Part 2: Supergravity

8 Superspace geometry

8.1 Curved Superspace

gravity = Dynamics of the geometry of space-time

geometry = Manifold + metric

or Manifold + metric + connection

(e.g. the unique torsion-free connection in which the metric is constant, i.e. the Riemannian connection)

or $\boxed{\text{Manifold + vielbein + connection}}$

we will generalize this definition of the geometry to superspace

recall what superspace is: $x, \theta, \bar\theta$

$\bar\theta$: conj. of θ

define an algebra of functions on the manifold

(in other words: $x, \theta, \bar\theta$ parameterize the manifold)

to simplify notation: $z^M = (x^m, \theta^\mu, \bar\theta_{\dot\mu})$

(note: this is different from our previous indices $a, \alpha, \dot\alpha$ for a good reason, see below)

Note also: making $\bar{\theta}_{\dot{\mu}}$ (with lower index) our basic coordinate simplifies certain contractions, e.g.

$$z^M \partial_M = x^m \partial_m + \theta^\mu \partial_\mu + \bar{\theta}_{\dot{\mu}} \partial^{\dot{\mu}} = x \cdot \partial + \theta\partial + \bar{\theta}\bar{\partial}$$

- We require invariance under reparametrizations:

$$z^M \longrightarrow z'^M = f^M(z)$$

(i.e. $x'^m = f^m(x,\theta,\bar{\theta})$; $\theta'^\mu = f^\mu(x,\theta,\bar{\theta})$; $\bar{\theta}_{\dot{\mu}} = \bar{f}_{\dot{\mu}}(x,\theta,\bar{\theta})$; where f^μ and $\bar{f}^{\dot{\mu}}$ are the complex conjugates of each other)

- The trfs. of general scalar SFs (not chiral!) are defined by $\phi'(z') = \phi(z)$

8.2 Vielbeins (also "vierbein" or "tetrad")

$$E_A = E_A{}^M(z)\,\partial_M = (E_a, E_\alpha, E^{\dot{\alpha}})$$

This is a set of vector fields labelled by "A".

(A "vector field" is defined as a linear differential operator mapping functions to functions by differentiation)

Note: We do not really need points, tangent spaces at each point, local bases at each point etc., although this GR-based intuition clearly underlies our construction

- In usual GR (where we have just $e_a = e_a{}^m \partial_m$) one now continues by <u>identifying</u> field configurations linked by the gauge symmetry

$$e_a \longrightarrow e_a' = \Lambda_a{}^b e_b \;; \quad \Lambda = \Lambda(x)$$

x-dependent Lorentz rotation

(local Lorentz-symm., Lorentz-group is the "structure group")

<u>in particular:</u> η_{ab} is an invar. tensor; the metric is defined by $g_{mn} = e^a{}_m e^b{}_n \eta_{ab}$, where $e^a{}_m$ is the inverse of the matrix $e_a{}^m$; g_{mn} describes "real", physical d.o.f. while the remainder of $e_a{}^m$ is gauge freedom.

- <u>In superspace:</u> We can not proceed by analogy since there is no obvious linear trf. mixing $x^m, \theta^\alpha, \bar\theta_{\dot\alpha}$. Instead, we do <u>literally the same</u>, i.e. we use again the Lorentz group as a structure group:

Define gauge symm. $E_A \to E_A' = \Lambda_A{}^B E_B$

where
$$\Lambda_A{}^B = \left(\begin{array}{c|c|c} \Lambda_a{}^b & 0 & 0 \\ \hline 0 & N(\Lambda)_\alpha{}^\beta & 0 \\ \hline 0 & 0 & -\bar N(\Lambda)^{\dot\alpha}{}_{\dot\beta} \end{array}\right) ; \quad \Lambda = \Lambda(x,\theta,\bar\theta)$$

Note: N is the spinor trf. corresponding to Λ (recall that $SO(1,3)$ & $SL(2,\mathbb{C})$ can be identified near $\mathbb{1}$)

$$\psi_\alpha \to N_\alpha{}^\beta \psi_\beta$$

$$\bar\psi_{\dot\alpha} \to \bar N_{\dot\alpha}{}^{\dot\beta} \bar\psi_{\dot\beta}$$

$$\bar\psi^{\dot\alpha} \to \bar N^{\dot\alpha\dot\beta} \bar\psi_{\dot\beta} = -\bar N^{\dot\alpha}{}_{\dot\beta} \bar\psi^{\dot\beta}$$

→ As in GR, many of the d.o.f. of $E_A{}^M$ now become unphysical. However, the metric does not play the dominant role it has in GR.

- Let us also introduce the inverse vielbein $E^A{}_M$:

$$E_A{}^M E^B{}_M = \delta_A{}^B \; ; \; E_A{}^M E^A{}_N = \delta^M{}_N$$

⇒ We can now always switch between

<u>Lorentz index A</u> & <u>Einstein index M</u>

(tangent space index) (coordinate index)

in particular, vector fields can be written with a Lorentz index:

$$V = V^M \partial_M = \underbrace{V^M E^A{}_M}_{V^A} \underbrace{E_A{}^N \partial_N}_{E_A} = V^A E_A$$

$\Rightarrow$ Instead of the basis ∂_M we can use the basis $E_A = E_A{}^M \partial_M$.

