This leads to a factor Z on the external leg which, together with the explicit $Z^{-1/2}$ from LSZ gives a final overall factor $Z^{1/2}$ per leg. Eventually, this corresponds to one-half of the effect of the diagram in Fig. 36, with on-shell fermion momentum.

Figure 36: Quark self energy.

To understand in detail how the infrared divergences cancel between the three different types of terms goes beyond what we can do here. Suffice it to say that the 'trick' of writing an apparent dim.-reg. zero as

$$0 = \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \tag{7.7}$$

is useful. Here, of course, $\epsilon_{UV} = \epsilon_{IR}$, but labelleling the divergences differently helps as a bookkeeping device. More details can be found in [1], in the book by Ellis/Sterling/Webber [22] and in my lecture notes on 'Perturvative QCD" in the ITP library. The much simpler QED analogue of such infrared cancellations can be promoted to an 'all-orders finiteness result', the **Kinoshita-Lee-Nauenberg theorem**, cf. [13].

For completeness, we recall the final formula,

$$\sigma_{e^+e^-\to hadrons} = \sigma_{e^+e^-\to q\bar{q}} \cdot \left(1 + \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2)\right) . \tag{7.8}$$

We also note that, if one keeps the result differential in the angular variables of the three outgoing particles, one predicts the 3-jet differential cross section (cf. Fig. 37). The collinear divergences correspond to the region where the gluon jet becomes nearly

[17]Obviously, in our case a collinear divergences arises only if one takes $m_q = 0$. This, however, is appropriate in the present situation since, at least for the light quarks, $m_q \ll \Lambda_{QCD}$.

parallel with one of the quark jets are now cut off by the so-called ‘jet-definitions’. For example, the jet defintion will in general say that, if quark and gluon are ‘too’ parallel, they have to be counted as a single jet. Then the 3-jet cross section above will become finite but dependent on this jet definition.

Figure 37: 3-jet final state in detector.

This can be extended to N^{n}LO with $n > 1$. Finiteness will continue to hold for the fully inclusive cross section.

7.3 Operator product expansion

For this particular observable of $e^+e^- \to hadrons$, the OPE represents a conceptually ‘cleaner’ approach. Indeed, while finiteness is reassuring, our loop and phase-space integrations go over momentum regions which are definitely too soft to trust perturbative QCD. (Recall that $\alpha_s(\mu)$ blows up at Λ_{QCD}.) One may wonder whether the naive perturbative analysis outlined above is justified. As the OPE shows, this is indeed the case.

Let us first recall the **optical theorem** of QM. Adopting it to our notation, we have an S-matrix $S \equiv \mathbb{1} + iT$ and, by unitarity,

$$\mathbb{1} \;=\; (\mathbb{1}+iT)^\dagger(\mathbb{1}+iT) \;=\; \mathbb{1} + iT - iT^\dagger + T^\dagger T\,. \tag{7.9}$$

Making the indices of the matrix T explicit, we have

$$i(T_{i'i} - iT^*_{ii'}) = \sum_f T^*_{fi'} T_{fi}\,. \tag{7.10}$$

After setting $i = i'$, one finds

$$2\mathrm{Im}T_{ii} = \sum_f |T_{fi}|^2\,. \tag{7.11}$$

Interpeting the r.h. side as a ’total cross’ section, we morally recognize the famous statement that the *imaginary part of the forward scattering amplitude is related to the total cross section.* Proper proofs in QM can be found in many textbooks and the generalization to QFT is straightforward. The only addition to our naive ‘derivation’ is a proper treatment of plane-wave normalization and phase space (see e.g. [23]). The result, already applied to our case of interest, reads

$$\sigma_{tot}(e^+e^- \to hadrons) = \frac{1}{s}\,\mathrm{Im}\,\mathcal{M}(e^+e^- \to e^+e^-)\,. \tag{7.12}$$

Crucially, on the r.h. side only hadronic intermediate states are allowed (since this is how we defined our T before squaring), cf. Fig. 38.

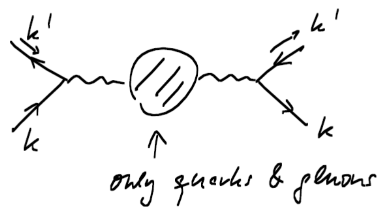


Figure 38: Forward e^+e^- scattering with only hadronic intermediate states.

Explicitly, we have

$$i\mathcal{M} = (-ie)^2\bar{u}(k)\gamma_\mu v(k')\frac{-i}{s}\left(i\Pi_h^{\mu\nu}(q)\right)\frac{-i}{s}\bar{v}(k')\gamma_\nu u(k)\,, \tag{7.13}$$

where $\Pi_h^{\mu\nu}$ is the hadronic (i.e. quark-antiquark plus higher orders in α_s) contribution to the vacuum polarization. Using $s = q^2 = (k+k')^2$ and

$$\Pi_h^{\mu\nu}(q) \equiv (q^2\eta^{\mu\nu} - q^\mu q^\nu)\Pi_h(q^2)\,, \tag{7.14}$$

one easily derives

$$\sigma_{tot}(e^+e^- \to hadrons) = -\frac{4\pi\alpha_{em}}{s}\,\mathrm{Im}\,\Pi_h(s)\,. \tag{7.15}$$

The leading diagrams contributing to Π_h are displayed in Fig. 39.

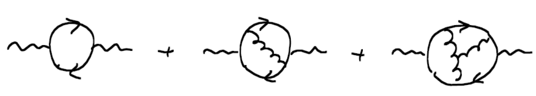


Figure 39: Leading diagrams contributing to Π_h.

Now, all diagrams start with the two photons coupling to four quarks via the $\gamma\bar{q}q$-vertex. Thus, the expectation value of two A_μ's can be replaced by the expectation values of two $\bar{q}q$'s. The latter explicitly enter in the same combination as in the electromagnetic current,

$$j^\mu = \sum_f Q_f\bar{q}_f\gamma^\mu q_f\,. \tag{7.16}$$

We can hence write

$$i\Pi_h^{\mu\nu}(q) = -e^2\int d^4x\, e^{iqx}\langle 0|Tj^\mu(x)j^\nu(0)|0\rangle\,. \tag{7.17}$$

The way in which the difference x of the space-time arguments of the currents is integrated over comes from the definition of $\Pi_h(q)$ in Fourier space. To check the correctness of our proposed identity, evaluate both sides at zeroth order in α_s, i.e. at the order of the first diagram in Fig. 39. It should then be clear that the identity continues to hold also including more and more gluon and quark loops.

We are now finally ready to apply the **operator product expansion**. It states that for two local operators $A(x)$ and $B(y)$ one has

$$A(x)B(y) \;\sim\; \sum_C F_C^{AB}(x-y)\cdot C(y)\,, \tag{7.18}$$

where the sum is over a suitable set of local operators $C(y)$ and the coefficients F are complex functions of the separation $x-y$ of A and B. These functions are in general singular in the limit $x-y\to 0$. The symbol '$\sim$' means equality of to non-singular terms, i.e. writing

$$f(x,y)\sim g(x,y) \tag{7.19}$$

is equivalent to the statement that

$$f(x,y)-g(x,y) \quad \text{is analytic at the point } x=y\,. \tag{7.20}$$

Thus, at the qualitative level, the OPE can be viewed as a Laurent expansion of $A(x)B(y)$ with operator-valued coefficients.[18]

Next, we focus on the behaviour of F as $x-y\to 0$. In this limit, the high mass-scale $|x-y|^{-1}$ dominates any other mass parameter possibly present in the theory. Thus, assuming that $|x-y|$ is the only dimensionful parameter in the problem, we have by dimensional analysis

$$F_C^{AB}(x-y)\simeq \Big(\#\text{ depending on } A, B \text{ and } C\Big)\cdot |x-y|^{\,[C]-[A]-[B]}\,. \tag{7.21}$$

Here $[A]$, $[B]$ and $[C]$ are the mass-dimensions of the corresponding operators and the power of $|x-y|$ is assumed to be negative. We note that in perturbative QFTs or QFTs defined by a lagrangian[19] (and we limit ourselves to this class in the present course), there is a distinguished type of local operators: the elementary fields φ, A_μ, ψ etc. We think of A, B, C as built from such fields. Hence $[C]$ grows with the complexity of C. The most singular terms in the OPE are associated with the simplest C's.

The OPE can be given a precise formulation (involving in particular the regularization and the renormalization of the relevant operators) and then proven. We refer e.g. to [12, 13] for details and only illustrate the issue.

[18]There is an ambiguity in the selection of operators C, similar to the basis-choice ambiguity one encounters in the description of vector spaces. One particular option that one has and which we will always use here is to always consider *normal-ordered* operators.

[19]As a side-remark, the OPE is also very useful in QFTs without a lagrangian. In such situations, the operators appearing in the OPE become an essential part of the definition of the theory and the OPE plays an even more central role.

The fact that $A(x)B(y)$ is singular at $x = y$ is a generic feature of QFTs. The simplest example is $\varphi(x)\varphi(y)$ in a free scalar theory. We know that

$$\langle 0|T\varphi(x)\varphi(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \cdot \frac{e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon}\,, \tag{7.22}$$

which clearly shows the singularity at $x-y=0$. If the theory is interacting, the propagator will receive loop corrections leading to additional divergences. In 'nice' theories (the scalar in $d = 4$ is not a good example of this), the divergences will be logarithmic and hence induce logarithmic corrections to the singular behaviour of the coefficients F_C^{AB}:

$$\frac{1}{|x-y|^\alpha} \quad \to \quad \frac{1}{|x-y|^\alpha} \ln\Big((x-y)^2\mu^2\Big)\,. \tag{7.23}$$

We now press ahead with our QCD application. Since we are only interested in the vacuum expectation value, $\langle 0|\cdots|0\rangle$ of our $j_\mu j_\nu$ operator product, we may restrict $\sum_C$ to Lorentz scalars. The simplest such operators (i.e. those with lowest mass dimension) are $\mathbb{1}$, $\bar{q}q$ and $\mathrm{tr}F^2$. Hence

$$j_\mu(x)j_\nu(0) \;\sim\; C^{\mathbb{1}}_{\mu\nu}(x)\cdot\mathbb{1} + C^{\bar{q}q}_{\mu\nu}(x)\cdot\bar{q}q(0) + C^{F^2}_{\mu\nu}(x)\cdot\mathrm{tr}F^2(0)\,. \tag{7.24}$$

Now, in the massless limit QCD has a so-called **chiral symmetry**, i.e. the action is invariant under

$$q \to e^{i\gamma^5\alpha}q\,. \tag{7.25}$$

This follows from

$$\bar{q}\gamma^\mu q \to \Big(e^{i\gamma^5\alpha}q\Big)^\dagger \gamma^0\gamma^\mu e^{i\gamma^5\alpha}q = q^\dagger e^{-i\gamma^5\alpha}\gamma^0\gamma^\mu e^{i\gamma^5\alpha}q = \bar{q}\gamma^\mu q\,, \tag{7.26}$$

while the breaking by the mass term follows from

$$\bar{q}q \to q^\dagger e^{-i\gamma^5\alpha}\gamma^0 e^{i\gamma^5\alpha}q = \bar{q}e^{2i\gamma^5\alpha}q \neq \bar{q}q\,. \tag{7.27}$$

Thus, $\bar{q}q$ can appear in the OPE of $j_\mu j_\nu$ only if $m \neq 0$.

On dimensional grounds we then have

$$C^{\mathbb{1}}_{\mu\nu} \propto x^{-6}\;; \qquad C^{\bar{q}q}_{\mu\nu} \propto m x^{-2}\;; \qquad C^{F^2}_{\mu\nu} \propto x^{-2}\,, \tag{7.28}$$

where the symbol '$\propto$' means proportionality at small $|x|$. Fourier transforming and using current conservation one finds

$$\begin{aligned} -e^2\int d^4x\, e^{iqx} j_\mu(x)j_\nu(0) \;\;\sim\;\; & -ie^2(q^2\eta_{\mu\nu} - q_\mu q_\nu)\;\times \\ & \times\Big\{C^{\mathbb{1}}(q^2)\mathbb{1} + C^{\bar{q}q}(q^2)\, m\bar{q}q(0) + C^{F^2}(q^2)\,\mathrm{tr}F^2(0) + \cdots\Big\} \end{aligned} \tag{7.29}$$

with

$$C^{\mathbb{1}} \propto 1 \quad; \qquad C^{\bar{q}q} \propto C^{F^2} \propto \frac{1}{(q^2)^2} \tag{7.30}$$

at large q^2 by dimensional analysis. The exact prefactors of C^1 etc. are *defined* by the requirement that (7.29) holds.

Now we apply time ordering and appeal to the relation between the two-current-correlator and $\Pi_h^{\mu\nu}$. Factoring out the transverse tensor structure in q^μ, this gives

$$\Pi_h(q^2) \;\sim\; -e^2 \left[C^1(q^2) + C^{\bar{q}q}(q^2) m \langle \bar{q}q \rangle + C^{F^2}(q^2) \langle \mathrm{tr} F^2 \rangle + \cdots \right] . \tag{7.31}$$

We note that $\langle \bar{q}q \rangle \neq 0$ even in the limit $m \to 0$ since the chiral symmetry of QCD is spontaneously broken in the vacuum. In other words, while $\hat{H}$ is invariant, the vacuum $|0\rangle$ is not. This is not in contradiction with our previous use of the chiral symmetry argument since the OPE respects even symmetries which are spontaneously broken (cf. [13]).

At the level of principle, we are now almost done. Indeed, the (imaginary part of the) the l.h. side of (7.31) gives our desired cross section. The expectation values on the r.h. side are trivial (first term) or can be obtained from the lattice (or fitted the data). The coefficients C can, and this is essential, be obtained in perturbation theory:

Indeed, we can 'sandwich' (7.29) not with the proper vacuum, but with any state we like. For example, we can take the 'perturbative' vacuum of the free theory, i.e. the Fock space vacuum. In this vacuum, only the unit operator contributes on the r.h. side. On the l.h. side, no external lines except the two currents (i.e. the two photons) will appear after evaluation in terms of Feynman diagrams. This is illustrated in Fig. 40, where we also switched left and right and suppressed all prefactors.

