

5 BRST Symmetry & Physical Hilbert Space

If, as we explained in Sect. 5, we really only want to calculate correlation fcts. of gauge-inv. operators, the treatment of Sect. 5 is completely sufficient. We don't need to understand in detail what the Hilbert space of non-abel. gauge theories looks like. But, as we also explained in Sect. 5, it sometimes useful to think of "scattering gluons".

(Imagining, e.g., that we live on a T^3 which is so small that confinement effects don't play a role.) Even though gauge-invariance of such amplitudes can only be established order-by-order in g , they are nevertheless useful technical objects to discuss, e.g., LHC experiments. For this, and for gaining more conceptual insight, we need to know how to characterize phys. gluon states. The outcome will be what we naively expect: As we learned from Gupta-Bleuler in QED, the ext. gluons have to be transversely polarized. (& no ext. ghosts are allowed).

5.1 BRST symmetry (Becchi, Rouet, Stora, '76 - Ann. Phys. Tyutin '75 - Lebedev Inst. preprint.)

Let us rewrite our gauge-fixed lagrangian with ghosts as

$$\mathcal{L} = -\frac{1}{2} \text{tr } F^2 + \frac{1}{2\lambda} B^a B^a - B^a \partial_\mu A^{a\mu} - \bar{C}^a \partial^\mu D_\mu C^a + \bar{\psi} (i \not{D}_m) \psi$$

introducing also a further integration over an adjoint boson B . This boson is an auxiliary field (no kinetic term) and the integral can be done trivially, in analogy to

$$\int dx e^{-\frac{d}{2}x^2 - \beta x} = \int dx e^{-\frac{d}{2}(x - \beta/d)^2} e^{-\beta^2/2d}$$

$$= \int dx^1 e^{\frac{d}{2}x^1{}^2} e^{-\beta^2/2x} \sim e^{-\beta^2/2x}.$$

Promoting $\int dx^1$ to a path int. $\int D\delta$ and with $\beta \rightarrow i\partial A$; $x \rightarrow \frac{i}{\lambda}$, our original Lagrangian follows.

Note: A simpler (equivalent) way to deal with such auxiliary fields is to "integrate them out" by solving their EOMs and plugging the result back into the Lagrangian:

$$\delta \left(\frac{\delta^2}{2\lambda} - \delta \partial A \right) = \frac{\delta}{\lambda} - \partial A \stackrel{!}{=} 0 \Rightarrow \delta = \lambda \partial A$$

$$\frac{\delta^2}{2\lambda} - \delta \partial A \xrightarrow{\delta = \lambda \partial A} -\frac{\lambda}{2} (\partial A)^2$$

This "integrating out" also works for fields which are non-dynamical because they are too heavy

We now make the crucial claim that the tr. δ_ϵ (with a Grassmann-parameter ϵ)

$$\delta_\epsilon A_\mu = \epsilon D_\mu c$$

$$\delta_\epsilon \psi = -ig \epsilon c \psi \quad ((c\psi)_i = c^a T_{ij}^\alpha \psi_j)$$

$$\delta_\epsilon c^a = \frac{1}{2} g \epsilon f^{abc} c^b c^c$$

$$\delta_\epsilon \bar{c} = -\epsilon b$$

$$\delta_\epsilon b = 0$$

is a symmetry of our action (in the form given above).

We now check this claim:

- Observe first that the infinites. gauge-hf. of a gauge field

$$\delta_X A_\mu = \partial_\mu X - i[X, gA_\mu]$$

can be written as

$$\delta_X A_\mu = D_\mu X \quad (\text{treating } X \text{ as a field transforming in the adjoint repres.}).$$

- The fund.-repres. fermions transform as $\psi \rightarrow e^{-igX} \psi$, i.e.

$$\delta_X \psi = -igX\psi.$$

- Hence the first two lines of the BRST hf. are just gauge-hfs. of A_μ & ψ with $X = gec$. The F^2 & $\bar{\psi}(iD_m)\psi$ terms are thus obviously invariant.

- The $\frac{1}{2\lambda} B^a B^a$ - term is trivially invariant.

- Now we only need to check the invariance of

$$B^a \partial_\mu A^a + \bar{c}^a \partial^\mu D_\mu c^a = 2\text{tr}(B \partial_\mu A + \bar{c} \partial^\mu D_\mu c).$$

- Since $\delta_e A_\mu = e D_\mu c$ & $\delta_e \bar{c} = -eB$, the variations of A_μ in the first term and of \bar{c} in the second term obviously cancel.
- We now only need to consider

$$\delta_e(D_\mu c).$$

Note that, equivalently the formulae given above,

$$\begin{aligned} \delta_e c &= -i\frac{1}{2}g\epsilon\{c, c\} \quad (\text{This is not zero since} \\ &= -ig\epsilon c^2 \quad \{c, c\} = \{T^a c^a, T^b c^b\} = [T^a, T^b] c^a c^b) \end{aligned}$$

- Now: $\delta_\epsilon(D_\mu c) = ig[\delta_\epsilon A_\mu, c] + D_\mu \delta_\epsilon c$

$$= ig[\epsilon D_\mu c, c] - ig\epsilon D_\mu c^2$$

$$= ig\epsilon (\{D_\mu c, c\} - [D_\mu c, c + c(D_\mu c)])$$

$$= 0 \quad \checkmark$$

Problem: Do this last calculation explicitly (keeping the f^{abc} of $\delta c^a = \frac{1}{2}g\epsilon f^{abc}c^bc^c$).