8.3 Connection

Motivation: vector field basis: $\partial_M = \frac{\partial}{\partial z^M}$

dual basis: dz^M ("1-forms")

(basis of dual space)

Crucial relation: $\partial_N (dz^M) = \delta_N{}^M$ (note: this is in general not $dz^M(\partial_N)$ because of (anti-) commutation)

exterior differentiation: d: function $\rightarrow$ 1-form

1-form $\rightarrow$ 2-form

etc.

e.g. • $d(z^M) = dz^M$; $df(z) = dz^M \partial_M f(z)$

• let ω be a 1-form: $\omega = \omega_M dz^M$

$$d\omega = d(\omega_M dz^M) = d\omega_M \wedge dz^M$$

$$= dz^N (\partial_N \omega_M) \wedge dz^M$$

this implies antisymm.,

e.g. $dx^m \wedge dx^n = - dx^n \wedge dx^m$

Everything here works as in usual differential geometry, but with extra "−"-signs for the commutation of fermionic variables:

$$z^M z^N = (-)^{\varepsilon(M)\,\varepsilon(N)} z^N z^M$$

$$\left[\, \varepsilon(m) = 0 \,,\quad \varepsilon(\alpha) = \varepsilon(\dot\alpha) = 1 \,\right]$$

$$dz^M \wedge dz^N = -(-)^{\varepsilon(M)\,\varepsilon(N)} dz^N \wedge dz^M$$
$$dz^M \cdot z^N = (-)^{\varepsilon(M)\,\varepsilon(N)} z^N dz^M \qquad \text{etc.}$$

The need for a <u>connection</u> arises from the need for a covariant exterior differentiation:

$$v_A \longrightarrow \Lambda_A{}^B v_B \qquad (\Rightarrow v_A \text{ is a vector})$$
$$dv_A \longrightarrow d(\Lambda_A{}^B) v_B + \Lambda_A{}^B dv_B \qquad (\Rightarrow dv_A \text{ is } \underline{\text{not}} \text{ a vector})$$

$\Rightarrow$ We want to define "D" such that Dv_A is a vector. In addition, "D" should have all the usual properties of "d".

<u>Ansatz</u>: $$Dv_A = dv_A + \underbrace{\Omega_A{}^B}_{\text{1-form with values in } sl(2,\mathbb{C})} v_B$$

$$\underbrace{\Omega = dz^M \Omega_M}_{\Omega_M:\ \text{matrix}} \;; \quad (\Omega_M)_A{}^B = \begin{pmatrix} (\Omega_M)_a{}^b & & \\ & (\Omega_M)_\alpha{}^\beta & \\ & & -(\bar\Omega_M)^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}$$

<u>Trf. of Ω under local Lorentz rotations:</u>

$$Dv \longrightarrow \Lambda Dv = D'v' = D'(\Lambda v)$$
$$\Rightarrow \Lambda dv + \Lambda \Omega v = d(\Lambda v) + \Omega' \Lambda v$$

$$\Rightarrow \Omega' \Lambda + d\Lambda = \Lambda \Omega \quad \Rightarrow \quad \underline{\Omega' = \Lambda \Omega \Lambda^{-1} - (d\Lambda)\Lambda^{-1}}$$

- In analogy to $d = dx^M \partial_M$, we have

$$D = dx^M D_M = dx^M E_M{}^A E_A{}^M D_M = E^A D_A$$

basis of 1-forms, covariant derivatives: both with Einstein index

basis of 1-forms, covariant derivatives: both with Lorentz index!

- Specifically, we can write

$$D_A v_B = \partial_A v_B + (\Omega_A)_B{}^C v_C \quad \text{or} \quad D_A = E_A + \Omega_A$$

(recall that $\partial_A = E_A{}^M \partial_M = E_A$)

<u>In summary:</u>

$$\boxed{\text{Connection} \;\hat{=}\; \text{operation } D \;\hat{=}\; sl(2,\mathbb{C}) \text{ valued 1-form } \Omega}$$

operation D: completely basis-independent formulation

1-form Ω: given by $(\Omega_A)_B{}^C$, i.e. a basis dependent set of numbers.

Furthermore:

$$\boxed{\text{In our context, a "geometry" is given by } E_A, \Omega_A}$$

E_A: set of vector fields

Ω_A: set of $sl(2,\mathbb{C})$ matrices

8.4 Torsion and Curvature

Definitions:

- **Torsion**: $-T^A = DE^A = dE^A + \Omega^A{}_B \wedge E^B$

  ($E^A = E^A{}_M dz^M$ is a 1-form basis or a vector-valued 1-form.)

  T^A is a Lorentz-vector-valued 2-form

- **Curvature**:
  - $R = d\Omega + \Omega \wedge \Omega$

    matrix-multiplication and wedge product of forms

    R is an $sl(2,\mathbb{C})$-valued 2-form
  - alternatively, R can be defined as the operator $R = D \circ D$ mapping Lorentz-tensors to Lorentz-tensor-valued 2-forms.
  - alternatively, both R and T can by defined by

$$\boxed{[D_A, D_B\} = T_{AB}{}^C D_C + R_{AB}}$$

$sl(2,\mathbb{C})$-valued tensor

Here $T^C = \frac{1}{2} E^A E^B T_{AB}{}^C$; $R = \frac{1}{2} E^A E^B R_{AB}$

T & R of our previous definitions.

- Some consistency checks:

① $\frac{1}{2} E^A E^B [D_A, D_B\} = \frac{1}{2} E^A E^B T_{AB}{}^C D_C + \frac{1}{2} E^A E^B R_{AB}$

here the exterior product of forms is tacitly assumed

$$\Rightarrow \quad E^A E^B D_A D_B = T^A D_A + R$$

$$= -(DE^A) D_A + R$$

$$= -E^B (D_B E^A) D_A + R$$

$$\Rightarrow R = E^A E^B D_A D_B + E^A (D_A E^B) D_B$$

$$R = (E^A D_A)(E^B D_B)$$

$R = D \cdot D$, as required by the previous definition of R

② $D \cdot D \cdot v = D(dv + \Omega v) = \underbrace{d \cdot dv}_{=0} + \Omega dv + d(\Omega v) + \Omega\Omega v$

some Lorentz-vector

$$= \cancel{\Omega dv} + d\Omega v - \cancel{\Omega dv} + \Omega\Omega v$$

$$= (d\Omega + \Omega\Omega) v = R v$$

$\Rightarrow R = d\Omega + \Omega\Omega$, as required by the first definition of R