Figure 40: Perturbative evaluation of the leading operator coefficient.

Similarly, we can take a one-quark or one-gluon state to sandwich (7.29), giving rise to perturbative expressions for $C^{\bar{q}q}$ and C^{F^2} respectively (cf. Fig. 41). Thus, once the non-perturbative information about the expectation values of the operators is provided, the calculation appears to be finished.

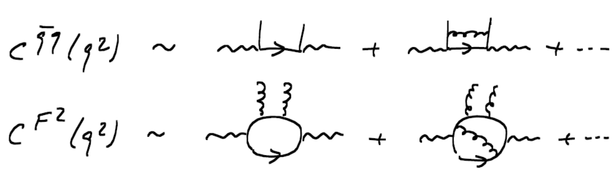


Figure 41: Perturbative evaluation of the subleading coefficients.

7.4 Analytic continuation

However, this is not the case for the following reason: We have seen in principle how the OPE separated perturbatively calculable quantities from the unavoidable non-perturbative input about the QCD vacuum. but we have not checked that the perturbative parts are not after all secretly sensitive to non-perturbatively small-momentum regions.

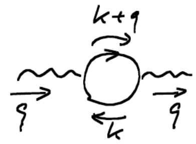


Figure 42: Leading perturbative diagram.

To check this, consider as an example the simplest diagram, cf. Fig. 42. Suppressing the numerator structure, the relevant integral reads

$$\int d^4k \, \frac{1}{(k^2+i\epsilon)((k+q)^2+i\epsilon)} \,. \tag{7.32}$$

Since $q^2>0$ in our case of interest, we can go to a frame where $q=(q_0,\vec{0})$, finding the following poles of the two propagators:

$$(1) \qquad k_0 \;=\; \pm\sqrt{\vec{k}^2-i\epsilon} \;\simeq\; \pm(|\vec{k}|-i\epsilon') \tag{7.33}$$

and, analogously,

$$(2) \qquad k_0+q_0 \;\simeq\; \pm(|\vec{k}|-i\epsilon') \quad \Rightarrow \quad k_0 \;\simeq\; -q_0 \pm (|\vec{k}|-i\epsilon')\,. \tag{7.34}$$

The position of the poles is illustrated in Fig. 43. It is apparent that, for $|\vec{k}|<q_0$, the integration contour is 'pinched' between the poles in such a way that Wick rotation to imaginary k_0 is impossible. In physical terms this means that the on-shell region of the quark propagators is important.

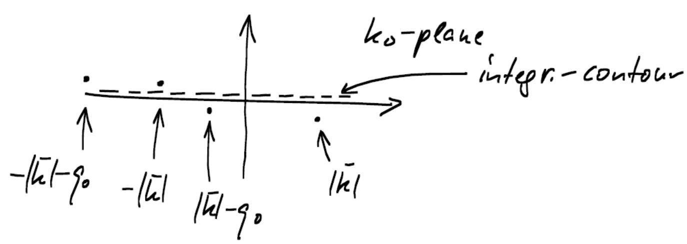


Figure 43: Position of poles in the complex k_0 plane.

By contrast, if $q^2<0$ we can choose a frame where $q=(0,\vec{q})$. Now, while the first pair of poles is as above, the second pair is at

$$k_0 \;\simeq\; \pm(|\vec{k}+\vec{q}|-i\epsilon')\,. \tag{7.35}$$

Hence, in this case the first and third quadrant are free of poles. Wick rotation is possible and the resulting euclidean integral is dominated by the energy scale $Q^2 = -q^2$. Perturbation theory is then justified as long as $Q^2 \gg \Lambda^2_{QCD}$.

So let us now assume that we can evaluate $\Pi_h(q^2)$ at q^2 large and negative using perturbative QCD together with the expectation values $\langle 0|\bar{q}q|0\rangle$, $\langle 0|\mathrm{tr}F^2|0\rangle$ etc. Does that help us to determine $\sigma_{e^+e^-}(q^2)$ at large and positive q^2? In fact, this can be achived using the concept of the spectral density from Sect. 6 of QFT I. Recall that

$$\langle T\varphi(x)\varphi(y)\rangle = \int_0^\infty dm'^2\, D_F(x-y, m'^2)\,\sigma(m'^2)\,, \tag{7.36}$$

or, in momentum space,

$$\frac{1}{q^2 - m_0^2 - \Pi(q^2)} = \int_0^\infty dm'^2\, \frac{1}{q^2 - m'^2 + i\epsilon}\,\sigma(m'^2)\,. \tag{7.37}$$

The qualitative form of $\sigma(q^2)$ is illustrated in Fig. 44. It implies that (7.37) has the pole structure shown in Fig. 45 (now ϵ is set to zero). In other words, the expression is analytical in q^2 except for poles and cuts on the real positive q^2-axis. They correspond to physical states that give a non-zero contribution if inserted between $\varphi(x)$ and $\varphi(y)$.

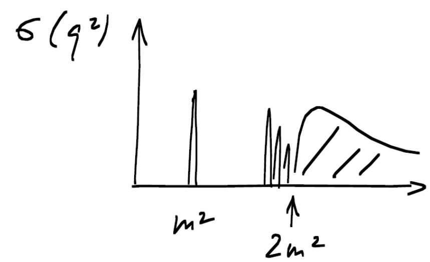


Figure 44: Spectral density of a scalar field theory.

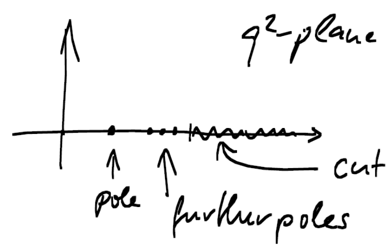


Figure 45: Pole structure of the Fourier transform of the full propagator.

The crucial point is now that all of this, including the derivation presented in QFT I, applies equally well to the time-ordered expectation value of two currents,

$$\langle 0|Tj_\mu(x)j_\nu(y)|0\rangle \qquad \text{and hence to} \qquad \Pi_h(q^2)\,. \tag{7.38}$$

With the understanding that $\Pi_h(q^2)$ is an analytic function of q^2 with poles and cuts only on the real positive axis (as in Fig. 45), we can make progress as follows:

Consider the so-called 'moments' (with $n \geq 1$) of our desired cross section,

$$I_n \equiv \int_0^\infty \frac{ds}{\pi} \frac{s\,\sigma_{e^=e^-}(s)}{(s+Q^2)^{n+1}} = -4\pi\alpha \int \frac{dq^2}{2\pi i} \cdot \frac{1}{(q^2+Q^2)^{n+1}} \cdot 2i\mathrm{Im}\,\Pi_h(q^2)\,. \tag{7.39}$$

This is somewhat reminiscent of the well-known concept of Mellin-transforms and Mellin-moments.

Now we note that, according (7.37), the cut or 'branch cut' (you are familiar with this feature of complex functions from the logarithm or the square root) can be thought of as a superposition of poles. For a pole, we have

$$\frac{1}{(q^2+i\Delta)-m'^2} = \left[\frac{1}{(q^2-i\Delta)-m'^2}\right]^*\,, \tag{7.40}$$

for real q^2 and Δ. In other words, approaching the real axis from above or below is related to complex conjugation. Hence, $2i\mathrm{Im}\Pi_h(q^2)$ is just the difference of the values of Π_h above and below the cut, the so-called 'disconituity'. We can write

$$I_n = -4\pi\alpha \int \frac{dq^2}{2\pi i} \cdot \frac{1}{(q^2+Q^2)^{n+1}}\ \text{Discontinuity}\{\Pi_h(q^2)\},. \tag{7.41}$$

Next, the integral of the discontinuity can is equivalent to the integral above the cut minus the integral below the cut. The latter contour can be completed can be closed at infinity (cf. Fig. 46) to give

$$I_n = -4\pi\alpha \oint \frac{dq^2}{2\pi i} \cdot \frac{1}{(q^2+Q^2)^{n+1}} \cdot \Pi_h(q^2) = -4\pi\alpha\,\mathrm{Res.}\left\{\frac{\Pi_h(q^2)}{(q^2+Q^2)^{n+1}}\right\}_{q^2=-Q^2}\,. \tag{7.42}$$

Expanding Π_h in a Taylor series,

$$\Pi_h(q^2) = \Pi_h(-Q^2) + (q^2+Q^2)\frac{d}{dq^2}\Pi_h(q^2)\Big|_{q^2=-Q^2} + \cdots\,, \tag{7.43}$$

we see that I_n is only sensitive to the nth term:

$$I_n = -\frac{4\pi\alpha}{n!}\left(\frac{d}{dq^2}\right)^n \Pi_h(q^2)\Big|_{q^2=-Q^2}\,. \tag{7.44}$$

Crucially, in this last expression Π_h appears only evaluated at negative q^2, i.e. in the regime where we can Wick rotate and where large Euclidean momentum corresponds to small euclidean distance (between the two currents). Hence, we apply (7.31) and have a clean, first principles prescription for calculating the moments of the cross section $\sigma_{e^+e^-}$. The structure of the result is

$$I_n = \int_0^\infty ds\,\frac{s\sigma_{e^+e^-}(s)}{(s+Q^2)^{n+1}} \quad = \quad \frac{4\pi\alpha}{n(Q^2)^n} \cdot \Sigma_f Q_f^2 + \mathcal{O}(\alpha_s(Q^2)) + \mathcal{O}\left(\frac{1}{(Q^2)^3}\right)\,. \tag{7.45}$$

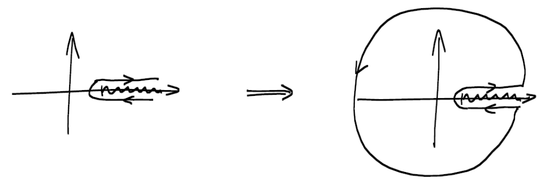


Figure 46: Closing the integral around the cut at infinity.

The first two terms on the r.h. side are just the moments of the LO result and of the α_s corrections. The last term are so called power corrections associated with higher terms in the OPE and sensitive to nonperturbative physics. Our last result is also known as the ITEP sum rule (Voloshin/Vainshtein/Novikov/Shifman '78). Extracting $\sigma_{e^+e^-}(s)$ from the moments is still non-trivial, but discussing this goes beyond what we can do here.

8 Parton distribution functions (PDFs)

8.1 Factorization of hard processes in hadronic collisions

In the last section, we learned that, up to corrections of relative size $\sim 1/(Q^2)^2$, the cross section $\sigma(e^+e^- \to \mathit{hadrons})$ is perturbatively calculable in spite of the relevant physics obviously being in part hadronic or 'soft'. The key is the 'hard' scale Q^2 and the 'inclusiveness' of the observable in question. (Indeed, the outcome would be much less favorable if we asked, e.g., for the number ratio between pions and ρ-mesons in the final state of this process.)

Our example was exceptionally nice in the sense that the suppression of non-perturbative physics was by $1/(Q^2)^2$, but more generally the perturbative calculability up to $1/Q^2$ corrections is expected for sufficiently inclusive observables with non-hadronic initial state.

However, sometimes one is interested in hard processes with hadronic initital state. Just at this very moment, with the proton-proton-collider LHC running and searching or new physics at the highest energy scales, this is a particularly timely issue to discuss.

The classical example is the Drell-Yan process, i.e. the production of an e^+e^- pair in a hadron-hadron collision. The LO diagram at the so-called partonic level is shown in Fig. 47. It may be thought of as the 'hard' part of the more complete process shown in Fig. 48. Here, we collide two protons but focus only on two quarks taken from each of the protons. Those get involved in a hard (governed by an energy scale $Q \gg \Lambda_{QCD}$) process for which we envision a perturbative order-by-order calculation using the smallness of $\alpha_s(Q^2)$ and possible further small couplings.

The key statement allowing us to draw Fig. 48 is that the cross section 'factorizes',

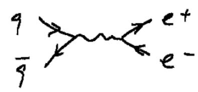


Figure 47: Leading partonic diagram for Drell-Yan process.

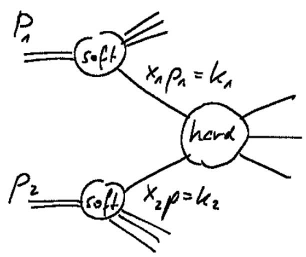


Figure 48: Pictorial representation of the factorization assumption.

i.e., it can be given as a convolution of PDFs f_i and a partonic cross section:

$$\sigma = \sum_{i,j} \int_0^1 dx_1 \int_0^1 dx_2\, f_i(x_1,\mu^2)\, f_j(x_2,\mu^2)\, \hat{\sigma}_{ij}(k_1,k_2,\alpha_s(\mu^2),Q^2,\mu^2)\,. \tag{8.1}$$

The PDF quantifies, e.g., the probability for finding, in a (highly relativistic) proton with momentum p_1, a quark or gluon with certain spin/polarization (specified by the index i) and with momentum $k_1 = x_1 p_1$ (where $x_1 \in (0,1)$). The physical picture is that, at very high resolution specified by $1/\mu \ll 1/\Lambda_{QCD} \sim R_{proton}$, a proton beam corresponds to a beam of quarks and gluons with certain a certain distribution of momenta. The PDF provides the translation between the respective luminosities. Clearly, we could modify (8.1) to make both σ and $\hat{\sigma}$ differential in some additonal final state variables (in addition to Q) – the assumption is that factorizations still holds. What is more, the final state may even contain hadronic jets – we still claim that (under certain very reasonable technical assumptions, such as high $p_\perp$) these do not interact with the so-called 'proton-remnants' and factorization continues to hold. Crucially, the PDFs are a feature of the hadron, independent of $\hat{\sigma}$.