5.2 BRST-operator

- We have defined δ_ϵ as a class. symm. of a Lagrangian system. This implies a corresponding symm. of the equivalent Hamiltonian system. This symm. is generated by a certain observable via the Poisson bracket. We call this observable Q and its operator version after quantization \hat{Q} . (They are known as BRST operator or BRST charge.)
- Equivalently, we can appeal to the Noether theorem and define a current j^μ associated with δ_ϵ . Q is then given by $\int d^3x j^0$, as usual.

Note: We will always write Q and let the reader to think of " \hat{Q} " whenever appropriate.

- It is, in principle, interesting and important to define Q explicitly in terms of the creation/annihilation operators of the canonically quantized theory corresponding to $S[A_\mu, \psi, \bar{\psi}, \varsigma, \bar{\varsigma}, b]$. A lot of what follows can then be done more carefully and correctly. We refer the reader to Kugo/Ojima, Progr. Theor. Phys. 60: 1863 ('78) & 61: 284 ('79) and Baulieu, Phys. Rept. 129: 1 ('85).

- A quick way to make the operator Q defined above more explicit goes as follows:
 - Let $O_t |0\rangle$ (with O_t some operator in the Heisenberg picture) be a state in the Hilbert space.
 - The overlap with $O_t' |0\rangle$ is given by

$$\int D\varphi e^{iS} O_t'[\varphi] O_t[\varphi] / \int D\varphi e^{iS},$$
 where $\varphi = \{A_\mu, \psi, \bar{\psi}, c, \bar{c}, \epsilon\}.$
 - We now define the corresponding amplitude for the transformed state $(1+i\in Q) O_t |0\rangle$ by

$$\int D\varphi e^{iS} O_t'[\varphi] (1 + \delta_\epsilon) O_t[\varphi] / \int D\varphi e^{iS}.$$
 - Hence " $i\in Q \equiv \delta_\epsilon$ " and it is sufficient to know how δ_ϵ acts on classical fields, which we have already defined.
 (As an example, think of spatial translations by $\bar{\epsilon}$ acting on $\psi(t, \bar{x}) |0\rangle$. The result is clearly $\psi(t, \bar{x} + \bar{\epsilon}) |0\rangle$, in other words, we just let the transformation act on the Heisenberg-picture-field generating the state.)

$$\boxed{\text{Crucial fact: } Q^2 = 0 \quad (\text{Q is nilpotent.})} \quad \boxed{(\text{Both with \& without "n"})}$$

According to the above, it is sufficient to check that $\delta_\epsilon \delta_\epsilon^\dagger = 0$:

- $\delta_\epsilon \delta_\epsilon^\dagger = 0 \Rightarrow \delta_{\epsilon'} \delta_\epsilon^\dagger \delta_\epsilon = 0 \quad \checkmark$
- $\delta_\epsilon \bar{c} = -\epsilon c \Rightarrow \delta_{\epsilon'} \delta_\epsilon^\dagger \bar{c} = \delta_{\epsilon'} (-\epsilon c) = -\epsilon \delta_{\epsilon'}^\dagger c = 0 \quad \checkmark$
- $\delta_\epsilon c = -ig \epsilon c^2 \Rightarrow \delta_{\epsilon'} \delta_\epsilon^\dagger c = -ig \epsilon \delta_{\epsilon'}^\dagger c^2 = (-ig)^2 \epsilon [(\epsilon' c^2) c + c (\epsilon' c^2)] = (-ig)^2 \epsilon \epsilon' [c^3 - c^3] = 0 \quad \checkmark$

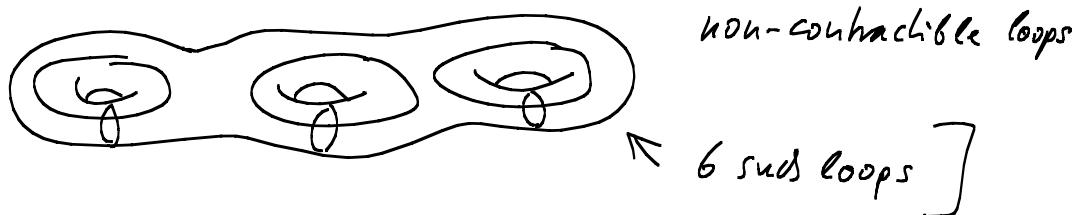
Problem: Complete this analysis, including the proof of $\{Q^2 \varphi = 0 \Rightarrow Q \varphi^n = 0\}$.