8.5 The flat superspace limit

recall: a geometry is given by the set (E, Ω)

definition: A supergeometry is called flat if $\exists$ a coordinate choice & choice of Lorentz basis such that $(\Omega_A)^B{}_C = 0$ and

$$E_A{}^M = \begin{pmatrix} \delta_a{}^m & 0 & 0 \\ i(\sigma^m \bar\theta)_\alpha & \delta_\alpha{}^\mu & 0 \\ i(\bar\sigma^m \theta)^{\dot\alpha} & 0 & \delta^{\dot\alpha}{}_{\dot\mu} \end{pmatrix}.$$

To understand the significance, calculate the covariant derivatives:

$$D_A = E_A{}^M \partial_M$$

$$D_a = \delta_a{}^m \partial_m$$

$$D_\alpha = \delta_\alpha{}^\mu \partial_\mu + i(\sigma^m \bar\theta)_\alpha \partial_m \quad \leftarrow$$ This is precisely our covariant derivative of global SUSY

$$\bar D^{\dot\alpha} = \delta^{\dot\alpha}{}_{\dot\mu} \partial^{\dot\mu} + i(\bar\sigma^m \theta)^{\dot\alpha} \partial_m$$

Here we have switched from our previous convention

$$\partial^{\dot\mu} \equiv \varepsilon^{\dot\mu\dot\rho} \partial_{\dot\rho} = \varepsilon^{\dot\mu\dot\rho} \frac{\partial}{\partial\bar\theta^{\dot\rho}}$$

to the new convention

$$\partial^{\dot\mu} \equiv \frac{\partial}{\partial\bar\theta_{\dot\mu}} = \frac{\partial}{\partial(\varepsilon_{\dot\mu\dot\rho}\bar\theta^{\dot\rho})} = \varepsilon_{\dot\mu\dot\rho} \frac{\partial}{\partial\bar\theta^{\dot\rho}} = -\varepsilon^{\dot\mu\dot\rho}\frac{\partial}{\partial\bar\theta^{\dot\rho}}$$

i.e. $(\partial^{\dot\mu})_{new} = -(\partial^{\dot\mu})_{old}$

$$\Downarrow$$

$$\bar D_{\dot\alpha} = -\delta_{\dot\alpha}{}^{\dot\mu} \partial_{\dot\mu} + i\varepsilon_{\dot\alpha\dot\beta}(\bar\sigma^m)^{\dot\beta\nu}\theta_\nu \partial_m$$

$$\bar{D}_{\dot\alpha} = -\delta_{\dot\alpha}{}^{\dot\mu}\partial_{\dot\mu} - i(\bar\sigma^m)_{\dot\alpha\nu}\,\theta^\nu\partial_m = -\delta_{\dot\alpha}{}^{\dot\mu}\partial_{\dot\mu} - i(\theta\sigma^m)_{\dot\alpha}\,\partial_m$$

This is again precisely our covar. derivative of global SUSY

- In GR, the flat metric η_{mn} is invariant under a subset of reparametrizations: The Poincare group
- What is the analogue for superspace?
- Consider $z^M \to z'^M = f^M(z)$
- The infinitesimal version is $\delta z^M = z'^M - z^M = \underset{\substack{\uparrow \\ \text{pure convention}}}{-}K^M(z)$
- How does the vielbein transform under this?
- By definition (for a coordinate-scalar) we have

$$E'_A{}^M(z')\,\partial'_M = E_A{}^M(z)\,\partial_M$$

- Apply this operator to $z'^N = z^N - K^N(z)$

$$\Rightarrow \underbrace{E'_A{}^N(z')}_{} = E_A{}^N(z) - E_A{}^M(z)\,\partial_M K^N(z)$$

$$= E'_A{}^N(z) + \delta z^M \partial_M E_A{}^N(z)$$

$$\Rightarrow \delta E_A{}^N = K^M\partial_M E_A{}^N - E_A{}^M\partial_M K^N \quad \Big|\cdot\partial_N$$

- Using $D_A = E_A{}^N\partial_N$ (since Ω-terms vanish) we get

$$\delta E_A{}^N\partial_N = [K, D_A] \overset{!}{=} 0 \qquad (\text{where } K \equiv K^M\partial_M)$$

- Recalling that $D_a = \partial_a$ and that D_α, $\bar D^{\dot\alpha}$ anticommute with Q_α, $\bar{Q}^{\dot\alpha}$, we see that

$$K = k^a \partial_a + k^\alpha Q_\alpha + \bar{k}_{\dot\alpha} \bar{Q}^{\dot\alpha}$$

fulfills our requirements. (Note that in flat superspace we can identify $a, \alpha, \dot\alpha$ & $m, \mu, \dot\mu$ indices)

$\Rightarrow$ global SUSY-trfs., generated by Q & $\bar{Q}$, interpreted as reparametrizations of superspace $z = (x, \theta, \bar\theta)$, <u>leave flat superspace invariant.</u>
(global SUSY is analogous to Poinc. symm. for superspace)

9 Supergravity Constraints

9.1 General Idea

- $\{E, \Omega\}$ still contain far too many d.o.fs.
  In usual GR, we continue by imposing the constraint $T = 0$. As a result, Ω can be expressed through the metric.
- In SG, $T = 0$ can not be imposed since $T \neq 0$ already in the flat case. (Recall that $\{D_\alpha, D_{\dot\alpha}\} \sim D_a$)
- Thus, we need to find other constraints. The aim is to reduce the physical d.o.f. to g_{mn} and its superpartner (the gravitino) (Recall that our analysis of SUSY representations included one case with a spin 2 and a spin 3/2 particle.)
- The constraints have to respect the fund. symmetries
  $\Rightarrow$ homogeneous constraints (like $T = 0$) can only be imposed on T & R (which are tensors), not on Ω (which transforms inhomogeneously under local Lorentz rotations).