Factorization can be promoted to a theorem an 'proven' (at a physics, not math level of rigor). The relevant authors in this context are Collins, Soper, an Sterman (see also [22, 24]). It holds only at leading order in $1/Q^2$. (This is sometimes referred to as 'leading twist', with higher terms – of the type discussed in the context of the OPE, called 'higher-twist' terms. The terminology derives from the higher angular momentum representations generally associated with higher-dimension operators.)

8.2 Alteralli-Parisi or DGLAP evolution

The appearance of the energy scale μ as an argument of the f_i is non-obvious. To understand this, recall that we calculate $\hat{\sigma}$ in dimensional regularization, which is used for both the UV and IR divergences. (One could use two separate ϵ's and separate μ's for these different types of divergences, but will not bother.) Now, while the IR (more precisely soft and collinear) divergences cancelled among themselves in $e^+e^- \to \textit{hadrons}$, here they cancel only if the full QFT description of the incoming hadrons as bound state of quarks and gluons is taken into acccount. But in attempting to do work this out one necessarily leaves the realm of perturbative QCD. Thus, the solution is to 'make clean cut' (actually, to interpret μ as providing such a cut) between non-perturbative information about the hadron (f_i) and the hard process ($\hat{\sigma}$).

Some of the relevant diagrams in the Drell-Yan process are shown in Fig. 49. Focus on the second diagram, where a one of the incoming quarks radiates a gluon before the hard process. Clearly, one can not objectively say whether this radiation is part of the proton dynamics or part of the hard scattering. Subjectively, this can be determined after the so-called factorization scale μ is introduced in the course of dimensional regularization.[20] Similarly, the exchange of the a soft or collinear gluon in the third diagram can be thought of occurring before or after the partons 'leave the proton'. Again, the in principle arbirary scale μ provides the cut.

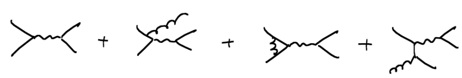


Figure 49: Some higher-order diagrams relevant for the DY process.

At the end, with the same argument used in UV renormalization, μ is chosen to be of the order of Q. Thus the relevant scale at which the PDFs are evaluated is the scale of the hard process. The PDFs can be either measured or extracted (using a slightly more compicated version of the OPE) from lattice data. 'Measuring' means extracting them from a hard process where both $\hat{\sigma}$ and is calculated and σ is measured with sufficient precision.

At a very rough level, one may view the PDF's as being a 'IR input quantities' just like the coupling constant $g(\mu^2)$ is a 'UV input quantity'. For the latter, we explained that its μ-dependence must precisely cancel the μ-dependence introduced by the UV-divergences loop-corrections. This allowed us to derive an RGE for $g(\mu^2)$. Analogously, the μ-dependence of $f_i(x,\mu)$ cancels the μ-dependence of $\hat{\sigma}$ introduced by IR divergences related to the initial state. This leads to 'kind-of' an RGE for the PDFs, known as Altarelli-Parisi or Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations.

For example, the μ^2-dependence of the quark-distribution $f_q(x,\mu^2) \equiv q(x,\mu^2)$ has a piece depending on the quark distribution at higher x:

$$\frac{dq(x,\mu^2)}{d\,\ln\mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z}\,P(z)\,q(y,\mu^2) \qquad \text{where} \qquad x \equiv z\cdot y\,. \tag{8.2}$$

Here $P(z)$ is a perturbatively calculable and in general highly singular function (distribution) and the above equation can be interpreted as describing the radiation of a gluon

[20] Actually, but seeing this explicitly requires more work, this scale can be viewed as separating partons with $|k_\perp| \ll \mu$ from those with $|k_\perp| \gg \mu$, only the latter being part of the hard process.

(with momentum fraction $1 - z$) by an incoming quark before it engages in the hard scattering, cf. Fig. 50.

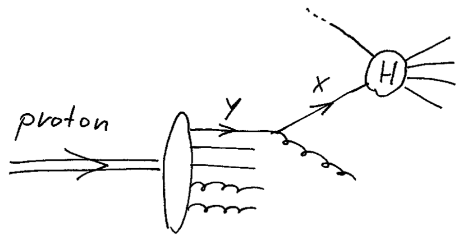


Figure 50: A diagram relevant to the evolution of the quark distribution.

The equation above is an integro-differential equation and the structure on the r.h. side is known as a convolution and often denoted by $\otimes$, i.e.

$$\frac{dq}{d\ln\mu^2} = \frac{\alpha_s}{2\pi}\, P \otimes q\,. \tag{8.3}$$

However, it is immediately clear that this can be only part of the story and that an incoming gluon can similarly contribute by 'splitting' into two quarks. Hence, the r.h. side should also have a term depending on $f_g \equiv g$. Furthermore, the gluon distribution has its own, very similar evolution equation. In total, one has

$$\frac{d}{d\ln\mu^2}\begin{pmatrix} q \\ g \end{pmatrix} = \frac{\alpha_s}{2\pi}\begin{pmatrix} P_{qq} & P_{qg} \\ P_{gq} & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} q \\ g \end{pmatrix}\,. \tag{8.4}$$

Our explicit example was clearly just the P_{qq} piece of this more complete expression.

8.3 Deep inelastic scattering (DIS)

Historically, all of this was first developed and tested in great detail in 'deep inelastic scattering' (cf. Fig. 51 and 52). This is the scattering of an electron off a proton. The interesting part for our purposes is the subprocess in which a virtual photon (γ^*) scatters off a parton – at LO always a quark. The situation (including the factorization proof) is simpler than in the hadron-hadron collisions since only one side involves a hadron. In particular, OPE methods can be used.

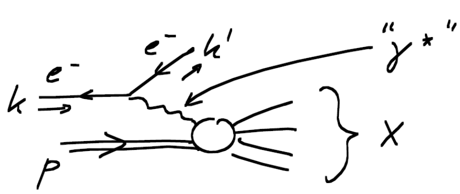


Figure 51: Deep inelastic scattering.

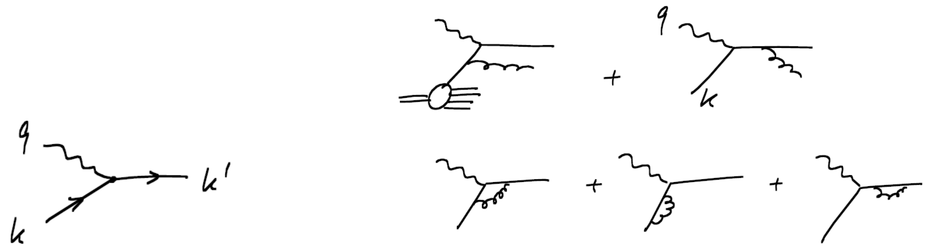


Figure 52: The leading leading partonic process in DIS (left) and the real and virtual corrections (right).

For a more detailed discussion, including at least a partial derivation of the DGLAP equations and of the explicit 'splitting functions' P_{ij}, see the handwritten notes of my older QFT II course.

9 Anomalies

An anomaly is the breakdown of a symmetry of a classical system after quantization.

9.1 Chiral anomaly – functional integral approach

Consider a model with a massless Dirac fermion, charged under a $U(1)$ gauge symmetry:

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \bar{\psi} i \not{D} \psi \,. \tag{9.1}$$

In addition to the gauged $U(1)$ symmetry $\psi \to \exp(i\alpha)\psi$, this model possesses a global, 'chiral' $U(1)$ symmetry, $\psi \to \exp(i\alpha\gamma_5)\psi$. This symmetry will turn out to be anomalous.

To see this, let us analyse the transformation properties of the path intergral measure $D\psi\, D\bar{\psi}$ under the change of variables (i.e. under the field-redefinition)

$$\psi(x) \;\to\; \psi'(x) = \int d^4y\, M(x,y)\psi(y)\,. \tag{9.2}$$

From our discussion of the fermionic path integral, especially the Gaussian example, it easily follows that

$$D\psi'\, D\bar{\psi}' \;=\; (\det M)^{-1} (\det \bar{M})^{-1} D\psi\, D\bar{\psi}\,. \tag{9.3}$$

Here the appearance of the inverse determinant (rather than of the determinant itself) can be easily understood by recalling that the fermionic path integral gives the determinant rather than the inverse determinant of the matrix in the exponential. Furthermore, the matrix

$$\bar{M} \equiv \gamma^0 M^\dagger \gamma^0 \tag{9.4}$$

appears due to the fact that we use $\bar{\psi}$ rather than $\psi^\dagger$ as our second variable.

Our interest will be specifically in the two field redefinitions

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \qquad \text{and} \qquad \psi(x) \to e^{i\alpha(x)\gamma^5}\psi(x)\,. \tag{9.5}$$

The corresponding matrices in the above formula can be given as

$$M(x,y) = U(x)\,\delta^4(x-y) \qquad \text{with} \qquad U(x) = \left\{ \begin{array}{c} e^{i\alpha(x)} \\ e^{i\alpha(x)\gamma^5} \end{array} \right. . \tag{9.6}$$

Inserting this form in (9.3), we encounter the expression

$$U\bar{U} = U\gamma^0 U^\dagger \gamma^0 = \mathbb{1} \tag{9.7}$$

in the first and

$$U\bar{U} = U\gamma^0 U^\dagger \gamma^0 = e^{i\alpha\gamma^5}\gamma^0 e^{-i\alpha\gamma^5}\gamma^0 = U^2 \tag{9.8}$$

in the second case. Hence for the first, 'vector-like' $U(1)$ transformation the measure is invariant.[21] By contrast, the measure is not invariant under the chiral transformation. Instead,

$$D\psi\, D\bar{\psi} \;\to\; (\det M)^{-2} D\psi\, D\bar{\psi}\,. \tag{9.9}$$

Next, we focus on infinitesimal ($\alpha \ll 1$) chiral rotations, such that

$$M(x,y) \simeq \mathbb{1} + i\alpha(x)\gamma^5\delta^4(x-y)\,. \tag{9.10}$$

We can then write

$$\begin{aligned} \det M &= \exp \ln \det M = \exp \operatorname{tr} \ln M = \exp \operatorname{tr} \ln(\mathbb{1} + (M - \mathbb{1})) \\ &\simeq \exp \operatorname{tr}(M - \mathbb{1}) = \exp i \int d^4x\, \alpha(x) \operatorname{tr}(\gamma^5)\delta(x-x)\,. \end{aligned} \tag{9.11}$$

Introducing the 'anonaly function' $\mathcal{A}$, defined by

$$-\frac{1}{2}\mathcal{A}(x) = \operatorname{tr}(\gamma^5)\delta(x-x)\,, \tag{9.12}$$

we find the transformation of the measure

$$D\psi\, D\bar{\psi} \;\to\; \exp\left(i\int d^4x\, \alpha(x)\mathcal{A}(x)\right) D\psi\, D\bar{\psi}\,. \tag{9.13}$$

We see that $\mathcal{A}$ is formally given by a product of zero ($\operatorname{tr}(\gamma^5)$) and infinity ($\delta^4(0)$), such that a regularized calculation is required. Such a calculation will be supplied momentarily.

Let us however first consider what the implication of a possibly non-zero $\mathcal{A}$ would be. Indeed, our considerations have proven the following identity:

$$\begin{aligned} &\int D\psi'\, D\bar{\psi}'\, DA_\mu \mathcal{O}(A_\mu) \exp\left\{ i\int \mathcal{L}(\psi', A_\mu)\right\} \\ &\qquad = \int D\psi\, D\bar{\psi}\, DA_\mu \mathcal{O}(A_\mu) \exp\left\{ i\int \left(\mathcal{L}(e^{i\alpha\gamma^5}\psi, A_\mu) + \alpha(x)\mathcal{A}(x)\right)\right\}\,, \end{aligned} \tag{9.14}$$

[21] The expression vector-like originates in the current being $\sim \bar{\psi}\gamma^\mu\psi$, i.e. a vector, as opposed to the pseudovector $\sim \bar{\psi}\gamma^5\gamma^\mu\psi$ appearing in the chiral case.

where $\mathcal{O}(A_\mu)$ is some observable depending on the gauge field. This identity provides a very clear understanding of the relation between classical and quantum symmetry: The classical symmetry condition is simply that the factor $e^{i\alpha\gamma^5}$ should drop out of $\mathcal{L}$ on the r.h. side. In our example this is indeed the case for constant α. If the path integral measure were invariant, i.e. if $\mathcal{A}$ were zero, this would imply that the classical symmetry can be promoted to the path-integral level, i.e. it becomes a symmetry of the quantum theory. In this case the path integral formula would simply be invariant under the transition from ψ to ψ'.

If, however, $\mathcal{A}$ is non-vanishing, this invariance is broken. Indeed, the term $\sim \alpha\mathcal{A}$ on the r.h. side can be viewed as an extra or 'anomalous' transformation of $\mathcal{L}$ which arises because $\mathcal{L}$ is placed under the path integral:

$$\mathcal{L}(\psi, A_\mu) \to \mathcal{L}(e^{i\alpha\gamma^5}\psi, A_\mu) + \alpha(x)\mathcal{A}(x) \quad \text{for } \psi \to e^{i\alpha\gamma^5}\psi \text{ under the path integral}\,. \tag{9.15}$$

Now calculate $\mathcal{A}$ by writing it in the following regularized form:

$$\mathcal{A}(x) = -2\,\mathrm{tr}\left(\gamma^5 f(\not{D}_x^2/m^2)\right)\, \delta^4(x-y)\big|_{y=x}\,. \tag{9.16}$$

Here $f(s)$ is some smooth function (e.g. e^{-s} or $1/(1+s)$) with

$$f(0) = 1 \qquad \text{and} \qquad \lim_{s\to\infty} f(s) = 0\,. \tag{9.17}$$

Here m provides the UV energy scale associated with the regularization. Indeed, in the limit $m \to \infty$ we can replace f by $f(0) = 1$ by assumption, such that formally the original expression is recovered. Furthermore, one might hope that the expression with finite m is finite since the typical fast variation of δ^4 near zero leads to large values of $\not{D}^2$, which are suppressed due to the behaviour of f. This, of course, has to be checked in the explicit calculation.

