# 4 Path integral quantization of (non-abelian) gauge theories

## 4.1 General setting

- Recall first the basic structure introduced in Ch. 12 of QFT I:

  - Let $\psi(x)$ be some field transforming in some repr. of some matrix group $G$ (e.g. the fund. of $SU(N)$)

    $$\psi(x) \to U(x)\psi(x).$$

  - $D_\mu\psi \to U D_\mu \psi$ if $D_\mu \equiv \partial_\mu + iA_\mu$

    where $A_\mu \in \mathrm{Lie}(G)$ and $iA_\mu \to U(\partial_\mu U^{-1}) + U iA_\mu U^{-1}$.

  - We then introduced a field strength $F_{\mu\nu} \equiv \frac{1}{i}[D_\mu, D_\nu]$ transforming as $F_{\mu\nu} \to U F_{\mu\nu} U^{-1}$.

  - The lagrangian for $A_\mu$ is $\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$ (coupled to fermions through $\mathcal{L}_\psi = \bar\psi(i\not{D} - m)\psi$.)

  - In the non-abelian case, even the theory without $\psi$ is non-trivial (in contrast to ED). To see this, write

    $$A_\mu = A_\mu^a T^a \;;\quad F_{\mu\nu} = F_{\mu\nu}^a T^a \quad \& \text{ rescale } A_\mu \to gA_\mu.$$

- It follows that

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} \;;\quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - f^{abc} A_\mu^b A_\nu^c$$

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int.}$$

where $\mathcal{L}_0 = -\frac{1}{4}\underbrace{[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a]}_{\text{like ED, but with dim}(G)\text{ fields}}[\cdots]$ ; $\mathcal{L}_{int} \supset A^3$ & $A^4$ terms.

- Thinking of $A_\mu^a$ as of a set of bosonic fields, we are tempted to appeal to our detailed discussion of the scalar field and to write

$$\langle T A_\mu(x_1) A_\nu(x_2) \cdots \rangle = \frac{\int DA\, A_\mu(x_1) A_\nu(x_2) \cdots e^{iS}}{\int DA\, e^{iS}}$$

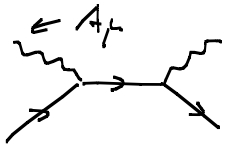
$$\left( S = \int d^4x \mathcal{L} \right).$$

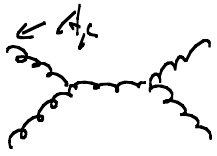
From this, we would like to derive scattering amplitudes of the non-abelian gauge bosons (e.g. gluons in the case $G = SU(3)$).

- An immediate problem that we are facing is gauge invariance: Phys. scatt. amplitudes are gauge-inv., while $T A_\mu(x_1) \cdots$ isn't. In ED, we dealt with this by looking only at the scattering of phys. photons:
  - Fourier-transforming w.r.t. $x_1$
  - Setting the mom. introduced in this way "on shell" ($k^2 = 0$)
  - contracting with a phys. polarization:

$$\tilde{A}_\mu(k) \to \varepsilon^\mu(k) \tilde{A}_\mu(k).$$

- In QCD, this is also done, but it's not quite sufficient: It is necessary to set <u>all other</u> particles in a given amplitude on shell & to contract with phys. polarizations to ensure gauge-inv. w.r.t. a given ext. gluon.

(cf. <u>QED</u>: 

$+ \cdots$ ; <u>QCD</u>: 

$+ \cdots$)

This is a result of more complicated structure of diagrams (gluon-gluon-interactions).

- Even if this is done, gauge-inv. is only ensured "to leading order":

- Recall: $iA_\mu \longrightarrow U(\partial_\mu U^{-1}) + U i A_\mu U^{-1}$

  $\Downarrow$ infinitesimally, with $U = e^{-i\chi} \approx 1 - i\chi$

  $A_\mu \longrightarrow A_\mu + \partial_\mu \chi - i[\chi, A_\mu]$

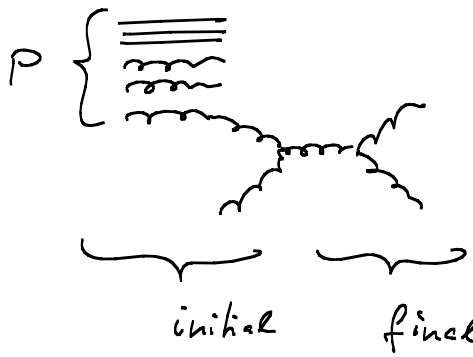
  $\Downarrow$ in pert. theory, it is natural to rescale $A_\mu \to g A_\mu$ (and correspondingly $\chi \to g\chi$)

  $A_\mu \longrightarrow A_\mu + \partial_\mu \chi - ig[\chi, A_\mu]$

  This extra (w.r.t. QED), higher-order piece does not disappear by going on shell & using phys. polarizations.