9.2 Representation preserving constraint

We want our smallest SUSY repres. (the chiral SF) to generalize to curved superspace. Thus, we need a

"covariantly chiral SF"

satisfying $\bar{D}_{\dot\alpha} \phi = 0$ (with $\bar{D}_{\dot\alpha}$ the covar. derivative)

• Recall that $[D_A, D_B\} = T_{AB}{}^C D_C + R_{AB}$
and that R_{AB} vanishes on a scalar field.

Thus $\bar{D}_{\dot\alpha}\phi = 0 \Rightarrow \{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\}\phi = (T_{\dot\alpha\dot\beta}{}^a D_a + T_{\dot\alpha\dot\beta}{}^\gamma D_\gamma)\phi = 0$

$\Rightarrow$ We should demand

$$\boxed{T_{\dot\alpha\dot\beta}{}^c = T_{\dot\alpha\dot\beta}{}^\gamma = 0 \quad \big(\underbrace{= T_{\alpha\beta}{}^c = T_{\alpha\beta}{}^{\dot\gamma}}_{\text{by compl. conjugation}}\big)}$$

<u>A useful interpretation of this constraint:</u>

$D_A = E_A + \Omega_A \quad , \quad E_A = E_A{}^M \partial_M$

<u>Definition:</u> $\quad [E_A, E_B\} \equiv \underbrace{C_{AB}{}^C}_{\text{"anholonomy coefficients"}} E_C$

• Applying $[D_A, D_B\}$ to a scalar, we find:

$$\Big(C_{AB}{}^C E_C + [\Omega_A, E_B\} + [E_A, \Omega_B\} + \underbrace{[\Omega_A, \Omega_B\}}_{\text{vanishes on scalar}}\Big)\phi = T_{AB}{}^C E_C\,\phi$$

$$\Rightarrow \Big(C_{AB}{}^C E_C + \Omega_{AB}{}^C E_C - (-)^{\varepsilon(A)\varepsilon(B)}\Omega_{BA}{}^C E_C\Big)\phi = T_{AB}{}^C E_C\,\phi$$

$$\Rightarrow C_{AB}{}^C = T_{AB}{}^C - \Omega_{AB}{}^C + (-)^{\varepsilon(A)\varepsilon(B)}\Omega_{BA}{}^C$$

Apply this to the cases $A, B, C = \alpha, \beta, a$
and $A, B, C = \alpha, \beta, \dot\alpha$.

Use $T_{\alpha\beta}{}^c = 0 \quad ; \quad T_{\alpha\beta}{}^{\dot\alpha} = 0$ (repres. pres. constraint) and $\Omega_{\alpha\beta}{}^{c} = 0$; $\Omega_{\alpha\beta}{}^{\dot\alpha} = 0$ (feature of the superspace connection)

to find $\underline{C_{\alpha\beta}{}^{a} = C_{\alpha\beta}{}^{\dot\alpha} = 0}$

This implies $\boxed{\{E_\alpha, E_\beta\} = C_{\alpha\beta}{}^{\gamma} E_\gamma}$, which means

that <u>undotted & dotted spinorial derivatives (with Lorentz index) form separate algebras.</u>

(This is a useful <u>equivalent formulation</u> of the repr.preserv. constraints)

<u>9.3 Conventional constraints I</u>

We demand that the flat-space SUSY algebra of covariant derivatives remains unchanged in curved superspace:

$$\boxed{\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} = -2i(\sigma^a)_{\alpha\dot\alpha}\, \mathcal{D}_a}$$

curved-superspace covariant derivatives

$$\Rightarrow \boxed{T_{\alpha\dot\alpha}{}^{a} = -2i(\sigma^a)_{\alpha\dot\alpha} \; ; \; T_{\alpha\dot\alpha}{}^{\beta} = T_{\alpha\dot\alpha}{}^{\dot\beta} = 0 \; ; \; (R_{\alpha\dot\alpha})_a{}^{b} = 0}$$

As a result, E_a & Ω_a are expressible in terms of

$$E_\alpha, \Omega_\alpha \;\&\; E_{\dot\alpha}, \Omega_{\dot\alpha}$$

(We see that we begin to achieve our goal of reducing the d.o.f.s contained in (E, Ω).)

<u>9.4 Conventional constraints II</u>

Following the experience of usual GR, we want to be able to express Ω_α in terms of E_α.

Idea: Calculate Ω_α from

$$C_{AB}{}^{C} = T_{AB}{}^{C} - \Omega_{AB}{}^{C} + (-1)^{\varepsilon(A)\varepsilon(B)}\,\Omega_{BA}{}^{C}.$$

For this to be possible, we will have to demand that certain components of T vanish.