A crucial point is that we use $\not{D}$ rather than $\not{\partial}$. This is necessary since, originally, everything derived from the operator M acting on the space of spinorial functions ψ. These are subject to the gauged vector-like $U(1)$ invariance which must be maintained in the regularization.

We now calculate $\mathcal{A}$ by first going to Fourier space,

$$\mathcal{A}(x) = -2\int \frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left(\gamma^5 f(\not{D}_x^2/m^2)\right)\, e^{ik(x-y)}\Big|_{y=x}\,. \tag{9.18}$$

Next, thinking of $f(\not{D}_x^2/m^2)$ in terms of a Taylor expansion and using

$$\not{D}_x\, e^{ik(x-y)}(\cdots) \;=\; e^{ik(x-y)}(i\not{k} + \not{D}_x)\,(\cdots)\,, \tag{9.19}$$

we find

$$\begin{aligned} \mathcal{A}(x) &= -2\int \frac{d^4k}{(2\pi)^4}\, e^{ik(x-y)}\mathrm{tr}\left(\gamma^5 f\left((i\not{k}+\not{D}_x)^2/m^2\right)\right)\Big|_{y=x} \\ &= -2\int_k \mathrm{tr}\left(\gamma^5 f\left((i\not{k}+\not{D}_x)^2/m^2\right)\right)\,. \end{aligned} \tag{9.20}$$

In the last expression, we were able to set $y = x$ without encountering a divergence. Note also that ∂_x contained in $\not{D}_x$ remains non-trivial since it can act on $A_\mu(x)$ factors which are implicitly present. We can however drop the index x since this is now the only space-time variable.

Let us now rescale $k \to km$ and subsequently use

$$(i\not{k} + \not{D}/m)^2 = -k^2 + \frac{2ik_\mu D^\mu}{m} + \left(\frac{\not{D}}{m}\right)^2 , \tag{9.21}$$

such that

$$\mathcal{A}(x) = -m^4 \int_k \operatorname{tr}\left(\gamma^5 f\left(-k^2 + \frac{2ik\cdot D}{m} + \left(\frac{\not{D}}{m}\right)^2\right)\right) . \tag{9.22}$$

Expanding f in powers of $1/m$, one can make the following crucial observation: If more than 4 factors of $1/m$ appear, one gets zero for $m \to \infty$. If fewer than 4 factors of $\not{D}$ appear, one gets zero because of the trace with γ^5. Thus, a contribution can only come from the second power of the term $(\not{D}/m)^2$:

$$\mathcal{A}(x) = -\int_k f''(-k^2)\operatorname{tr}(\gamma^5\not{D}^4) . \tag{9.23}$$

The k-integral factors off and, after Wick rotation $k^0 \to ik^0_E$, gives

$$\begin{aligned} i\int d^4k_E f''(k_E^2) &= i\int_0^\infty 2\pi^2k^3dk\, f''(k^2) = i\pi^2\int_0^\infty s\,ds\, f''(s) \\ &= -i\pi^2\int ds f'(s) = -i\pi^2 f(s)\Big|_0^\infty = i\pi^2 . \end{aligned} \tag{9.24}$$

Here we also used that $f'(s)$ vanishes sufficiently fast at infinity, which is however easy to realize.

Finally, we have

$$\begin{aligned} \not{D}^2 &= D_\mu D_\nu\gamma^\mu\gamma^\nu = D_\mu D_\nu\left(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu]\right) \\ &= D_\mu D^\mu + \frac{1}{4}[D_\mu, D_\nu]\cdot[\gamma^\mu,\gamma^\nu] = D_\mu D^\mu + \frac{i}{4}F_{\mu\nu}[\gamma^\mu,\gamma^\nu] . \end{aligned} \tag{9.25}$$

Thus,

$$\operatorname{tr}(\gamma^5\not{D}^4) = -\frac{1}{16}F_{\mu\nu}F_{\rho\sigma}\operatorname{tr}\left(\gamma^5[\gamma^\mu,\gamma^\nu][\gamma^\rho,\gamma^\sigma]\right) . \tag{9.26}$$

Now, the commutation relations of γ matrices immediately imply that the trace is $\sim \epsilon^{\mu\nu\rho\sigma}$. The prefactor is easily fixed by also recalling that in our connventions $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and working out the case of $\{\mu,\nu,\rho,\sigma\} = \{0,1,2,3\}$. One finds

$$\operatorname{tr}(\gamma^5\not{D}^4) = -\frac{1}{16}F_{\mu\nu}F_{\rho\sigma}\, i\cdot 16\,\epsilon^{\mu\nu\rho\sigma} \tag{9.27}$$

and hence

$$\mathcal{A}(x) = \frac{-i\pi^2}{(2\pi)^4} \cdot i \cdot F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} \; = \; \frac{1}{16\pi^2} \, F\tilde{F} \,. \tag{9.28}$$

This is a fundamental result. Our method of derivation is due to Fujikawa. The anomaly we obtained is also known as the Adler-Bell-Jackiw anomaly and their (earlier) Feynman-diagramatic method of derivation will be discussed later on.

9.2 Anomalous current non-conservation

The anomaly implies a violation of the of the conservation of the Noether current j_5^μ associated with the global symmetry $\psi \to e^{i\alpha\gamma^5}\psi$.

The two symmetries $\psi \to e^{i\alpha}\psi$ and $\psi \to e^{i\alpha\gamma^5}\psi$ have associated Noether currents

$$j^\mu = \bar{\psi}\gamma^\mu\psi \qquad \text{and} \qquad j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi \,. \tag{9.29}$$

Both conservation laws are proven in complete analogy, following the general derivation of the Noether theorem.

Now consider the transformation $\psi \to e^{i\alpha(x)\gamma^5}\psi$ where, crucially, we allow for a non-constant transformation parameter $\alpha = \alpha(x)$. Let us consider the (in general non-vanishing) variation of the action induced by this transformation:

$$\begin{aligned} \delta S \;&=\; \delta \int d^4x \, \bar{\psi} i \partial\!\!\!/ \psi = \int d^4x \left[\overline{(\psi + i\alpha\gamma^5\psi)} i\partial\!\!\!/ (\psi + i\alpha\gamma^5\psi) - \bar{\psi} i\partial\!\!\!/ \psi \right] \\ &=\; \int d^4x \, \bar{\psi} i \gamma^\mu (\partial_\mu \alpha) i \gamma^5 \psi = -\int d^4x j_5^\mu \, \partial_\mu \alpha = \int d^4x \, (\partial_\mu j_5^\mu) \, \alpha(x) \,. \end{aligned} \tag{9.30}$$

As a (very important!) side remark, the above provides an elegant alternative derivation of the Noether theorem. Indeed, if our underlying field configuration obeys the EOMs, then the action must be stationary. Thus, $\delta S = 0$ irrespective of whether the field transformation is a symmetry. In other words, the r.h. side is zero for any function $\alpha(x)$, which clearly implies (classical) current conservation, $\partial_\mu j_5^\mu = 0$.

Now we apply the field transformation discussed above to the whole path integral. While we now clearly must allow for field configurations not satisfying the EOMs, the result is still zero since the transformation is just a change of variables in an integral:

$$0 = \delta \int D\psi D\bar{\psi} \, e^{iS} = \int D\psi D\bar{\psi} \, e^{iS + i\int d^4x \left[\alpha\mathcal{A} + (\partial_\mu j_5^\mu)\alpha\right]} - \int D\psi D\bar{\psi} \, e^{iS} \,. \tag{9.31}$$

Expanding in α, we see that the relation

$$\partial_\mu j_5^\mu = -\frac{1}{16\pi^2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} \tag{9.32}$$

holds under the path integral, i.e. for operators in the QFT. This is know as the anomalous non-conservation of the so-called axial-vector current (the name derives from the fact that $\bar{\psi}\gamma^\mu\gamma^5\psi$ is an axial vector). It generalizes in several ways:

First, we recall the relation between (general, left-handed and right handed) Dirac spinors and Weyl spinors, with the latter characterized by an index 'w':

$$\psi = \begin{pmatrix} \psi_w \\ \bar{\chi}_w \end{pmatrix} \quad , \qquad \psi_L = \begin{pmatrix} \psi_w \\ 0 \end{pmatrix} \quad , \qquad \psi_R = \begin{pmatrix} 0 \\ \bar{\chi}_w \end{pmatrix} . \tag{9.33}$$

We introduce the operation of 'charge conjugation' (characterized by an upper index 'c'), which exchanges the two Weyl spinors contained in a Dirac spinor:

$$\psi^c = \begin{pmatrix} \chi_w \\ \bar{\psi}_w \end{pmatrix} \quad , \qquad (\psi_L)^c = \begin{pmatrix} 0 \\ \bar{\psi}_w \end{pmatrix} \quad , \qquad (\psi_R)^c = \begin{pmatrix} \chi_w \\ 0 \end{pmatrix} . \tag{9.34}$$

In particular, we note that the charge-conjugate of a l.h. (r.h.) spinor is r.h. (l.h.) spinor. Also, the name 'charge conjugation' is clearly justified since any phase rotation of ψ corresponds to an opposite phase rotation of ψ^c.[22]

Now, any Dirac spinor can be decomposed in its l.h. and r.h. parts,

$$\psi = \frac{1-\gamma^5}{2}\psi + \frac{1+\gamma^5}{2}\psi \equiv \psi_L + \psi_R \, . \tag{9.35}$$

Furthermore, the physics described by a r.h. spinor ψ_R can equivalently be described by the l.h. spinor $(\psi_R)^c$. This is just a change of variables, like interchanging ϕ and ϕ^* in the scalar case. In particular, by observing that

$$\bar{\psi} i \not{D} \psi = \bar{\psi}_L i \not{D} \psi_L + \overline{(\psi_R)^c} i \not{D} (\psi_R)^c \tag{9.36}$$

we are able to describe a theory with a Dirac spinor ψ in terms of two l.h. spinor fields ψ_L and $(\psi_R)^c$.

From the above definitions, it easy to see how our l.h. spinors transform under the two $U(1)$ symmetries (the vector-like, V, and the chiral or axial-vector-like, A):

	ψ_L	$(\psi_R)^c$
$U(1)$-V charge	$+1$	-1
$U(1)$-A charge	-1	-1

. (9.37)

Now comes the crucial point: In considering the anomaly or anomalous current-non-conservation, it was not important to actually carry out the path integral over A_μ. However, with A_μ being non-dynamical (i.e. a classical background field), our model is just the sum of two completely independent theories with the l.h. spinor fields ψ_L and $(\psi_R)^c$. Thus, the anomaly (under the A-transformation, as before) must be the sum of the two anomalies induced by two path intergal measures, i.e.

$$\mathcal{A} = \mathcal{A}_L + \mathcal{A}_{R^c} \, . \tag{9.38}$$

[22]The operation of charge conjugation can also be defined by $\psi^c \equiv C\bar{\psi}^T$, with the matric C obeying the relation $C^{-1}\gamma_\mu C = -\gamma_\mu^T$. This relation ensures that ψ^c is again a Dirac spinor. An explicit representation is given by $C = i\gamma_2\gamma_0$.

Moreover, the form the $U(1)_A$ perspective the two fields are equivalent. From the $U(1)_V$ perspective their charges are opposite, but since the gauge field enters quadratically this sign can not enter the anomaly. Thus,

$$\mathcal{A} = \mathcal{A}_L + \mathcal{A}_{R^c} = 2\mathcal{A}_L \,. \tag{9.39}$$

The axial-vector current receives contributions from both fields,

$$j_5^\mu = j_L^\mu + j_{R^c}^\mu \tag{9.40}$$

and, by the same logic as above, its anomalous non-conservation must be due to two equal contributions:

$$\partial_\mu j_5^\mu = \partial_\mu j_L^\mu + \partial_\mu j_{R^c}^\mu = -\frac{1}{16\pi^2} F\tilde{F} = -\frac{1}{32\pi^2} F\tilde{F}\Big|_{\text{from } \psi_L} - \frac{1}{32\pi^2} F\tilde{F}\Big|_{\text{from } (\psi_R)^c} . \tag{9.41}$$

Our final result is that, in a theory with a single l.h. fermion (or, equivalently, a single Weyl fermion), the anomalous current non-conservation reads

$$\partial_\mu j^\mu = -\frac{1}{32\pi^2} F\tilde{F} \,. \tag{9.42}$$

Here we not have to distinguish between V and A current since (up to possible signs and normalization conventions) there is only one $U(1)$ current in such a theory. In particular, a $U(1)$ (quantum) gauge theory with a single Weyl fermion is inconsistent – one can not gauge a transformation that is not a symmetry. One simple more technical argument for this is as follows: For a theory being renormalizable 'by power counting', boson propagators have to behave as $1/p^2$. Thus, gauge invariance is needed to bring the gauge-boson propagator to the form $\eta^{\mu\nu}/p^2$. If that fails, renormalizability is lost.

9.3 Non-abelian generalization

The second important generalization is to the non-abelian case. As above, we work with l.h. fermions only, allowing for as many of them as desired:

$$\psi_L \to \psi_{L\,i} \,. \tag{9.43}$$

Here i runs over the basis of the relevant representation of some non-abelian symmetry group (e.g. $SU(3)\times SU(2)\times U(1)$ in the case of the Standard Model). We emphasize that this group can contain many 'simple' factors like $SU(N)$ and many $U(1)$s. No matter how large and complicated the group, we will treat all its generators on equal footing and call them T_a. Familiar examples include $T_a = \lambda_a/2$ with $a = 1, \cdots, 8$ for $SU(3)$ or $T_a = \mathbb{1}$ for a $U(1)$ under which all fermions transform univerally, $\psi_{L\,i} \to e^{i\alpha}\psi_{L\,i}$.