5.3 The physical Hilbert space

- Let \mathcal{H} be the full Hilbert space arising in the quantization of $\mathcal{L}(A_\mu, \Psi, c, \bar{c}, b)$.
- Define $\text{Ker } Q \subset \mathcal{H}$ by $|\Psi\rangle \in \text{Ker } Q \Leftrightarrow Q|\Psi\rangle = 0$.
(it is a lin. subspace)
- Define $\text{Im } Q \subset \mathcal{H}$ by $|\Psi\rangle \in \text{Im } Q \Leftrightarrow \exists |\Psi'\rangle \text{ s.t. } |\Psi\rangle = Q|\Psi'\rangle$.
(also a lin. subspace).
- $Q^2 = 0 \Rightarrow \text{Im } Q \subset \text{Ker } Q$.
- $\mathcal{H}_{\text{phys.}} \equiv \text{Ker } Q / \text{Im } Q$ (Quotient space)

[This construction, known as the "cohomology of Q " works for any lin. operator which squares to zero. In particular, $d : n\text{-forms} \rightarrow (n+1)\text{ forms}$; $d^2 = 0$ gives the n -th cohomology H^n of the underlying compact manifold. It characterizes the topology of the manifold, e.g.

$$\dim H^1 \text{ (2-dim. manifold = Riem. surface)} = \# \text{ of "indep." ,}$$



- In more detail:

$$\mathcal{H}_{\text{phys.}} = \{ \text{equivalence classes of elements of } \text{Ker } Q \}$$

$$\text{where } |\Psi_1\rangle \sim |\Psi_2\rangle \Leftrightarrow |\Psi_1\rangle - |\Psi_2\rangle \in \text{Im } Q.$$

We define the lin. structure (adding & multiplying by compl. numbers) & scalar prod. using "representatives" of the relevant classes.

One needs to proof that these operations do not depend on the choice of the representative. For the linear structure this follows from the linearity of Q . For the scalar product, we need that Q is hermitian:

$$\text{let } |\psi_1\rangle \in \ker Q \quad \& \quad |\psi_2\rangle \in \text{Im } Q. \text{ Then } \langle \psi_1 | \psi_2 \rangle = \\ = \langle \psi_1 | Q\psi' \rangle = \langle \psi_1 Q | \psi' \rangle = 0 \text{ since } Q|\psi_1\rangle = 0.$$

Note: The hermiticity of Q is a slightly controversial issue, related to the hermiticity properties of the ghosts and the "Nakanishi-Lautrup" field θ . In my opinion, the most careful treatment is that of Kugo/Ojima and Weinberg (vol. II, ch. 15.7). In an alternative perspective, one also has an "anti-BRST" tf. (related to the herm. conj. of Q). See Baulieu & the textbook of Pokorný for this perspective.

- Thus, we have: States from $\text{Im } Q$ have vanishing overlap with states from $\ker Q$ (\equiv phys. states). This implies, of course, that they have zero norm. Thus, they must correspond to residual gauge freedom.

- One can also show

$$\delta_{\text{change-of-gauge-fixing}} \langle f | i \rangle = \langle f | \{ Q, \dots \} | i \rangle \\ = 0 \text{ if } |f\rangle \& |i\rangle \text{ are phys.}$$

This is consistent with our definition " $\{\text{phys. states}\} = \ker Q$ ".

[cf. Weinberg II and Polchinski I, ch. 4.2, and Kugo/Uehara NPB 197
p. 378, '82.]

- Finally, we would like to understand our phys. states more explicitly, in terms of creation/annih. operators of the various fields:

- To do so, we first note that:

$$\text{class.: } \delta_{\epsilon} A_{\mu} = \epsilon \partial_{\mu} c \quad \Rightarrow \quad \text{quant.: } i [Q, A_{\mu}] = \partial_{\mu} c \\ \text{etc.} \qquad \qquad \qquad \text{etc.}$$

- Thus, we can continue to use our class. BRST-hgr., thinking of the operator fields acting on the vacuum. (Alternatively, we may think of $\int D\varphi e^{iS} \dots f[\varphi] \sim \langle 0 | \dots f[\varphi] | 0 \rangle$ with Q acting classically on φ .)
- Furthermore, we focus on $t \rightarrow \pm \infty$ where we assume $g \rightarrow 0$:

$$\left[\begin{array}{lll} \delta_{\epsilon} A_{\mu} = \epsilon \partial_{\mu} c & \delta_{\epsilon} c = 0 & \delta_{\epsilon} b = 0 \\ \delta_{\epsilon} \varphi = 0 & \delta_{\epsilon} \bar{c} = -\epsilon b \end{array} \right]$$

- We see that $Qf = 0$ obviously excludes anti-ghosts.
- Furthermore, thinking in momentum space, we have

$$Q \epsilon^{\mu}(k) A_{\mu}(k) = \epsilon^{\mu}(k) k_{\mu} c(k)$$

Hence, unphys. polarizations ($\epsilon^{\mu}(k) k_{\mu} \neq 0$) are also forbidden.