- In fact, this gauge-inv. problem of scattering amplitudes with gluons has a deeper phys. origin: gluons are not the right incoming/outgoing states; they are <u>confined</u> in color neutral states (nucleons, mesons, glueballs).

- Pragmatically speaking, the LHC (& other such machines) are nevertheless (to a very large extent) machines for scattering gluons. However, one needs to characterize the incoming protons as collections of very energetic gluons & sum over all final states (in a certain well-defined way, e.g. defining jets).



- Quantitatively, this is encoded in the so-called "gluon distribution function" (e.g. of a proton), to be defined in more detail later. This quantity replaces the free incoming gluon and has to be defined (and, in principle, calculated) order by order in $g$. It is only in this procedure that gauge-invariance for incoming gluons is eventually fully recovered.

- Anticipating this procedure of
  - convoluting cross-sects. with quark/gluon (parton) distributions
  - & using jet definitions / fragmentation fcts. / complete final state sums for the outgoing partons,

we will now simply assume that it is sufficient for us to understand how to calculate

$$\langle 0|\, T\, O[\hat{A}]\, |0\rangle = \frac{\int DA\; O[A]\, e^{iS[A]}}{\int DA\; e^{iS[A]}},$$

where $O$ is a gauge-inv. expression (an observable) of our non-abelian gauge-th. [We will apply this to $T A_{\mu_1}(x_1) A_{\mu_2}(x_2) \cdots$, which is justified with the caveats explained above.]

- A field-theoretically simpler and cleaner approach would be to directly consider the scattering of color neutral objects,

i.e. $T A_{\mu_1}(x_1) A_{\mu_2}(x_2) \cdots \longrightarrow T\, C(x_1) C(x_2) \cdots$

($C$: some operator built from $A$'s which is gauge-inv. ...)

... and has non-vanishing overlap with a glueball:

$$\langle 0 | C(x) | \text{glueball} \rangle \neq 0.$$

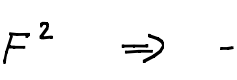
- One can in principle properly normalize this operator (calculate the $Z$-factor, as we did for the fundamental fields) and extract scatt. amplitudes from green-fcts. for $C(x)$ à la LSZ. This might even be practically doable for certain heavy mesons ($b\bar{b}$-bound states). It is certainly not doable for glueballs, protons etc. since for such "light" objects pert. theory breaks down. Thus, even though

$$O[A] = T\, C(x_1)\, C(x_2) \cdots$$

is a nice way to use our (to be described) procedure for calculating non-abelian functional integrals, it is not the most relevant one practically.

- Let us finally give an even simpler (and, with a few important exceptions, even more useless) gauge-inv. observable for testing non-abelian gauge theories:

$$-\text{ consider } \mathcal{L}_{\text{gauge-th.}} + \underbrace{\mathcal{L}_\varphi}_{\text{free real scalar}} + \mathcal{L}_{\text{int}}$$

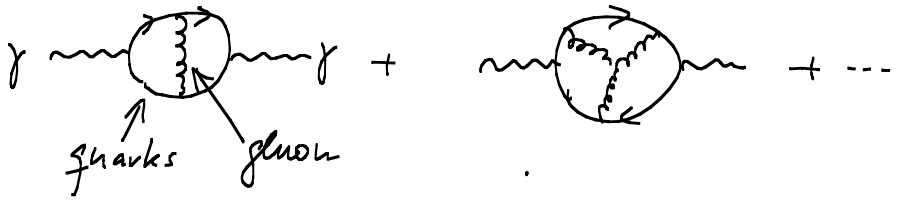
with, e.g., $\mathcal{L}_{\text{int.}} = \lambda \varphi\, \mathrm{tr}\, F^2 \Rightarrow$  + other vertices

- There are now obviously gauge-inv. observables corresponding to $\varphi$-$\varphi$-scattering, induced by the coupling of $\varphi$ to gluons:

$$\langle T \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle \sim$$ $+ \cdots$

with $\langle T\varphi_1 \cdots \varphi_4 \rangle \sim \int \mathcal{D}\varphi \mathcal{D}A\, \varphi_1 \cdots \varphi_4\, e^{iS} \Big/ \int \mathcal{D}\varphi \mathcal{D}A\, e^{iS}$.