Now pick out one specific generator, say T_a for some fixed a, and consider the $U(1)$ transformation generated by this T_a. We calculate its anomaly exactly as before, but for all fermions at once. Crucially, our starting point was a phase rotation with γ^5, which can now be different for the different fermions involved. This information is encoded in T_a:

$$\left[\psi \to e^{i\alpha\gamma^5}\psi\right] \quad \Rightarrow \quad \left[\psi_i \to \left(e^{i\alpha\gamma^5 T^a}\right)_i^{\ j}\psi_j\right]. \tag{9.44}$$

Later, we regularized $\mathrm{tr}(\gamma^5)\delta(0)$ by introducing an appropriate function of $\not{D}$. The analogous expression now reads

$$\mathrm{tr}\left[\gamma^5 T_a f(\not{D}^2/m^2)\right] \quad \text{with} \quad D_\mu = \partial_\mu + iA^b_\mu T_b\,. \tag{9.45}$$

It is an easy excercise to follow the algebra from that point onward to discover that eventually, in addtion to the Dirac traces to be worked out as before, a trace of three generators is left: The first is the T_a explicitly introduced by above, the other two come from the $\not{D}$-algebra and are contracted with the two field-strength tensors:

$$\mathrm{tr}[T_a T_b T_c] F^b_{\mu\nu}\, F^c_{\rho\sigma}\,. \tag{9.46}$$

We leave it to the reader to work through this carefully and to reconsider every step. The outcome is that all goes through as before, giving

$$\partial_\mu j^\mu_a = -\frac{1}{32\pi^2}\, D_{abc}\epsilon^{\mu\nu\rho\sigma} F^b_{\mu\nu} F^c_{\rho\sigma} \quad \text{with} \quad D_{abc} \equiv \frac{1}{2}\mathrm{tr}[T_a\{T_b, T_c\}]\,. \tag{9.47}$$

Here we used the symmetry w.r.t. the interchange of $F^b\tilde{F}^c$ to introduce the anti-commutator $\{T_b, T_c\}$. The D_{abc}'s form a tensor which characterizes the representation of the symmetry group one considers (be careful - normalization conventions vary). As opposed to the f^{abc}, it is totally symmetric (this is easy to check using $\mathrm{tr}(AB) = \mathrm{tr}(BA)$).

Much more could be said, in particular, anomalies have an interesting topological interpretation which is worth pursuing. Unfortunately, we have to leave it to the reader to explore this further, see e.g. [25, 26].

An important application of the above is the Standard Model. Writing all fields as left-handed spinors, the transformation properties can be summarized as follows (this shorthand notation has been explained in QFT I):

$$\begin{array}{c|ccc} & SU(3) & SU(2) & U(1) \\ \hline \begin{pmatrix} u_L \\ d_L \end{pmatrix} & 3 & 2 & \frac{1}{6} \\ (u_R)^c & \bar{3} & 1 & -\frac{2}{3} \\ (d_R)^c & \bar{3} & 1 & \frac{1}{3} \\ \begin{pmatrix} \nu_e \\ e_L \end{pmatrix} & 1 & 2 & -\frac{1}{2} \\ e_R & \bar{1} & 1 & 1 \end{array}\,. \tag{9.48}$$

Note that there are different conventions concerning the normalization (an extra factor 2 is sometimes introduced) and sign of the $U(1) \equiv U(1)_Y$ charges. Our convention is such that the electric charge obeys $Q = T_3 + Y$, with T_3 being the diagonal $SU(2)$ generator. Using the above table, you should be able to prove that all anomalies associated with gauge group genrators vanish, making the Standard Model consistent in highly non-trivial way.

As a final remark, one can of course also pick a symmetry generator which is not gauged (examples are the U(1) transformations defining baryon (B) and lepton number (L) in the Standard Model). The anomaly of the corresponding current follows as above, from evaluating the trace together with all pairs of gauged generators. Now, however, a non-zero anomaly does not imply the inconsistency of the model. Indeed, in the Standard Model $B + L$ is anomalous while $B - L$ is conserved also at the quantum level.

9.4 The chiral anomaly in the Feynman-diagram approach – 2d example

Consider QED in $d = 2$, i.e.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} i \not{D} \psi \tag{9.49}$$

with

$$\mu, \nu \in \{0, 1\} \; ; \quad \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \; ; \quad \gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; ; \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} . \tag{9.50}$$

As always in even dimensions, chiral spinors can be defined using γ^5, in this case

$$\gamma^5 = \gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{9.51}$$

and the Dirac spinor above contains two one-component Weyl spinors:

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} . \tag{9.52}$$

The V and A currents are precisely as in 4d: $j^\mu = \bar{\psi}\gamma^\mu\psi$ and $j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$. However, compared to the 4d case a crucial simplification arises due to the identity

$$\gamma^\mu \gamma^5 = -\epsilon^{\mu\nu} \gamma_\nu \qquad\qquad (\text{with } \epsilon^{01} \equiv 1) . \tag{9.53}$$

This implies

$$j_5^\mu = -\epsilon^{\mu\nu} j_\nu . \tag{9.54}$$

Let us calculate $\langle j^\mu \rangle$ in the presence of a background field A_μ at leading order in e. This is most easily done by appealing to our very basic formula (from QFT I) for evaluating time-ordered expectation values of interacting fields in terms of free fields:

$$\langle \, \bar{\psi}(x) \gamma^\mu \psi(x) \, \rangle = \frac{\langle \, T \, \bar{\psi}_0(x) \gamma^\mu \psi_0(x) \, \exp\left(i S_{int}[\psi_0, \bar{\psi}_0, A_\mu] \right) \rangle}{\langle \, T \, \exp\left(i S_{int}[\psi_0, \bar{\psi}_0, A_\mu] \right) \rangle} . \tag{9.55}$$

Recall that the vacuum on the r.h. side is free and that the denominator is just there to cancel vacuum bubbles. It is immediately clear that the $\mathcal{O}(e)$ contribution will involve just two fermion propagators and that they will be connected as shown in the diagram in Fig. 53.

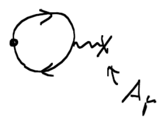


Figure 53: Electromagnetic current at leading order. Note that the wavy line and the cross are used to symbolize the appearance of a (classical) background field A_μ in the vertex – no gauge boson propagator is present.

Working this expression out in detail and Fourier transforming both A_μ and j^μ, one can bring it to the form

$$\langle j^\mu(q)\rangle = (-ie)^{-1} i\Pi^{\mu\nu}(q)\, A_\nu(q)\,. \tag{9.56}$$

Here $\Pi^{\mu\nu}$ is the familiar one-loop expression for the vacuum polarization, which contains the trace of the two propagators and the loop integral. The advantage of this form of writing our expression is that we already calculated $\Pi^{\mu\nu}$ for general d. Unfortunately, at the time we took its trace a few lines too early. So let us supply some details to have a reasonably complete calculation. With Wick rotation already in place, one finds

$$\Pi^{\mu\nu}(q) = -4e^2\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\,\mathrm{tr}(\mathbb{1})\,\frac{\left(1-\frac{2}{d}\right)\eta^{\mu\nu}k^2 - 2x(1-x)q^\mu q^\nu + \eta^{\mu\nu}x(1-x)q^2}{(k^2+\Delta)^2}\,, \tag{9.57}$$

where

$$\Delta \equiv -x(1-x)q^2\,. \tag{9.58}$$

Crucially, the $1/d$ term comes from the replacement

$$k^\mu k^\nu \;\to\; \frac{1}{d}\,k^2\,\eta^{\mu\nu} \tag{9.59}$$

under the integral.

After performing the k-integration, we have

$$\begin{aligned}\Pi^{\mu\nu}(q) \;&=\; -e^2\int_0^1 dx\,\frac{\mathrm{tr}(\mathbb{1})}{(4\pi)^{d/2}}\cdot\frac{1}{\Delta^{2-d/2}}\cdot\Bigg\{\left(1-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)(-\eta^{\mu\nu}\Delta) \\ &\qquad +\Gamma\left(2-\frac{d}{2}\right)\left[-\eta^{\mu\nu}\Delta - 2x(1-x)q^\mu q^\nu\right]\Bigg\}\,.\end{aligned} \tag{9.60}$$

The first part in the curly bracket comes from the k^2 in the numerator, which naively gives a quadratic divergence at $d=4$, i.e. a pole at $d=2$. However, we see that the prefactor cancels this pole,

$$\left(1-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right) = \Gamma\left(2-\frac{d}{2}\right)\,, \tag{9.61}$$

such that only a pole at $d = 4$ and hence log-divergence in 4d is left.

We then have

$$\begin{aligned} \Pi^{\mu\nu}(q) &= -2\,\mathrm{tr}(\mathbb{1})\, e^2 \int_0^1 dx\, \frac{x(1-x)}{(4\pi)^{d/2}} \cdot \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}} \cdot (q^2\eta^{\mu\nu} - q^\mu q^\nu) \\ &= \left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) \frac{e^2}{\pi}\,, \end{aligned} \tag{9.62}$$

where in the last line we have set $d = 2$, such that $\mathrm{tr}(\mathbb{1}) = 2$ and the x-integration became trivial,

$$\int_0^1 dx\, \frac{x(1-x)}{-x(1-x)q^2} = -\frac{1}{q^2}\,. \tag{9.63}$$

As a side-remark, note that this does not vanish at $q^2 \to 0$. As a result, the quantity Π defined by $\Pi_{\mu\nu} = (\eta_{\mu\nu}q^2 - q_\mu q_\nu)\Pi$, has a pole: $\Pi \sim 1/q^2$. If you recall how this corrects the photon propagator, you see that the photon gets a mass in 2d. This is an interesting fact, maybe not notally unexpected given that e has positive mass dimension and hence the theory becomes strongly coupled at low energies. We will not pursue this.

Our central result now reads

$$\begin{aligned} \langle j_5^\mu \rangle &= -\epsilon^{\mu\nu}\langle j_\mu \rangle = -\epsilon^{\mu\nu}(-ie)^{-1} i \Pi_{\nu\rho} A^\rho \\ &= -\epsilon^{\mu\nu}(-ie)^{-1} i \left(\eta_{\nu\rho} - \frac{q_\nu q_\rho}{q^2}\right) \frac{e^2}{\pi} A^\rho = \epsilon^{\mu\nu} \frac{e}{\pi} \left(A_\nu - \frac{q_\nu q^\rho}{q^2} A_\rho\right)\,. \end{aligned} \tag{9.64}$$

Thus, while vector current conservation, $q_\mu \langle j^\mu \rangle = 0$, obviously holds, the axial current is not conserved:

$$q^\mu \langle j_5^\mu \rangle = \frac{e}{\pi}\, \epsilon^{\mu\nu}\, q_\mu A_\nu(q) \neq 0\,. \tag{9.65}$$

After Fourier transformation, this reads

$$\partial_\mu j_5^\mu = \frac{e}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}\,, \tag{9.66}$$

which is our main result in this subsection.

Note first that this is structurally similar to what we found in 4d: The r.h. side is a combination of ϵ tensor and field strength. Indeed, anomalies arise in all even dimensions. This is related to the fact that an extra, independent γ-matrix ('γ^5') exists only in even dimensions. For example, in 5d the spinor continues to be our familiar four-component spinor and the Clifford algebra reads

$$\{\gamma^M, \gamma^N\} = 2\eta^{MN}\,, \quad M, N \in \{0,1,2,3,5\}\,, \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,, \tag{9.67}$$

with precisely the same γ^5 we used in 4d to define chirality. To define chirality in 5d by analogy to 4d, one would need

$$\text{`}\gamma^6\text{'} \sim \gamma^0\gamma^1\gamma^2\gamma^3\gamma^5\,, \tag{9.68}$$

but this is of course $\sim \mathbb{1}$. Thus, indeed, chirality does not exits in 5d (and in all other odd dimensions). By contrast, in even dimensions our chirality and anomaly story goes through and the general structure (in the abelian case) is

$$\sim \epsilon^{\mu_1\mu_2\cdots\mu_{d-1}\mu_d} F_{\mu_1\mu_2} \cdots F_{\mu_{d-1}\mu_d} . \tag{9.69}$$

The 2d case provides the simplest instance for discussing how anaomalies arise in the Feynman diagram approach at the conceptual level: Classical $U(1)$ invariance, chiral or not, always implies classical current conservation. As shown in QED for the non-chiral case, this symmetry can be used to derive Ward identities for the Green's functions and hence to establish current conservation also at the level of correlation functions, i.e. for operators or 'under the path integral',

$$q_\mu \langle j_5^\mu(q) \rangle \text{ `} = \text{' } 0 . \tag{9.70}$$

However, the manipulations allowing one to achieve this are at a certain level *formal* in that they use divergent and hence undefined integral expressions.

This is where the anomaly comes in: In certain cases the explicit UV regularization is inconsistent with the manipulations one would need to do to establish current conservation. We have seen this UV sensitivity very clearly in the Fujikawa method. In our present analysis, it arose in the way in which the naively expected pole at $d = 2$ disappeared in dimensional regularization.

We can make this more explicit by noting that $\Pi_{\mu\nu}$ contains two structurally different integrals,

$$\int d^2k \, \frac{k^2 \eta^{\mu\nu}}{(k^2+\Delta)^2} \qquad \text{and} \qquad \int d^2k \, \frac{q^\mu q^\nu}{(k^2+\Delta)^2} . \tag{9.71}$$

The first one is divergent, but the divergence disappeared due to the prefactor $(1-d/2)$,

$$(1-d/2)\Gamma(1-d/2) \sim \text{finite} . \tag{9.72}$$

The second is simply finite. Thus, while we found

$$\Pi^{\mu\nu} \sim (A\eta^{\mu\nu} - Bq^\mu q^\nu / q^2) , \tag{9.73}$$

with $A = B$, we have to remember that A is finite but regularization dependent. Invariance under $U(1)_V$ requires $A = B$ and dimensional regularization respects this. This is a general feature of dimensional regularization, which respects vector $U(1)$'s but has trouble with chiral symmetries. (We will see that even more clearly below.)