(1) $A, B, C = \alpha, \beta, \gamma$; demand $T_{\alpha\beta\gamma} = 0$

$$\Rightarrow \quad C_{\alpha\beta\gamma} = -\Omega_{\alpha\beta\gamma} - \cancel{\Omega_{\beta\alpha\gamma}}$$

$$C_{\alpha\gamma\beta} = -\Omega_{\alpha\gamma\beta} - \cancel{\Omega_{\gamma\alpha\beta}}$$

$$-C_{\beta\gamma\alpha} = +\cancel{\Omega_{\beta\gamma\alpha}} + \cancel{\Omega_{\gamma\beta\alpha}}$$

$$\underline{\underline{\Omega_{\alpha\beta\gamma} = -\tfrac{1}{2}\left(C_{\alpha\beta\gamma} + C_{\alpha\gamma\beta} - C_{\beta\gamma\alpha}\right)}}$$

add these equations and use the fact that $\Omega_{..\alpha\beta} = \Omega_{..\beta\alpha}$

Proof of the fact that $\Omega_{..\alpha\beta} = \Omega_{..\beta\alpha}$

let $A_\alpha{}^\beta \in sl(2,\mathbb{C}) \Rightarrow A \cdot \varepsilon = 0$ since ε is an inv. tensor

$$\varepsilon_{\alpha\beta} \xrightarrow{A} A_\alpha{}^\gamma \varepsilon_{\gamma\beta} + A_\beta{}^\gamma \varepsilon_{\alpha\gamma} = 0$$

$$-A_{\alpha\beta} + A_{\beta\alpha} = 0 \quad \checkmark$$

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(2) We also need $\Omega_{\alpha\dot\beta\dot\gamma}$ and $\Omega_{\alpha bc}$.

Note: It would be <u>too naive</u> to conclude from the general form

$$\Lambda_A{}^B = \begin{pmatrix} \Lambda_a{}^b & & \\ & N(\Lambda)_\alpha{}^\beta & \\ & & -\bar N(\Lambda)^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}$$

of local Lorentz rotations that $\Omega_{\alpha\beta\gamma} = 0$ implies $\Omega_{\alpha\dot\beta\dot\gamma} = 0$ and $\Omega_{\alpha bc} = 0$. The reason is, roughly speaking, that for $\Lambda_A{}^B = (e^K)_A{}^B$

we have $K^{ab} = (\sigma^{ab})_{\alpha\beta} K^{\alpha\beta} - (\bar\sigma^{ab})_{\dot\alpha\dot\beta} K^{\dot\alpha\dot\beta}$.

$(\to$ B/K, (5.232))

Since we are forced to always work with the complexified Lie-algebras, $K^{\alpha\beta} = 0$ does not imply $K^{ab} = K^{\dot\alpha\dot\beta} = 0$ (while $K^{ab} = 0$, as a complex quantity, does imply $K^{\alpha\beta} = K^{\dot\alpha\dot\beta} = 0$

We start from the eq. for $A, B, C = \alpha, b, c$:

$$C_{\alpha bc} = T_{\alpha bc} - \Omega_{\alpha bc} - \underbrace{\Omega_{b\alpha c}}_{=0}$$

We want this quantity.

<u>Strategy</u>: first calculate $C_{\alpha bc}$ from other information.

<u>Technique</u>: Introduce <u>"semi-covariant vielbein" $\check{E}_A$</u>

defined by $\check{E}_\alpha = E_\alpha$, $\check{\bar{E}}_{\dot\alpha} = \bar{E}_{\dot\alpha}$, $\check{E}_{\alpha\dot\alpha} = \frac{i}{2}\{\check{E}_\alpha, \check{\bar{E}}_{\dot\alpha}\}$

This is an equivalent way of writing a vector index "a" using the definition <u>$v_{\alpha\dot\alpha} = (\sigma^a)_{\alpha\dot\alpha} v_a$</u>

[The definition of $\check{E}_{\alpha\dot\alpha}$ is modelled after the relation $D_{\alpha\dot\alpha} = \frac{i}{2}\{D_\alpha, \bar{D}_{\dot\alpha}\}$.]

- In the context of $\check{E}_A$, all anholonomy coefficients are known once E_α, $\bar{E}_{\dot\alpha}$ are given, e.g.

$$[\check{E}_\alpha, \check{E}_b] = \check{C}_{\alpha b}{}^c \check{E}_c + \cdots$$

- However, $C_{\alpha b}{}^c$ differs from $\check{C}_{\alpha b}{}^c$. To see this, apply $D_{\alpha\dot\alpha} = \frac{i}{2}\{D_\alpha, \bar{D}_{\dot\alpha}\}$ to a scalar field:

$$E_{\alpha\dot\alpha}\phi = \frac{i}{2}\{E_\alpha, \bar{E}_{\dot\alpha}\}\phi + \frac{i}{2}\{\Omega_\alpha, \bar{E}_{\dot\alpha}\}\phi + \frac{i}{2}\{E_\alpha, \bar\Omega_{\dot\alpha}\}\phi$$

$$\Rightarrow E_{\alpha\dot\alpha} = \check{E}_{\alpha\dot\alpha} + \frac{i}{2}\Omega_{\alpha\dot\alpha}{}^{\dot\beta}\bar{E}_{\dot\beta} + \frac{i}{2}\Omega_{\dot\alpha\alpha}{}^{\beta}E_\beta \qquad (*)$$

Take the commutator with E_α:

$$[E_\alpha, E_{\beta\dot\beta}] = [E_\alpha, \check{E}_{\beta\dot\beta}] + \frac{i}{2}[E_\alpha, \Omega_{\beta\dot\beta}{}^{\dot\gamma}\bar{E}_{\dot\gamma}] + \frac{i}{2}[E_\alpha, \Omega_{\dot\beta\beta}{}^{\gamma}E_\gamma]$$

Expand both sides in E_α, $\bar{E}_{\dot\alpha}$, E_a and focus on the coefficient of E_a:

$$\boxed{C_{\alpha,\beta\dot\beta}{}^c E_c = \check{C}_{\alpha,\beta\dot\beta}{}^c E_c - \Omega_{\beta\dot\beta}{}^{\dot\gamma}E_{\alpha\dot\gamma}}$$

(Here we used (*)).