- This very simple situation does in fact arise in practice since, e.g., QCD interactions correct the photon self-energy (and many processes involving photons):




<u>In summary</u>: It makes sense and has many phys. applications to calculate

$$\int \mathcal{D}A\, \mathcal{O}(A)\, e^{iS[A]}$$

for gauge-inv. functionals $\mathcal{O}$ of the gauge field $A_\mu(x)$.

## 4.2 Faddeev-Popov method

- We immediately anticipate a fundamental problem in the evaluation of funct. integrals of the type given above (i.e. $\int \mathcal{D}A$ with gauge-inv. integrand):
  - recall $iA_\mu \to iA_\mu^\alpha = U(\partial_\mu U^{-1}) + iUA_\mu U^{-1}$ with $U = e^{-ig\alpha}$
  - think of $\int \mathcal{D}A$ as $\int \underbrace{\mathcal{D}A}_{\text{all } A \text{ not related by a gauge trf.}} \mathcal{D}\alpha$.
  - obviously, $\int \mathcal{D}\alpha$ will be divergent since the integrand does not depend on $\alpha$.
  - also obviously, this problem is an artifact of pert. theory, since really the gauge group is

compact ($U(1) \sim S^1$, $SU(2) \sim S^3$, etc.), i.e. for sufficiently large $\alpha$ the gauge trf. is trivial (just like $e^{2\pi i} = 1$). However, in pert. theory we only think in terms of Taylor expansions of all fields about the (trivial) vacuum ($A_\mu = 0$). Thus, we never see the compactness of the gauge group.

- Note: In a non-pert. definition, e.g. on the lattice, the problem is not present and $\int DA$ can be (numerically) evaluated without worrying about the gauge freedom.

- Note: Technically, we already encountered the problem in QED: If we write

$$iS[A] = -A_\mu (D^{-1})^{\mu\nu} A_\nu ,$$

it turns out that $(D^{-1})^{\mu\nu}$ has zero-eigenfcts and is hence not invertible. We fixed this by choosing a gauge (and we will do so again, just in the path-int. approach).

- We will "fix the gauge" by demanding $G(A) = 0$ for some (non-gauge-inv.) fct. of $A$. A typical fct. is

$$G(A) \equiv \partial^\mu A_\mu - \omega$$

($G(A)$: this is Lie-alg. valued!; $\omega$: any Lie-alg.-valued fct.)

- Use $1 = \int D\alpha \, \delta[G(A^\alpha)] \left| \det \left( \frac{\delta G(A^\alpha)}{\delta \alpha} \right) \right|$

in analogy to $1 = \int dx\, \delta(f(x))\,|f'(x)|$.

(Note: By its definition via

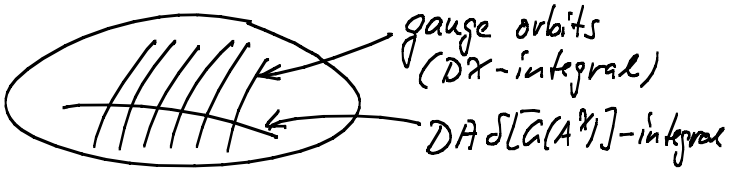
$$\delta G(A^\chi)(x) = \int d^4y \cdot \frac{\delta G(A^\chi)(x)}{\delta \chi(y)} \cdot \delta\chi(y) + \cdots$$

$\frac{\delta G(A^\chi)}{\delta \chi}$ is an operator in the space of Lie-alg.-valued fcts. and its det. is well-defined (with appropriate UV/IR-regularization).)