Here, we note that we found $\partial_\mu j^\mu = 0$ and $\partial_\mu j_5^\mu \neq 0$ precisely because we found $A = B$. Indeed, a different regularization could have given a different A, e.g. $A = 0$. In this case, as one can easily seen by recalling the calculation above, we would have found $\partial_\mu j_5^\mu = 0$. But, at the same time we necessarily would also have $\partial_\mu j^\mu \neq 0$, making the calculation plain inconsistent if j^μ is gauged.

9.5 The chiral anomaly in the Feynman diagram approach – 4d example

In $d = 4$, there is no analogue of the simple 2d relation $j_5^\mu = -\epsilon^{\mu\nu} j_\nu$. Thus, while we could calculate the vector current expectation value diagramatically, as in the last section, we can not derive the axial current from it. However, we can directly evaluate the expectation value of the axial current, cf. Fig. 54. The only difference is the γ^5 in the l.h. vertex and, of course, the number of dimensions.

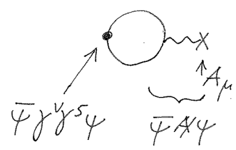


Figure 54: Axial current at leading order. As before (cf. Fig. 53) the wavy line and the cross are used to symbolize the appearance of a (classical) background field A_μ in the vertex – no gauge boson propagator is present.

The expectation value will be proportional to the Fourier transform of the external field, allowing us to define a matrix element $\mathcal{M}$ by

$$\langle j_5^\mu(q)\rangle\Big|_{\text{1st order in } A_\mu} \equiv \mathcal{M}^{\mu\nu}(q)\, A_\nu(q)\,. \tag{9.74}$$

However, it will become clear below that we will not find an anomaly.[23]

An anomaly arises only at 2nd order in A_μ, i.e. from a diagram with two external fields. This diagram is conventionally drawn as a triangle, cf. Fig. 55. The corresponding equation reads

$$\langle j_5^\mu(q)\rangle\Big|_{\text{2nd order in } A_\mu} \equiv \mathcal{M}^{\mu\nu\rho}(p,k)\, A_\nu(p) A_\rho(k)\,. \tag{9.75}$$

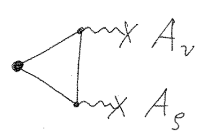


Figure 55: Axial current at 2nd order in A.

It is conventional to slightly reinterpret this calculation. Indeed, it is equivalent to think of a transition amplitude with one incoming vector boson which couples to the axial current and two outgoing vector bosons which couple to the vector current,

[23]The reason is that we would encounter a trace of four γ matrices with γ^5, producing an ϵ-tensor. But the number of independent momenta available at leading order in A_μ is too small to saturate all indices and produce a non-zero result.

cf. Fig. 56. This basically corresponds to 'adding' two photons with momenta p and k to the external field. Since we now specify which external momenta are present, a momentum conservation δ function also appears together with the matrix element:

$$\langle p,k|j_5^\mu(q)|0\rangle = (2\pi)^4\delta^4(p+k-q)\mathcal{M}^{\mu\nu\rho}(p,k)\,\epsilon_\nu^*(p)\epsilon_\rho^*(k)\,. \tag{9.76}$$

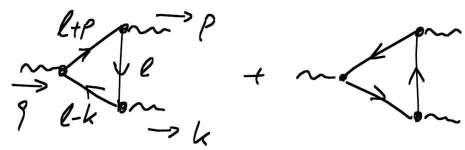


Figure 56: Scattering amplitude interpretation of the expectation value of the axial current at 2nd order in A (equivalent to Fig. 55).

We will discuss the explicit calculation only briefly, focussing on the conceptual points. Crucially, since we have 'split up' the external field in two photons, there are now two diagrams with different ordering of the photons along the fermion loop. The first diagram, already mutliplied with q_μ for current conservation, gives

$$\sim q_\mu \int_\ell \mathrm{tr}\left[\gamma^\mu\gamma^5\frac{1}{\slashed{\ell}-\slashed{k}}\gamma^\rho\frac{1}{\slashed{\ell}}\gamma^\nu\frac{1}{\slashed{\ell}+\slashed{p}}\right]\,. \tag{9.77}$$

Using

$$\slashed{q}\gamma^5 = (\slashed{\ell}+\slashed{p}-\slashed{\ell}+\slashed{k})\gamma^5 = (\slashed{\ell}+\slashed{p})\gamma^5+\gamma^5(\slashed{\ell}-\slashed{k}) \tag{9.78}$$

this gives

$$= \int_\ell \mathrm{tr}\left[\gamma^5\frac{1}{\slashed{\ell}-\slashed{k}}\gamma^\rho\frac{1}{\slashed{\ell}}\gamma^\nu \;+\; \gamma^5\gamma^\rho\frac{1}{\slashed{\ell}}\gamma^\nu\frac{1}{\slashed{\ell}+\slashed{p}}\right]\,. \tag{9.79}$$

Next, shifting the integration variable as $\ell\to\ell+k$ in the first term only and interchanging γ matrices, $\gamma^5\gamma^\rho = -\gamma^\rho\gamma^5$, in the second term, one has

$$= \int_\ell \mathrm{tr}\left[\gamma^5\frac{1}{\gamma^5\slashed{\ell}}\gamma^\rho\frac{1}{\slashed{\ell}+\slashed{k}}\gamma^\nu \;-\; \gamma^5\frac{1}{\slashed{\ell}}\gamma^\nu\frac{1}{\slashed{\ell}+\slashed{p}}\gamma^\rho\right]\,. \tag{9.80}$$

This last expression is antisymmetric under the exchange $\{p,\nu\}\leftrightarrow\{k,\rho\}$. But the second diagram of Fig. 56 follows from the first precisely under this substitution, i.e. under $\{p,\nu\}\leftrightarrow\{k,\rho\}$. Hence, the sum of two diagrams appears to be zero, suggesting that the axial current is conserved. However, this result is wrong precisely as explained at the end of last section: It is obtained by manipulations, in particular the shift of an integration variable, with divergent integrals.

So let us now repeat the analysis in a well-defined manner, using dimensional regularization. Clearly, one faces the problem of defining γ^5 in d dimensions. The correct procedure turns out to be the use of the so-called 't Hooft-Veltman prescription: One defines a formal object γ^5, extending the formally defined Clifford algebra and obeying the rules[24]

$$\begin{aligned}
\{\gamma^5, \gamma^\mu\} &= 0 \qquad \text{for } \mu = 0, \cdots, 3 && (9.81)\\
\left[\gamma^5, \gamma^\mu\right] &= 0 \qquad \text{for } \mu = 5, \cdots, d\,. && (9.82)
\end{aligned}$$

It can be shown that, together with the usual rules of dimensional regularization, this makes all loop integrals well-defined and finite and leads to results consistent with the conceptually better-justified Fujikawa method.

In concrete applications, it proves convenient to split the integration variable as

$$\ell = \ell_\| + \ell_\perp\,, \tag{9.83}$$

where the 'parallel' part stands for dimensions $0, \cdots, 3$ and the 'perpendicular' part for the rest. Our previous naiv manipulations are now replaced by

$$\begin{aligned}
\not{q}\gamma^5 &= (\not{\ell} + \not{p} - \not{\ell} + \not{k})\gamma^5 = (\not{\ell} + \not{p} - \not{\ell}_\| - \not{\ell}_\perp + \not{k})\gamma^5 \qquad (9.84)\\
&= (\not{\ell} + \not{p})\gamma^5 + \gamma^5(\not{\ell}_\| - \not{\ell}_\perp - \not{k}) = (\not{\ell} + \not{p})\gamma^5 + \gamma^5(\not{\ell} - \not{k}) - 2\gamma^5\not{\ell}_\perp\,.
\end{aligned}$$

Now, the first two terms in the final expression change sign between the two diagrams and cancel, exactly as before. By contrast, the new contribution encoded in the third and last term arises in both diagrams with the *same* sign. Thus, overall one finds

$$q_\mu \mathcal{M}^{\mu\nu\rho} \sim \int_\ell \text{tr}\left[\gamma^5 \not{\ell}_\perp \frac{1}{\not{\ell} - \not{k}} \gamma^\rho \frac{1}{\not{\ell}} \gamma^\nu \frac{1}{\not{\ell} + \not{p}}\right]\,. \tag{9.85}$$

We now rewrite all fractions in analogy to

$$\frac{1}{\not{\ell} - \not{k}} = \frac{\not{\ell} - \not{k}}{(\ell - k)^2} \tag{9.86}$$

and introduce Feynman parameters according to the (more general than what we had before) formula

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1 \cdots dx_n\, \delta\left(\sum x_i \,-\, 1\right) \frac{(n-1)!}{[x_1 A_1 + \cdots + x_n A_n]^n}\,. \tag{9.87}$$

Next, one shifts the integration variable, $\ell \to \ell + P$, such that the ℓ-integration takes the form

$$\int \frac{d^d\ell}{(2\pi)^d} \cdot \frac{\text{tr}[\gamma^5 \not{\ell}_\perp (\not{\ell} + \not{P} - \not{k})\gamma^\rho(\not{\ell} + \not{P})\gamma^\nu(\not{\ell} + \not{P} - \not{p})]}{(\ell^2 - \Delta)^3}\,. \tag{9.88}$$

Here P is an appropriate linear combination of k and p (depending on the x_i) which ensures that the denominator has no piece linear in ℓ. Furthermore, Δ is a function of k, p and the x_i and the integration over the x_i still has to be performed in the end. Note

[24] We note that it is common to label indices as $\mu, \nu, \rho, \cdots = 0, 1, 2, 3, 5, 6, \cdots, d$, such that the 'natural' labelling $1, 2, 3, 4, 5, 6, \cdots, d$ emerges after Wick rotation, $x^0 \to ix^4$.

that in the above trace the term $\ell_\perp$ has not received a P-dependent piece since P is by definition a 4d momentum.

Now, to get a non-zero result under the $SO(1,d-1)$ symmetric integration, one of the other $\not\ell$ factors in the numerator must be replaced by $\not\ell_\perp$. The remaining matrices must be $\not k$, $\not p$, γ^ρ and γ^ν, such that a non-zero trace with γ^5 can be produced. (Of course, $\not k$ and $\not p$ may also come from $\not P$.)

Furthermore, since all these matrices only involve only 'parallel-direction' γ's, the two $\not\ell_\perp$ terms can be moved next to each other. Thus, one has to evaluate

$$\int_\ell \frac{\not\ell_\perp \not\ell_\perp}{(\ell^2-\Delta)^3} = \frac{d-4}{d}\,\mathbb{1}\int_\ell \frac{\ell^2}{(\ell^2-\Delta)^3} = \frac{-i}{2(4\pi)^2}\,, \tag{9.89}$$

where we set $d=4$ in the last expression. This was possible because the pole of the integral, which is $\sim \Gamma(2-d/2)$, was cancelled by the explicit $(d-4)$ prefactor.

Now, the trace can be evaluated and, as it turns out, the contributions with $\not P$ cancel. Putting everything together, one finds

$$q_\mu \mathcal{M}^{\mu\rho\nu} = \frac{2ie^2}{(4\pi)^2}\,\mathrm{tr}[\gamma^5 \not k \gamma^\rho \not p \gamma^\nu] = \frac{e^2}{2\pi^2}\epsilon^{\alpha\rho\beta\nu}k_\alpha p_\beta\,. \tag{9.90}$$

With this, we return to the formula for $\langle p,k|j_5^\mu(q)|0\rangle$ (Eq. (9.76)), multiply by q_μ, Fourier transform and set $x=0$. The result reads

$$\begin{aligned}\langle p,k|\partial_\mu j_5^\mu(0)|0\rangle &= -\frac{e^2}{2\pi^2}\,\epsilon^{\alpha\rho\beta\nu}(-ik_\alpha)\epsilon^*_\nu(p)(-ip_\beta)\epsilon^*_\rho(k)\\ &= -\frac{e^2}{16\pi^2}\,\langle p,k|\epsilon^{\alpha\nu\beta\rho}F_{\alpha\nu}F_{\beta\rho}(0)|0\rangle\,. \end{aligned}\tag{9.91}$$

Here the correctness of the last step is most easily checked by expressing the $F_{\mu\nu}$'s through A_μ's and using their decomposition in terms of creation and annihilation operators. The prefactor e^2 appears since, in contrast to our previous path integral derivation, we here use the convention with $D_\mu = \partial_\mu + ieA_\mu$. Apart from this factor, the results agree. We note without proof that anomalies are 'saturated at one loop', i.e. all higher-order corrections to this result vanish.

9.6 Final comments and generalizations

The calculation of the least subsection gives this ABJ anomaly the alternative name 'triangle anomaly', cf. Fig. 57. It should be clear from the discussion of the 2d case and the qualitative comments about higher-dimensional situations how this generalizes to dimensions other than four: The relevant diagram in $d=2n$ dimensions is a fermion loop with one axial-current vertex and n vector-current vertices.

Figure 57 also makes it very clear how the non-abelian anomaly formular arises: Each vertex carries the generator of the corresponding current, that the condition for not having an anomaly reads

$$\mathrm{tr}(T_a\{T_b,T_c\}) = 0\,. \tag{9.92}$$

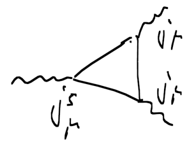


Figure 57: Another representation of the triangle or ABJ anomaly.

It is also immediately clear what the condition for a single $U(1)$ and a set of l.h. fermions labelled by i is:

$$\sum_i Q_i^3 = 0\,. \tag{9.93}$$

The calculation and the possible presence of a triangle anomaly also extends to fluctuations of the metric background, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where now $h_{\mu\nu}$ takes the role of the gauge background A_μ. The corresponding diagram is shown in Fig. 58 and the condition for not having an anomaly is

$$\sum_i Q_i = 0\,. \tag{9.94}$$

This is also satisfied for the $U(1)$ of the Standard Model.