- We now apply the relation $v_a = -\frac{1}{2}(\bar\sigma_a)^{\dot\alpha\alpha}v_{\alpha\dot\alpha}$

[This is the inverse of $v_{\alpha\dot\alpha} = (\sigma^a)_{\alpha\dot\alpha}v_a$, as can be seen from $v_a = -\frac{1}{2}(\bar\sigma_a)^{\dot\alpha\alpha}(\sigma_b)_{\alpha\dot\alpha}v^b = -\frac{1}{2}(-2\eta_{ab})v^b$]

$$\Rightarrow -\frac{1}{2}C_{\alpha,\beta\dot\beta,}{}^c(\bar\sigma_c)^{\dot\gamma\gamma}E_{\gamma\dot\gamma} = -\frac{1}{2}\check{C}_{\alpha,\beta\dot\beta}{}^c(\bar\sigma_c)^{\dot\gamma\gamma}E_{\gamma\dot\gamma} - \Omega_{\beta\dot\beta}{}^{\dot\gamma}\delta_\alpha{}^\gamma E_{\gamma\dot\gamma}$$

$$\Rightarrow C_{\alpha,\beta\dot\beta,\gamma\dot\gamma} = \check{C}_{\alpha,\beta\dot\beta,\gamma\dot\gamma} + 2\,\Omega_{\beta\dot\beta\dot\gamma}\,\varepsilon_{\gamma\alpha}$$

- let us compare this to our previously derived formula (now written with $c \to \gamma\dot\gamma$ etc.)

$$C_{\alpha,\beta\dot\beta,\gamma\dot\gamma} = T_{\alpha,\beta\dot\beta,\gamma\dot\gamma} - \Omega_{\alpha,\beta\dot\beta,\gamma\dot\gamma}$$

- Next, we eliminate C and use

$$\Omega_{\alpha,\beta\dot\beta,\gamma\dot\gamma} = 2\,\Omega_{\alpha\beta\gamma}\,\varepsilon_{\dot\beta\dot\gamma} + 2\,\Omega_{\alpha\dot\beta\dot\gamma}\,\varepsilon_{\beta\gamma}$$

(This can be verified by demanding that $\psi\sigma^a\bar\chi$ is rotated consistently with the rotation of ψ_α and $\bar\chi_{\dot\alpha}$.)

$$\Rightarrow \check{C}_{\alpha,\beta\dot\beta,\gamma\dot\gamma} = T_{\alpha,\beta\dot\beta,\gamma\dot\gamma} - 2\,\Omega_{\alpha\beta\gamma}\,\varepsilon_{\dot\beta\dot\gamma} - 2\,\Omega_{\alpha\dot\beta\dot\gamma}\,\varepsilon_{\beta\gamma}$$

$$- 2\,\Omega_{\beta\dot\beta\dot\gamma}\,\varepsilon_{\gamma\alpha}$$

$\boxed{\text{We want } \Omega_{\alpha\dot\beta\dot\gamma}\text{ !}}$

This term can be eliminated by contracting with $\varepsilon^{\beta\gamma}$.

$$\Rightarrow \check{C}_{\alpha,\beta\dot\beta,}{}^{\beta}{}_{\dot\gamma} = T_{\alpha,\beta\dot\beta,}{}^{\beta}{}_{\dot\gamma} + 4\,\Omega_{\alpha\dot\beta\dot\gamma} - 2\,\Omega_{\alpha\dot\beta\dot\gamma}$$

We can symmetrize in $\dot\beta,\dot\gamma$ without affecting $\Omega_{\alpha\dot\beta\dot\gamma}$.
Then, imposing $\boxed{T_{\alpha,\beta(\dot\beta,}{}^{\beta}{}_{\dot\gamma)} = 0}$ we find

$$\Omega_{\alpha\dot\beta\dot\gamma} = \frac{1}{2}\,\check{C}_{\alpha,\beta(\dot\beta,}{}^{\beta}{}_{\dot\gamma)}\,, \text{ as desired.}$$

9.5 Summary of the constraints of conformal supergravity

1) Consistency of $E_\alpha \phi = 0$ $\Rightarrow$ $\{E_\alpha, E_\beta\} = C_{\alpha\beta}{}^\gamma E_\gamma$

2) Demanding the flat-SUSY derivative algebra (which also ensures that E_α, Ω_α fix E_a, Ω_a) $\Rightarrow$ $\{D_\alpha, \bar{D}_{\dot\alpha}\} = -2i(\sigma^a)_{\alpha\dot\alpha} D_a$

3) To ensure that E_α fixes Ω_α
① $\Omega_{\alpha\beta\gamma}$ $\Rightarrow$ $T_{\alpha\beta\gamma} = 0$
② $\Omega_{\alpha\dot\beta\dot\gamma}$ $\Rightarrow$ $T_{\alpha,\beta(\dot\beta,}{}^{\beta}{}_{\dot\gamma)} = 0$

9.6 Constraints of Einstein supergravity

Define the torsion trace as $T_\alpha \equiv (-1)^{\varepsilon(B)} T_{\alpha B}{}^B$.

In addition to the conf. SUGRA constraints, demand

$$T_\alpha = 0.$$

With this additional constraint imposed,

$T_{\alpha,\beta(\dot\beta,}{}^{\beta}{}_{\dot\gamma)} = 0$ can be replaced by $T_{\alpha b}{}^c = 0$.

[It is clear that $T_{\alpha b}{}^c = 0$ implies $T_{\alpha,\beta(\dot\beta,}{}^{\beta}{}_{\dot\gamma)} = 0$.
The fact that conf. constraints + $T_\alpha = 0$ imply $T_{\alpha b}{}^c = 0$ will be demonstrated later.]