- So far, we are still allowed to have functionals f which depend on $\epsilon^{\mu}(k) A_{\mu}(k)$; $c(k)$; $b(k)$.
- Using EOM (which is Ok under the path-int.), we can replace b by $A_{\mu}(k) k^{\mu}$.
- Thus, the presence of a dependence on $\partial_{\mu} A^{\mu}$ & c remains allowed, but it can be removed by adding contributions from $\text{Im } Q$ (which contains ghosts & the b -field).

$\Rightarrow \text{Ker } Q / \text{Im } Q$ is the right definition of $\mathcal{H}_{\text{phys}}$.

It would be useful to go ahead and construct $\mathcal{H}_{\text{phys}}$ explicitly

Using creation ops. for c & A_μ and, in particular, to demonstrate that the norm is pos. definite on $\mathcal{H}_{\text{phys}}$. This can be done (in analogy to the Gupta-Bleuler construction in QED).

→ Kugo / Ojima '79

- The crucial implication of the above is that, since Q commutes with H , we can construct a phys. state in the far past (e.g. two transverse gluons, $g=0$).
- We then "switch on" g , let them scatter, and switch g off again. The resulting final state will again by physical since time-evolution commutes with the construction of $\mathcal{H}_{\text{phys}}$ in terms of \mathcal{H} .

Note: In principle, the same argument is needed in QED. However, in QED a shortcut exists which makes the BRST construction unnecessary:

- Observe that our statement (that the dynamics at $g \neq 0$ does not take us outside $\mathcal{H}_{\text{phys.}}$) is equivalent to the unitarity of $S_{g.f.}$ [\equiv gauge-fixed scattering matrix] restricted to the phys. states. Thus, if P is the projector on phys. states, we need to show that

$$(PS_{g.f.} P)(PS_{g.f.} P)^+ = P$$

- Indeed

$$(PS_{g.f.} P)(PS_{g.f.} P)^+ = PS_{g.f.} P S_{g.f.}^+ P \quad (P^\dagger = P, P^2 = P)$$

↑
focussing on a matrix element M^μ with
one ext. photon, this means

$$\sum_{i=1,2} \sum_{\nu} \epsilon_{i\nu}^+ \epsilon_{i\nu} M^\mu M^{*\nu}$$

- Choosing $k' = (k^0, \vec{0}_\perp, \vec{k})$, we have

$$\begin{aligned} -g_{\mu\nu} M^\mu M^{*\nu} &= -(M^0)^2 + (M^1)^2 + (M^2)^2 + (M^3)^2 \\ &= (M^1)^2 + (M^2)^2 \quad (\text{using } k_\mu M^\mu = 0) \\ &= \sum_{i=1,2} \varepsilon_{i\mu}^* \varepsilon_{i\nu}^* M^\mu M^{*\nu}. \end{aligned}$$

- Hence the P between $S_{g.f.}$ & $S_{g.f.}^+$ drops out
and $(P S_{g.f.} P)(P S_{g.f.} P)^+ = P S_{g.f.} S_{g.f.}^+ P = P^2 = P$.

- The fact that $k_\mu M^\mu = 0$ is (a very simple special case of)
Ward or Ward-Takahashi identities* & follows from

$$\int DAD\bar{\psi} D\bar{\psi} A_\mu(x) \dots e^{iS} \sim \text{[diagram with a loop]};$$

* see e.g.
Peskin/Schroeder

$$= \int_y D_{\mu\nu}(x,y) \text{ [diagram]} = \int_y D_{\mu\nu}(x,y) \int DAD\bar{\psi} D\bar{\psi} \bar{\psi}(x)\gamma^\nu \psi(x) \dots e^{iS},$$

together with (classical and quantum) current conservation:

$$\partial_\mu j^\mu = \partial_\mu \bar{\psi}(x) \gamma^\mu \psi(x) = 0.$$

- This last statement requires just the global $U(1)$ -sym., which is clearly still Ok after gauge-fixing.
- It is also clear that this fails for non-abelian theories (e.g. because external gluons couple to internal gluons and not just to $\bar{\psi} \gamma^\mu T^a \psi$).
- However, similar, more involved identities can be derived using the BRST-sym. ("Slavnov-Taylor identities"). See e.g. Books of Polkovski & Zinn-Justin.