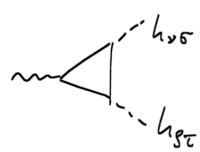


Figure 58: Gravitational anomaly.

Furthermore, we already noted that the QCD lagrangian with massless quarks has a chiral global symmetry with current

$$j_5^\mu = \bar{Q}\gamma^\mu\gamma^5 Q\ , \qquad \text{where} \quad Q = \begin{pmatrix} u \\ d \end{pmatrix} \tag{9.95}$$

stands for the pair of up- and down-type quark Dirac fermions. Analogously, QCD without quark masses has a global $SU(2)$ or Isospin symmetry with currents

$$j_5^{\mu\, a} = \bar{Q}\gamma^\mu\gamma^5\tau^a Q\,. \tag{9.96}$$

(Note that only the l.h. part of this is gauged in the Standard Model.) These currents have non-trivial overlap with the pions, which is quantified by the so-called pion decay constant f_π:

$$\langle 0|j_5^{\mu\, a}|\pi^b(p)\rangle = -ip^\mu f_\pi \delta^{ab} e^{-ipx}\,. \tag{9.97}$$

Here the three pion fields π^b are just an alternative basis w.r.t. the more commonly used fields π^0 and $\pi^\pm$. This overlap and hence the value of f_π can for example be determined from the charged-pion decay process $\pi^+ \to \mu^+\nu_\mu$, cf. Fig. 59.

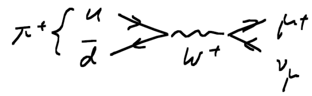


Figure 59: Charged pion decay.

Now focus on the current $j_5^{\mu\,0}$ corresponding to the neutral pion. Due to the anomaly, this current has a non-zero 3-point function with two photon fields. At the same time, as noted above, it overlaps with the pion. Hence, using the anomaly calculation and the pion decay constant, the decay rate $\pi^0 \to \gamma\gamma$ can be derived, cf. Fig. 60. This provides a successful experimental test of the anomaly calculation. For more details, see [1]. An understanding of chiral symmetry breaking and chiral perturbation theory in low-energy QCD is useful in this context, but we have no time for this in the present course.

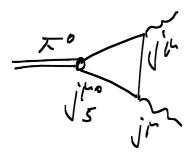


Figure 60: Neutral pion decay to photons via the chiral anomaly.

Finally, an entirely different but also very important anomaly is the 'scale-invariance' or 'trace' anomaly. The underlying symmetry is the change of all length scales of some experimental setup by some universal constant factor (rescaling). A theory would be symmetric if the experimental results do not change.

A theory which classically has this symmetry is QCD,

$$\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}\,, \tag{9.98}$$

which is obvious since no dimensionful constant is involved. If one includes quarks, they would have to be massless not to break the scale-inavariance explicitly.

However, we also know that $dg(\mu^2)/d\ln(\mu^2) = \beta(g) \neq 0$, such that the lagrangian to be used in a LO analysis of an experiment at energy scale μ is rather

$$\mathcal{L} = -\frac{1}{2g^2(\mu^2)}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu}\,. \tag{9.99}$$

Hence, scale invariance is broken in the quantum theory, i.e. one is dealing with an anomaly. This anomaly is sometimes referred to as 'trace anomaly' for the following reason: The energy momentum tensor can be defined by

$$T^{\mu\nu} \sim \frac{\delta}{\delta g_{\mu\nu}}\,S\,. \tag{9.100}$$

From this, one can derive that the 'trace' $T^\mu{}_\mu$ measures the non-invariance under rescalings, $g_{\mu\nu} \to c\, g_{\mu\nu}$. More specifically, the relevant statement in the quantum theory is $\langle T^\mu{}_\mu\rangle \neq 0$.

We also note that there exist field theories which are scale-invariant even after quantization. It can in many cases be shown that such quantum scale invariance implies the stronger property of conformal invariance (invariance under angle-preserving deformations).

10 Generating functionals, effective actions, spontaneous symmetry breaking, condensed-matter-QFTs

We can only give a brief overview of the above topic, each of which deserves a much more detailed study.

10.1 Generating functionals and effective actions

Recall that

$$Z[j] = \int D\varphi \, e^{iS[\varphi]+i\, j\varphi} \qquad \text{with} \qquad j\varphi \equiv \int d^4x \, j(x)\varphi(x) \tag{10.1}$$

is the generating functional for time-ordered Green's functions. We can give it a more physical interpretation by recalling in additon that

$$Z[j] = \langle 0|e^{-iHt}|0\rangle\Big|_{t\to\infty} \,, \tag{10.2}$$

where we view the source as an intrinsic part of H. Let us compare this to the thermodynamic 'partition function' which (and this is no accident) is also denoted by Z:

$$Z = \mathrm{tr}\, e^{-\beta H} = \sum_i \langle i|e^{-\beta H}|i\rangle \,, \qquad \text{where} \qquad \beta \equiv \frac{1}{T} \;,\;\; k_B \equiv 1 \,. \tag{10.3}$$

With the same methods that we used for amplitudes, this expression for Z can be rewritten in terms of a path or functional integral, both in QM and in QFT. The outcome will be a euclidean path integral and the time interval over which the particle trajectories or fields are integrated is finite and determined by β. Hence,

$$Z = \int D\varphi \, \exp\left[-\int_0^\beta d\tau \, d^3x \, \mathcal{L}_E(\varphi)\right] \,, \tag{10.4}$$

where the integral is over field configurations that are periodic in τ with period β. This accounts for the original presence of the trace and can be easily derived in detail by interpreting

$$\sum_i \langle i| \cdots |i\rangle \tag{10.5}$$

in the field basis (i.e. in a Hilbert space basis of eigenfunctions of the field operators $\varphi(0,\vec{x})$).

For fermions, one has to impose anti-periodic rather than periodic boundary conditions. The key idea behind this important statement can be summarized in the following short calculation (cf. our earlier discussion of fermionic coherent states):

$$\begin{aligned}\int d\bar{\eta}\, d\eta\, e^{-\bar{\eta}\eta}\langle -\eta|A|\eta\rangle &= \int d\bar{\eta}\, d\eta\, (1-\bar{\eta}\eta)\, (\langle 0| - \bar{\eta}\langle 1|)\, A\, (|0\rangle + |1\rangle\eta) \\ &= \langle 0|A|0\rangle + \langle 1|A|1\rangle = \mathrm{tr}\, A\,. \end{aligned} \tag{10.6}$$

The crucial sign is related to the necessary ordering prescription between Grassmann variables and fermionic states.

Due to the compactness of Euclidean time, the 'energy' or k_E^0 spectrum in Fourier space is now discrete ('Matsubara frequencies'). These are the first steps to towards 'finite temeperature field theory', at least in the equilibrium case. For more details see e.g. [27, 28].

We also note that, at $\beta \to \infty$ the distinction between the Euclidean and Minkowskian theory disappears. Indeed, it should be irrelevant whether $T \to 0$ or $T \to i0$. Hence our standard, Minkowskian partition function $Z[j]$ is identified with the zero-temperature limit of the thermodynamic partition function Z, with an external classical source field j present. This j is then completely analogous to an external magnetic field, as it is frequently used in the analysis of condensed matter systems at finite (or even zero) temperature. Thus, there is a close relation between thermodynamics and field theory.

Let us from now on stick with the Euclidean theory and keep the close analogy with thermodynamics in mind. It is then natural to define another generating functional,

$$W[j] \equiv -\ln Z[j]\,, \tag{10.7}$$

which is the analogue of the Helmholtz free energy,

$$F(B) \equiv -T\, \ln Z(B)\,, \tag{10.8}$$

with $B \widehat{=} j$ the external magnetic field.

We can recall from standard thermodynamics that $F = E - TS$ which implies that, at $T = 0$, one has $F = \langle \hat{H}\rangle$. In the field theory context, we demonstrate this as follows:

$$e^{-W} = Z = \int D\varphi\, e^{-S_E} = \mathrm{tr}\, e^{-\beta\hat{H}} \simeq \langle 0|e^{-\beta\hat{H}}|0\rangle = e^{-\beta\langle\hat{H}\rangle}\,, \tag{10.9}$$

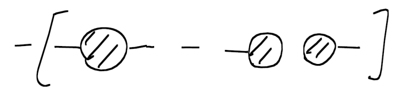
where in the last steps we used that we are interested in very large β and that the vacuum is an eigenstate of $\hat{H}$. Hence, writing ρ for the vacuum energy density, V_3 for the spatial volume and $V_4 \equiv \beta V_3$, we have

$$W \simeq \beta\langle\hat{H}\rangle = \beta V_3 \rho = V_4 \rho\,. \tag{10.10}$$

Thus, W/V_4 is the energy density of our field theory (always with some UV cutoff imposed). This is now a well-define statement purely in the 4d euclidean QFT, in a form in which the physically interesting limit $V_4 \to \infty$ can be taken.

Now comes the crucial new point to be made specifically in the field theory context: $-W[j]$ generates the connected Green's functions. We demonstrate this for special case of 2-point functions:

$$
\begin{aligned}
\frac{\delta^2 W}{\delta j_1 \delta j_2} &= -\frac{\delta}{\delta j_1}\frac{\delta}{\delta j_2}\ln Z = -\frac{\delta}{\delta j_1}\left(\frac{1}{Z}\frac{\delta}{\delta j_2}Z\right) \\
&= -\frac{1}{Z}\frac{\delta^2}{\delta j_1 \delta j_2}Z + \left(\frac{1}{Z}\frac{\delta}{\delta j_1}Z\right)\left(\frac{1}{Z}\frac{\delta}{\delta j_2}Z\right) \\
&=
\end{aligned}
$$

 . (10.11)

The last line provides a graphical representation of the line before. We see that the second term subtracts the disconnected part, confirming our claim.

This argument can be extended to n-point functions, i.e. to

$$
\frac{\delta^n W}{\delta j_1 \cdots \delta j_n}\,, \tag{10.12}
$$

cf. the book by Rivers [8]. An alternative proof is given in [13]: It starts with the assumption that $-W$ generated connected Green's functions and then demonstrates (which is merely an issue of combinatorics) that

$$
e^{-W} = 1 + (-W) + \frac{1}{2}(-W)^2 + \cdots \tag{10.13}
$$

generates all Green's functions.

We proceed by analogy to thermodynamics, where the Helmholtz free energy gives rise to the definition of the Gibbs free energy via a Legendre transformation. In our case this is functional Legendre transformation (or equivalently a multi-variable Legendre transformation, with $j(x)$ being an indpendent variable for each x). We define

$$
\Gamma[\varphi] \equiv W[j[\varphi]] - j[\varphi]\cdot\varphi \qquad \text{with} \qquad \varphi[j] \equiv \frac{\delta W[j]}{\delta j}\,. \tag{10.14}
$$

Recall that we use the shorthand notation

$$
j\cdot\varphi = \int d^4x\, j(x)\varphi(x)\,. \tag{10.15}
$$

In the QFT context, $\Gamma[\varphi]$ is known as the 'effective action', which plays a central role in many formal developments and applications. Its argument φ can be interpreted as the (in general space-time dependent) VEV of the fundamental quantum field. Indeed

$$
\varphi[j] = \frac{1}{Z}\frac{\delta}{\delta j}\int D\varphi'\, e^{-S[\varphi']+j\varphi'} = \langle\hat{\varphi}\rangle_j\,, \tag{10.16}
$$

is the VEV in the presence of a source $j(x)$.

As a general proporty of a Legendre transform, we also have

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi} = \frac{\delta W}{\delta j}\cdot\frac{\delta j}{\delta\varphi} - \frac{\delta j}{\delta\varphi}\cdot\varphi - j[\varphi] = -j[\varphi]\,, \tag{10.17}$$

where we used $\delta W/\delta j = \varphi$. We conclude that at stationary points of $\Gamma[\varphi]$, we have $j[\varphi] = 0$ and hence

$$\Gamma[\varphi] = W[j[\varphi]] = V_4\,\rho\,. \tag{10.18}$$

Here the last equality additionally assumes that $\varphi = \text{const}$.

More generally, we define

$$V_{eff}(\varphi) = \frac{1}{V_4}\,\Gamma[\varphi = \text{const.}]\,. \tag{10.19}$$

This 'effective potential' is the volume-normalized restriction of (minus) the effective action to constant field configurations. It corresponds to the Gibbs free energy for such configurations. Moreover, we have seen that at stationary points it measures the energy density. Minima of $\Gamma[\varphi]$ with constant field correspond to minima of V_{eff} and to Poincare-invariant vacua of the theory.

To interpret Γ as a generating functional, we make the following series of observations: First, we have

$$\frac{\delta\Gamma}{\delta\varphi} = -j \quad\Rightarrow\quad -\frac{\delta}{\delta j}\,\frac{\delta\Gamma}{\delta\varphi} = \mathbb{1}\,. \tag{10.20}$$

Written in somewhat more detail, the last equation reads

$$-\frac{\delta}{\delta j_1}\,\frac{\delta\Gamma}{\delta\varphi_2} = \mathbb{1}_{12} \qquad\text{with}\qquad \mathbb{1}_{12} \equiv \delta^4(x_1 - x_2)\,. \tag{10.21}$$

Next, since Γ depends of j only indirectly, through φ, we have

$$-\frac{\delta\varphi_3}{\delta j_1}\cdot\frac{\delta^2\Gamma}{\delta\varphi_3\,\delta\varphi_2} = \mathbb{1}_{12} \quad\Rightarrow\quad -\frac{\delta^2 W}{\delta j_1\,\delta j_3}\cdot\frac{\delta^2\Gamma}{\delta\varphi_3\,\delta\varphi_2} = \mathbb{1}_{12}\,. \tag{10.22}$$

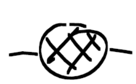
This can be rewritten as

$$\frac{\delta^2\Gamma}{\delta\varphi_1\,\delta\varphi_2} = -\left(\frac{\delta^2 W}{\delta j_1\,\delta j_2}\right)^{-1} \;\text{“}=\text{”}\; p^2 + m^2 + \Pi(p^2)\,. \tag{10.23}$$

Here in the last step we used switched to Fourier space (hence the equal sign is in quotation marks) and used our previously introduced notation for the resummed inverse propagator.