- Hence:

$$\langle T\, O(A)\rangle = \frac{\int D\chi \int DA\, \delta[G(A^\chi)]\left|\det\left(\frac{\delta G(A^\chi)}{\delta \chi}\right)\right| O(A)\, e^{iS[A]}}{\int D\chi \int DA\, \delta[G(A^\chi)]\left|\det\left(\frac{\delta G(A^\chi)}{\delta \chi}\right)\right| e^{iS[A]}}$$

Intuitive: Space of all gauge-field configs.:




- We now focus on the numerator (The denominator follows for $O(A) = 1$).
- Use $O(A)\, e^{iS[A]} = O(A^\chi)\, e^{iS[A^\chi]}$.
- Use $\int DA = \int DA^\chi \left|\det\left(\frac{\delta A^\chi}{\delta A}\right)\right|^{-1} = \int DA^\chi$

at linear order in $\chi$, this has the form $\mathbb{1} + M$, with $M$ an antisymm. matrix. Thus, $\det(\mathbb{1}+M) = 1$ at lin. order in $\chi$. For finite gauge-trfs. which can be built by many infinitesimal gauge-trfs. (and we consider only those!) it then follows that

$$\det\left(\frac{\delta A^\chi}{\delta A}\right) = 1 .$$

(Problem: Proof the above by calculating $\frac{\delta A^\chi}{\delta A}$ explicitly at lin. order in $\chi$.)

$$\Rightarrow \langle T\,O(A)\rangle = \frac{1}{Z[0]} \int DX \int DA^X\, \delta[G(A^X)] \underbrace{\left|\det\left(\frac{\delta G(A^X)}{\delta X}\right)\right|}_{= \left|\det\left(\frac{\delta G(A^{X+X'})}{\delta X'}\right)\right|_{X'=0}} O(A^X)\, e^{iS[A^X]}$$

- rename $A^X \to A$
- drop $\int DX$ (just a divergent prefactor)
- rename $X' \to X$

$$\Rightarrow \langle T\,O(A)\rangle = \frac{1}{Z[0]} \int DA\, \delta[G(A)] \left|\det\left(\frac{\delta G(A^X)}{\delta X}\right)\right|_{X=0} O(A)\, e^{iS[A]}$$

- <u>We are now, in principle, finished:</u>
  The gauge-freedom is fixed; Our path-int. is well-defined (as far as gauge-freedom is concerned).
- <u>At a technical level, two issues remain:</u>
  ① Working with fields satisfying $\partial A = \omega$ is inconvenient.
  ② $\det(\delta G/\delta X)$ has to be calculated.

① Since nothing really depends on $G = \partial A - \omega$ (and hence on $\omega$), we can multiply by $\int D\omega\; e^{-i\frac{\lambda}{2}\omega^2}$ ($\uparrow \int d^4x\, \omega(x)^2$).

The $\omega$-integration can then be performed using $\delta[\partial A - \omega]$, giving

$$\langle T\,O(A)\rangle = \frac{1}{Z[0]} \int DA \left|\det\left(\frac{\delta G(A^X)}{\delta X}\right)\right|_{X=0} O(A)\, e^{i\left(S[A] - \frac{\lambda}{2}(\partial A)^2\right)}$$

($\uparrow$ "gauge-fixing term")

② To evaluate the "Fadeev-Popov-determinant" $\det\left(\frac{\delta G}{\delta X}\right)$,

we use $\det M = \int D\bar\theta D\theta\, e^{-\bar\theta M\theta}$.

- We have $(A_\mu^\chi)^a = A_\mu^a + \partial_\mu \chi^a + g f^{abc}\chi^b A_\mu^c$ (check this!)

and hence

$$\frac{\delta G(A^\chi)^a}{\delta \chi^d} = \frac{\delta}{\delta \chi^d}\left(\partial A^a + \partial^2\chi^a + g f^{abc}\partial^\mu(\chi^b A_\mu^c)\right)$$

and $$M(x,y)^{ad} = \delta^{ad}\partial^2\delta^4(x-y) + g f^{adc}\partial_x^\mu(\delta^4(x-y)A_\mu^c(x)).$$

- Calling the (unphysical!) fermionic fields ("ghosts") introduced in this procedure $c$ & $\bar c$, we have

$$\det\left(\frac{\delta G}{\delta\chi}\right) = \int D\bar c Dc\, \exp\Big[-i\int d^4x d^4y\, \bar c^a(x)\{\delta^{ad}\partial^2\delta^4(x-y) + g f^{abc}\partial_x^\mu(\delta^4(x-y)A_\mu^c(x))\} \cdot c^b(y)\Big].$$