The Einstein-SUGRA constraints, defined by

$$\boxed{\text{conf. constraints} + T_\alpha = 0}$$

can be equivalently characterized by

$$T_\alpha = 0\,,\quad T_{\alpha\beta}{}^{A} = 0\,,\quad T_{\alpha\dot\alpha}{}^{B} = -2i(\sigma^c)_{\alpha\dot\alpha}\,\delta_c{}^{B}$$

$$R_{\alpha\dot\alpha}{}^{cd} = 0\,,\quad T_{\alpha b}{}^{c} = 0 \qquad (+\text{h.c.})$$

This requirement can also be replaced by constraints on T. [The stronger statement that R can be expressed through T and its derivatives is also known as the "Dragon theorem".]

10 Solving the constraints

10.1 General idea

Setting some components of T & R to zero leads to many further relations between other components of T & R.

Reason: T & R are not indep. quantities. They follow from E & Ω. Thus, constraining T & R leads to constraints on E & Ω. This restricted form of E & Ω leads to expressions for T & R which obey many further relations.

Result: all components of T & R can be expressed through a (relatively) small set of objects:

- scalar R with $\bar{D}_{\dot\alpha} R = 0$
- real vector G_a
- tensor $W_{(\alpha\beta\gamma)}$ ← symmetric
- vector T_α (see above; zero in Einstein case)

[**Note:** These objects are not totally independent. (Certain relations between them exist.)]

Important consequence: The whole algebra of D_A's can be expressed through this smaller set of variables.

e.g. $[D_A, D_B\} = T_{AB}{}^C D_C + R_{AB} \xrightarrow{\text{constraints}}$ $\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i D_{\alpha\dot{\alpha}}$

$$\{D_\alpha, D_\beta\} = -4\bar{R} M_{\alpha\beta}$$

(where $M_{\alpha\beta} = \frac{1}{2}(\sigma^{ab})_{\alpha\beta} M_{ab}$ and M_{ab} are the standard generators of $SO(1,3)$)

- An important tool in this procedure are the

10.2 Bianchi identities

① $DR = 0$

② $-DT^A = (R \cdot E)^A$

Proof:

① $DR = dR + \Omega \circ R$

abstract Lie-algebra action in the the appropriate representation, in this case the adjoint (commutator)

$$\Rightarrow DR = dR + \Omega \wedge R - R \wedge \Omega$$

$$= d(d\Omega + \Omega \wedge \Omega) + \Omega \wedge R - R \wedge \Omega$$

$$= d\Omega \wedge \Omega - \Omega \wedge d\Omega + \Omega \wedge d\Omega + \Omega \wedge \Omega \wedge \Omega - d\Omega \wedge \Omega - \Omega \wedge \Omega \wedge \Omega$$

$$= 0 \quad \checkmark$$

② $(R\cdot E)^A = (D\cdot D\cdot E)^A = D\cdot D E^A = D(-T^A) = -DT^A$ ✓

More useful is the <u>component form</u>:

① Writing R as $R = \frac{1}{2} E^A E^B R_{AB}$, we find

(form indices!)

$$0 = 2DR = E^A D_A E^B E^C R_{BC}$$
$$= E^A E^B E^C D_A R_{BC} - 2T^B E^C R_{BC}$$
$$= E^A E^B E^C D_A R_{BC} - E^A E^D T_{AD}{}^B E^C R_{BC}$$

$$\Rightarrow \boxed{D_A R_{BC} - T_{AB}{}^D R_{DC} + (\text{graded cyclic permutations}) = 0}$$

② First make "$(R\cdot E)^A$" explicit:

$R: E_A \longrightarrow R_A{}^B E_B$ (suppressing the form-indices of R)

Since $E^A E_A$ is a scalar, we have

$R: E^A \longrightarrow -E^B R_B{}^A$

Thus, we find that

$$-DT^A = (R\cdot E)^A \Rightarrow -E^B D_B \tfrac{1}{2} E^C E^D T_{CD}{}^A = -\tfrac{1}{2} E^B E^C R_{BCD}{}^A E^D$$

$$\Rightarrow E^B E^C E^D D_B T_{CD}{}^A - E^B E^E T_{BE}{}^C E^D T_{CD}{}^A = E^B E^C E^D R_{BCD}{}^A$$

$$\Rightarrow \boxed{R_{ABC}{}^D - D_A T_{BC}{}^D - T_{AB}{}^E T_{EC}{}^D + (\text{grad. cycl. perms.}) = 0}$$

Note: All this can also be obtained from

$$[D_A, [D_B, D_C\}\} + (\text{grad. cycl. perms.}) = 0 ,$$

by replacing $[D_B, D_C\}$ with $T_{BC}{}^D D_D + R_{BC}$.
(This avoids the use of external calculus.)

10.3 Systematically solving the constraints

Idea: Decompose the Bianchi identities in their contributions of different mass dimension.

Use, e.g., $[D_\alpha] = \frac{1}{2}$, $[D_a] = 1$;

$$[T_{\alpha\beta}{}^a] = 0, \ [T_{\alpha\beta}{}^\gamma] = \frac{1}{2}, \ \ldots, \ [T_{ab}{}^\alpha] = \frac{3}{2}, \ldots$$

- Start from lowest mass dim. and work upwards:

Dim 1/2: only ② contributes, e.g. by choosing

$$ABCD \to \alpha\beta\dot{\alpha}c$$

- $R \to 0$
- $$- D_\alpha T_{\beta\dot{\alpha}}{}^c - D_\beta T_{\dot{\alpha}\alpha}{}^c - D_{\dot{\alpha}} T_{\alpha\beta}{}^c$$
$$+ T_{\alpha\beta}{}^E T_{E\dot{\alpha}}{}^c + T_{\beta\dot{\alpha}}{}^E T_{E\alpha}{}^c + T_{\dot{\alpha}\alpha}{}^E T_{E\beta}{}^c = 0 \qquad (*)$$

now use: $T_{\beta\dot{\alpha}}{}^c = -2i(\sigma^c)_{\beta\dot{\alpha}}$; $T_{\beta\dot{\alpha}}{}^{\cdot\cdot} = T_{\dot{\alpha}\beta}{}^{\cdot\cdot}$
(because it comes from $\{D_\beta, \bar{D}_{\dot{\alpha}}\} = \ldots$)