We recall that $\Pi(p^2)$ is the self-energy, i.e. the one-particle irreducible (1PI) two-point function in Fourier space. (Note that we are in the Euclidean theory.) Also, the term $p^2 + m^2$ can be interpreted as the two-fold functional derivative of S_{free} w.r.t. φ. Hence

$$\frac{\delta^2\Gamma}{\delta\varphi_1\,\delta\varphi_2} = \frac{\delta^2 S_{free}}{\delta\varphi_1\,\delta\varphi_2} + \tag{10.24}$$

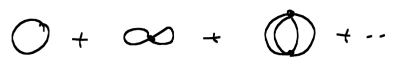


This statement as well as its derivation generalize straightforwardly to n-point functions: Γ is the generating functional of 1PI n-point functions.

Note that the free action is, by definition, quadratic in the fields and hence gives rise to a 1PI 2-point function. It is hence completely logical that it appears in the equation above. Similarly, if there is a φ^3 term in S_{int}, it will appear on the r.h. side of the corresponding equation for the 3rd derivative of Γ. The remainder on the r.h. side will be the 1PI 3-point function. And so on and so forth....

The corresponding relation without drivatives reads

$$V_{eff} = V_{free} + \qquad (10.25)$$



with the '1-loop term' given by

$$\sim \int d^4k_E \, \ln(k_E^2 + m^2)\,. \qquad (10.26)$$

This follows most directly from our earlier discussion of the generating functional Z or the partitition function of the free theory.

We finally also note that the vertices relevant for the calculation of Γ are clearly simply the corresponding functional derivatives of the free action. But these are the same vertices that appear in the Feynman rules for W and Z. Thus, we finally arrived at a particularly simple method to derive Feynman rules for vertices.

10.2 Spontaneous symmetry breaking

It can happen that both $S \equiv S_{class}$ and the resulting quantum theory (and hence Γ) have a certain symmetry, but the field configuration minimizing Γ is not invariant under this symmetry. Quantum mechanically, this means that $\hat{H}$ possesses a symmetry, but $|0\rangle$ is not invariant under it.

One of the simplest examples is provided by the complex scalar with

$$\mathcal{L}_E = |\partial\phi|^2 + m^2|\phi|^2 + \lambda|\phi|^4 \qquad \text{where} \qquad m^2 < 0\,. \qquad (10.27)$$

The potential is illustrated in Fig. 61. In this particular case, it is apparent that there exists a massless scalar (let's call it χ) which parametrizes the set of degenerate minima, in this case obviously an S^1. We can choose $\chi \equiv \arg\phi$. The scalar χ is called the 'Goldstone bosons' and its mass is zero precisely because of the exact symmetry of the quantum theory.

All that was said above generalizes to other 'spontaneously broken symmeties', with the statemant about the (set of) massless fields known as the 'Goldstone theorem'. If a symmetry group G is broken to $H \subset G$ in the vacuum, the scalars parametrize the coset manifold G/H. (This is easy to demonstrate, but we will not do so.)

As a side-remark, we note that the above behaviour, in particular the appearance of a non-zero VEV of ϕ, is stable under the introduction of a small non-zero temperature.

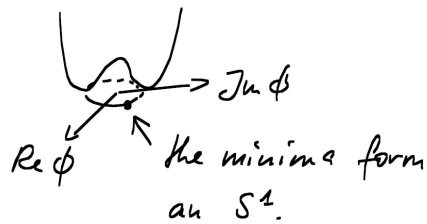


Figure 61: (Effective) potential of abelian Higgs model.

Indeed, we do not expect the effective potential, which we can also calculate at $T \neq 0$, to suddenly change if a very small T is introduced. However, it turns out that the situation is very different in $d \leq 2$ dimensions: Here, the 'Mermin-Wagner theorem' forbids spontaneous symmetry breaking at *any* $T \neq 0$. Roughly speaking, the reason is that fluctuations are so strong that vacuum configurations permanently 'explore' the whole S^1.

Let us return to $d = 4$ and to the simplest case of a $U(1)$ symmetry. If the original symmetry is gauged, then the gauge boson acquires a non-zero mass and thus 3 rather than 2 degrees of freedom. One says that the Goldstone boson is 'eaten' by the gauge field. No massless particle is left.

More explicitly (and we are now returning to Minkowskian notation),

$$|D\phi|^2 = D_\mu\phi(D^\mu\phi)^* = \partial_\mu\phi\partial^\mu\phi^* + iA_\mu\phi\partial^\mu\phi^* - iA_\mu\phi^*\partial^\mu\phi - A_\mu A^\mu|\phi|^2 \,, \tag{10.28}$$

where the last term gives A_μ a mass if $\langle\phi\rangle \neq 0$. Furthermore, we can write

$$\phi = \rho e^{i\chi} \,, \quad \text{such that} \quad \partial^\mu\phi = (\partial^\mu\rho)e^{i\chi} + i(\partial^\mu\chi)\phi \tag{10.29}$$

and

$$iA_\mu[\phi\partial^\mu\phi^* - (\partial^\mu\phi)\phi^*] = 2A_\mu\partial^\mu\chi|\phi|^2 \,. \tag{10.30}$$

We see that, while χ itself has no mass term, it mixes with A_μ via a kinetic or derivative term. The latter has a mass term and thus all three degrees of freedom of A_μ and χ become massive.

At a more detailed, technical level, one can work in the so-called 'unitary' gauge where $\chi \equiv 0$ or in the 'covariant' gauge, where χ remains a relevant field. In the former, only physical degrees of freedom propagate und unitarity is manifest. In the latter, the vector propagator is similar to what we saw in the unbroken case. In particular, the absence of power-divergences and renormalizability can be demonstrated. Going back and forth betweem the two types of gauges requires a gauge-invariant regularization as provided by 'dim.reg.'. This logic is crucial in establishing the consistency of spontaneously broken gauge-theories and hence of the Standard Model (as first explained by 't Hooft and Veltman).

Before closing, we want to provide a particularly important and widely used example of a covariant gauge: the R_ξ gauge. We parametrize the field as

$$\phi(x) = \frac{1}{\sqrt{2}}\Big((v + h(x)) + i\varphi(x)\Big) \,, \tag{10.31}$$

noting that at leading order $v + h$ and φ correspond to the fields ρ and χ used earlier. Now, if one chooses the gauge fixing function

$$G = \frac{1}{\sqrt{\xi}}(\partial_\mu A^\mu - \xi v \varphi)\,, \tag{10.32}$$

the gauge fixin lagrangian

$$\mathcal{L}_{g.f.} \sim G^2 \tag{10.33}$$

introduces a mixing between φ and ∂A. This mixing cancels the mixing between φ and ∂A present in the original lagrangian (we derived it as a mixing between χ and ∂A above). This cancellation makes the R_ξ gauge particularly convenient.

10.3 QFT in condensed matter theory

We can only briefly mention a few central ideas, following Altland/Simons [29]. An older classical textbook on this subject is Fetter/Walecka [30].

Consider a QM many-particle system. Each particle can be in a discrete set of states which we label by k. (Think of the discrete momenta of particles in a box or of phonons in crystal.)

The generic state can then be written as

$$|\Psi\rangle = |n_{k_1}\, n_{k_2} \cdots\rangle\,, \tag{10.34}$$

with n_{k_1} particles in state k_1 etc. The Fock space representation is

$$|\Psi\rangle = \prod_k \frac{(a_k^\dagger)^{n_k}}{\sqrt{n_k!}}|0\rangle \qquad \text{with} \quad [a_k, a_q^\dagger\} = \delta_{kq}\,, \tag{10.35}$$

for bosons or fermions (which we will treat in parallel in this subsection).

The most general 1-particle state is

$$|\psi\rangle = \sum_k \tilde{\psi}_k a_k^\dagger |0\rangle\,. \tag{10.36}$$

To be specific, let k, q be momenta. (Let space-time be $1+1$–dimensional, so we don't have to write arrows). Let furthermore

$$\psi(x) \equiv \sum_k e^{ikx}\tilde{\psi}_k \qquad \text{and} \qquad a(x) \equiv \sum_k e^{ikx} a_k\,, \tag{10.37}$$

such that we can also write

$$|\psi\rangle = \int_x \psi(x) a^\dagger(x)|0\rangle\,. \tag{10.38}$$

Next, take the Hamlitonian to be

$$\hat{H} = \sum_k \frac{k^2}{2m} a_k^\dagger a_k = \int_x a^\dagger(x) \frac{\hat{p}^2}{2m} a(x)\,, \tag{10.39}$$

with $\hat{p} = -i\partial_x$. We clearly simple have a somewhat cumbersome description of free 1-particle QM, realized in the 1-particle sector of our Fock space.

However, the above Hamiltonian clearly also acts in anatural way on the whole Fock space, including multi-particle states. It is the right Hamiltonian for the, still free, many-particle QM system. It is also easy to give all those particles a certain x-dependent potential $V(x)$:

$$\hat{H} = \int_x a^\dagger(x) \left[\frac{\hat{p}^2}{2m} + V(x)\right] a(x)\,. \tag{10.40}$$

It is easy to see check that this is the right Hamiltonian for a many-particle system of non-interacting particles, with each particle described by a standard Schrodinger equation with potential V.

Now, the most interesting generalization arises as one adds interactions, which in this approach is extremely easy: For example, let us allow give to each pair of particles a potential energy $V_2(|x - x'|)$, with $|x - x'|$ the distance between the particles. The Hamiltonian now reads

$$\hat{H} = \int_x a^\dagger(x) \left[\frac{\hat{p}^2}{2m} + V_1(x)\right] a(x) + \frac{1}{2}\int_x\int_{x'} V_2(x - x')a^\dagger(x)a^\dagger(x')a(x)a(x')\,. \tag{10.41}$$

Going back to 3 dimensions by $x \to \vec{x}$ and $\partial_x \to \nabla_x$, we now have a highly-non-trivial system of great practical interest. It calls for QFT and path integral methods.

Indeed, our system is a set of (bosonic or fermionic) harmonic oscillators, labelled by $\vec{k}$ or $\vec{x}$. Let us introduce the universal notation index i for these possible further lables (spin, different particle types, etc. Thus,

$$H = H(a, a^\dagger) \qquad \text{with} \qquad a \equiv \{a_i\} \tag{10.42}$$

is our (normal-ordered) Hamiltonian. When we discussed the path integral for fermions, we learned how to treat such a system (both bosonic and fermionic) in the coherent-state picture and to write down a path integral. We now simply have to repeat this step by step.

The coherent states are

$$|\psi\rangle = e^{a_i^\dagger \psi_i}|0\rangle \qquad \text{with} \qquad a_i|\psi\rangle = \psi_i|\psi\rangle\,. \tag{10.43}$$

We are interested in the thermodynamics and hence in the partition function (although we could of course also condsider vacuum expectation values, as before, or other amplitudes):

$$Z = \mathrm{tr}\, e^{-\beta(\hat{H}-\mu\hat{N})} = \sum_n \langle n|e^{-\beta(\hat{H}-\mu\hat{N})}|n\rangle\,. \tag{10.44}$$

Here, for generality, we also introduced a chemical potential and the particle number operator. The latter is of course also a normal-ordered expression in a and $a^\dagger$, so we can treat it as an additonal piece to be added to $\hat{H}$ and nothing in the path integral derivation changes.

The crucial steps in the analysis are now the splitting of $e^{\beta(\cdots)}$ in many factors $e^{\Delta\beta(\cdots)}$ and the insertion of

$$\mathbb{1} = \int D\bar{\psi}D\psi e^{-\sum_i \bar{\psi}_i\psi_i}|\psi\rangle\langle\bar{\psi}|\,. \tag{10.45}$$

Following precisely the earlier derivation (but being careful with the i's and signs since we are now in the Euclidean theory), we find

$$Z = \int D\bar{\psi}D\psi\, e^{-S}\,, \tag{10.46}$$

with periodic or antiperiodic boundary conditions, depending on the type of particle, and

$$S = \int_0^\beta d\tau\,[\bar{\psi}\partial_\tau\psi + H(\psi,\bar{\psi}) - \mu N(\psi,\bar{\psi})]\,. \tag{10.47}$$

We recall that H and N appearing above have to be read as the corresponding original functions of a and $a^\dagger$, with $a \to \psi$ and $a^\dagger \to \bar{\psi}$.

Splitting off the interaction part of H, we have

$$S = \int_0^\beta d\tau\,\left[\bar{\psi}\partial_\tau\psi - \frac{1}{2m}\bar{\psi}(-\nabla^2)\psi \,+\, H_{int} - \mu N\right]\,. \tag{10.48}$$

This is an example of an action for a non-relativistic QFT, in this case with the LO field equation being the Schrodinger equation. One sees this immediately by varying the above free part of S w.r.t. $\bar{\psi}$. Of course, it is the imaginary-time Schrodinger equation, but it returns to the familiar one upon $\tau \to -it$:

$$-\frac{\partial}{\partial\tau}\psi = -\frac{\nabla^2}{2m}\psi \quad\longrightarrow\quad i\frac{\partial}{\partial t}\psi = -\frac{\nabla^2}{2m}\psi \quad\text{or}\quad \partial_t\psi = -i\left(-\frac{\nabla^2}{2m}\right)\psi\,. \tag{10.49}$$

In the way derived this, it is an effective description of many-particle QM. Hence, as opposed to our fundamental QFTs, the interaction can be non-local, i.e. $V(|\vec{x}-\vec{x}'|)$ does not have to be $\sim\delta^3(\vec{x}-\vec{x}')$. But is is also possible and very interesting to derive an NR-QFT as the low-energy limit of our relativistic theories, see e.g. Sect. 7.6 of [9] and Sect. III.5 of [31].

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