- Thus, we have found

$$\mathcal{L}_{ghost} = \bar c^a\left[-\delta^{ab}\partial^2 - g f^{abc}\partial^\mu A_\mu^c\right]c^b$$

($\partial^\mu$ acting on both $A$ & $c$.)

$$= -\bar c\,\partial^\mu D_\mu c$$

(Here we abbreviate $\bar c^a c^a$ by $\bar c c$.)

$$D_\mu c \equiv \partial_\mu c + [iA_\mu, c]$$

[This is the proper covariant derivative for a "field in the adjoint repres.", like $c$. To see this, not that under gauge trfs. $c = c^a T^a \longrightarrow e^{-ig\chi} c\, e^{ig\chi} = c - ig[\chi, c] + \cdots$. The gauge trf. $A_\mu \to A_\mu + \partial_\mu\chi + \cdots$ compensates the term arising from $\partial_\mu$ acting on the $\chi$ of $[\chi, c]$. However, our ghost lagrangian is still not gauge-inv. because of the extra $\partial^\mu$.] Note: Using this concept of the covariant derivative of an adjoint field, one also has $A_\mu^\chi = A_\mu + D_\mu\chi$.

Problem: Show that the lagrangian $\mathcal{L} = -\frac{1}{2}\mathrm{tr}(D_\mu\varphi D^\mu\varphi)$ of a physical (bosonic) adjoint scalar is gauge-inv. and work out the Feynman-rules (vertices) for its coupling to $A_\mu^a$.

Finally:

$$\langle T\,O(A)\rangle = \frac{1}{Z[0]}\int DA\,D\bar{c}\,Dc\;O(A)\,e^{iS[A]+iS_{gf}[A]+iS_{gh}[\bar{c},c,A]}$$

where $\mathcal{L}_{gf} = -\frac{\lambda}{2}(\partial A)^2$ (covariant gauges, parameterized by $\lambda$)

& $\mathcal{L}_{gh} = -\bar{c}\,\partial^\mu D_\mu c$ .

## 4.3 Feynman rules

- We now introduce sources $j^{\mu a}$, $\bar{\eta}^a$, $\eta^a$ for $A_\mu^a$, $c^a$, $\bar{c}^a$:

$$S \to S + \int d^4x\left(A_\mu^a j^{\mu a} + \bar{\eta}^a c^a + \bar{c}^a\eta^a\right),$$

define a generating functional $Z[j^{\mu a}, \eta^a, \bar{\eta}^a]$

as $Z = \int DA\,D\bar{c}\,Dc\,\exp i\left[S + Aj + \bar{\eta}c + \bar{c}\eta\right]$,

write $S = S_0 + S_{int}$

($S_0$: quadratic in $A$ & $\bar{c}, c$; $S_{int}$: rest),

define $Z_0$ on the basis of $S_0$,

evaluate $Z_0[j,\eta,\bar{\eta}] = Z_0[0]\exp\left[-\frac{1}{2}AD^{-1}A - \bar{c}D_g c\right]$,

and derive Feynman rules for the calculation of green's fcts. (note that we will never need external ghosts!) from

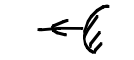
$$Z[j,\eta,\bar\eta] = \exp\left[iS_{int.}\left(\frac{\delta}{i\delta j}, \frac{\delta}{-i\delta\eta}, \frac{\delta}{i\delta\bar\eta}\right)\right] Z_0[j,\eta,\bar\eta].$$

$\uparrow$ instead of $A, \bar c, c$

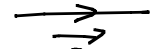
(Always remember that $c, \bar c$, and hence $\eta, \bar\eta$, are scalars & Grassmann variables ("fermions").

Note that this is not in contradiction with the Spin-Statistic-theorem since the ghosts are not phys. particles: We will always set $\eta, \bar\eta$ to zero in $Z$, so that ghosts appear only in loops.)

The result (including also fermions in the fund. repres.), given directly in momentum space, reads: (for SU(N))

- incoming fermion $u(p)_i$ ← "color" index
  outgoing $-\!\!-$ $\bar u(p)_i$
  incoming anti-ferm. $\bar v(p)_i$
  outgoing $-\!\!-$ $v(p)_i$
- inc./outg. gauge boson  $\varepsilon_\mu(k)^a$ / $\varepsilon_\mu^*(k)^a$

yet to be derived!

- fermion propagator  $\dfrac{i\delta_{ij}}{\not p - m + i\varepsilon}$
- gauge boson prop. $i\delta^{ab}\,\dfrac{-\eta_{\mu\nu} + (1-\xi)k_\mu k_\nu/k^2}{k^2+i\varepsilon}$

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Note!

$$-\frac{1}{2}\,\mathrm{tr}\,F^2 \;\to\; -\partial_\mu A_\nu\,\partial^\mu A^\nu + \partial_\mu A^\mu \partial_\nu A^\nu$$

$$-\frac{\lambda}{2}(\partial A)^2 \;\to\; -\lambda\,\partial_\mu A^\mu \partial_\nu A^\nu\,,$$

which combines to an inverse prop. with the structure

$$k^2\eta_{\mu\nu} - k_\mu k_\nu(1-\lambda)$$

with inverse $A\eta^{\mu\nu} + B k^\mu k^\nu$ ,

$$\Big( Ak^2 \eta_\mu{}^\nu + \big(k^2 B - A(1-\lambda) - k^2 B(1-\lambda)\big) k_\mu k^\nu \overset{!}{=} \eta_\mu{}^\nu$$

$$\Rightarrow Ak^2 = 1 \;\; ; \;\; \lambda k^2 B - A(1-\lambda) = 0 \Big)$$

i.e. $$\frac{\eta^{\mu\nu}}{k^2} + \left(\frac{1-\lambda}{\lambda}\right) \frac{k^\mu k^\nu}{(k^2)^2} \; .$$

With $\xi = \frac{1}{\lambda}$ this is the desired result.

( $\xi = 1$ – Feynman gauge ; $\xi = 0$ – Landau gauge )

Lorentz or covariant gauges

Note: Recall that $\lambda$ emerged from

$$\int D\omega \, e^{-i\frac{\lambda}{2}\omega^2} \; ,$$

where $\omega = \partial_\mu A^\mu$ was the gauge fixing condition.

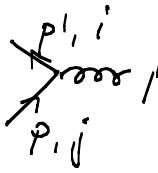
Hence $\lambda \to \infty$ (or $\xi \to 0$) corresponds to $\partial_\mu A^\mu = 0$. Thus, the Landau gauge is the only "truely gauge-fixed gauge" among the covar. gauges. The others are in some sense "averaged", because of the non-trivial averaging over $\omega$. This special role manifests itself in the fact that the Landau-gauge propagator is transverse:

$$(\eta^{\mu\nu} - k^\mu k^\nu / k^2) k_\mu = 0 \; ,$$

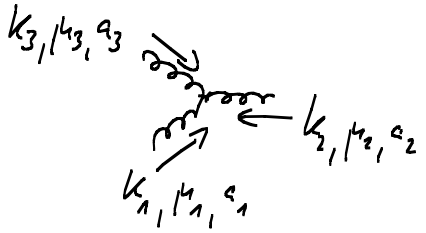
i.e. longit. gauge bosons don't propagate.

- ghost prop. $\quad --\gg-- \quad \dfrac{i\delta^{ab}}{p^2+i\varepsilon}$

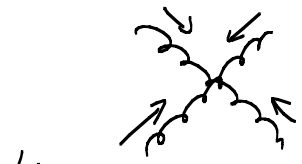
(remember the minus-sign for each fermion loop; derive this minus-sign in the path-int. approach!)

- fermion-gluon vertex 

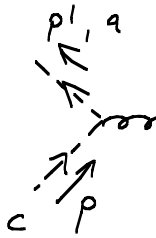
$-ig\gamma_\mu T^a_{ij}$

- 3-gluon-vertex 

$-gf^{a_1a_2a_3}\cdot$
$\{(k_1-k_2)_{\mu_3}\eta_{\mu_1\mu_2}$
$+ \text{cyclic.}\}$

- 4-gluon-vertex 

$\Big[ig^2 f^{a_1a_2b} f^{a_3a_4b}\cdot$
$\{\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3} - \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4}\}$
$+ \text{cyclic permutations in } 1,2,3\Big]$

- ghost-gluon-vertex 

$gf^{abc}\, p'_\mu$

Problem: Derive them, especially the 3- & 4-gluon vertices!

Note: All of this, including the ghosts, applies trivially to the abelian case. However, the ghosts don't interact ($no\ f^{abc}$'s!).