$D_\alpha(\sigma^c{}_{\beta\dot\gamma}) = 0$;

(because $\partial_\alpha(\sigma^c{}_{\beta\dot\gamma}) = 0$ and

$\Omega(\sigma^c{}_{\beta\dot\gamma}) = 0$ due to inv. tensor property)

$T_{\alpha\beta}{}^c = 0$; $T_{\alpha\beta}{}^\gamma = 0$; $T_{\alpha\beta}{}^{\dot\gamma} = 0$;

$T_{\alpha\beta}{}^c = 0$; $T_{\alpha\dot\beta}{}^\gamma = 0$; $T_{\alpha\dot\beta}{}^{\dot\gamma} = 0$

- $(*) \Rightarrow \quad T_{\beta\dot\alpha}{}^d T_{d\alpha}{}^c + T_{\dot\alpha\alpha}{}^d T_{d\beta}{}^c = 0$

$$(\sigma^d)_{\beta\dot\alpha} T_{d\alpha c} + (\sigma^d)_{\alpha\dot\alpha} T_{d\beta c} = 0$$

$\Rightarrow \quad \underline{\underline{T_{\beta\dot\alpha,\alpha,\gamma\dot\gamma} + T_{\alpha\dot\alpha,\beta,\gamma\dot\gamma} = 0}}$ (& compl. conj.)

<u>Dim 1</u> : again only ② contributes, e.g. by choosing

$$ABCD = \alpha\beta\gamma\delta$$

- since $T_{\alpha\beta}{}^E = 0$, all T-terms vanish

$\Rightarrow \quad \underline{\underline{R_{\alpha\beta\gamma\delta} + (\text{graded cyclic in } \alpha\beta\gamma) = 0}}$

(& compl. conj.)

$$ABCD = \alpha\beta\dot\gamma\dot\delta$$

$\Rightarrow \quad \underline{\underline{R_{\alpha\beta\dot\gamma\dot\delta} = 2i\,(T_{\alpha,\beta\dot\gamma,\dot\delta} + T_{\beta,\alpha\dot\gamma,\dot\delta})}}$ (& c.c.)

---> a number of further identities of dims. $1, \frac{3}{2}, 2$ follow (higher dims. not needed)

$\rightarrow$ BIK Sect. 5.3.2 // WB, Sect. $\overline{\text{XV}}$

10.4 Analysis in Lorentz representations

To proceed further, analyse each of the above identities in terms of irred. Lorentz representations into which T & R fall:

As an example, consider $R_{\alpha\beta\gamma\delta}$ (at dim. 1)

- obvious from definitions: $R_{\alpha\beta\gamma\delta} = R_{(\alpha\beta)(\gamma\delta)}$
- recall our $SL(2,\mathbb{C})$ discussion: $(\frac{1}{2},0)$ undotted, $(0,\frac{1}{2})$ dotted spinor

$$R_{\alpha\beta\gamma\delta} \subset \underbrace{[(\tfrac{1}{2},0)\times(\tfrac{1}{2},0)]}_{\text{symm.}} \times \underbrace{[(\tfrac{1}{2},0)\times(\tfrac{1}{2},0)]}_{\text{symm.}}$$

$$(\tfrac{1}{2},0)\times(\tfrac{1}{2},0) = \underset{\substack{\text{antisymm.}\\(\sim\varepsilon_{\alpha\beta})}}{(0,0)} + \underset{\substack{\text{symm.}\\(\sim f_{\alpha\beta}\text{ with } f_{\alpha\beta} = f_{(\alpha\beta)})}}{(1,0)}$$

$$R_{\alpha\beta\gamma\delta} \subset (1,0)\times(1,0) = \underbrace{(2,0)}_{\text{totally symm.}} + (1,0) + \underbrace{(0,0)}_{\text{singlet}}$$

① $(2,0)$ – due to total symmetry, adding the cyclic graded sum in $\alpha\beta\gamma$ does not change anything. Thus, $R_{\alpha\beta\gamma\delta}$ + (grad. cycl. in $\alpha\beta\gamma$) $= 0$ implies that the coefficient of the $(2,0)$-part vanishes

(2) $(1,0)$ — must depend on tensor $h_{\alpha\beta} = h_{(\alpha\beta)}$ (see above)
— must be symmetric in $\alpha\beta$ and $\gamma\delta$

$\Rightarrow$ unique possibility: $\underbrace{\varepsilon_{(\alpha(\gamma} h_{\delta)\beta)}}$
(symmetrized in $\gamma\delta$ & $\alpha\beta$)

plugging this into $R_{\alpha\beta\gamma\delta}$ + (grad. cycl. in $\alpha\beta\gamma$) $= 0$ implies $h_{\alpha\beta} = 0$.

(3) The unique singlet is $\sim (\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma})$

$$\Rightarrow \boxed{R_{\alpha\beta\gamma\delta} = -2\bar{R}\,(\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} + \varepsilon_{\alpha\delta}\varepsilon_{\beta\gamma})}$$

It follows from the D-algebra that R is chiral ($\bar{R}$ is antichiral).

This analysis can be continued to show that the whole D-algebra can be written using only

$$\boxed{R,\ G_a = \bar{G}_a,\ W_{\alpha\beta\gamma} = W_{(\alpha\beta\gamma)},\ T_\alpha}$$

- We do not display this algebra explicitly to avoid writing many complicated expressions (see BK).
- In the Einstein-case, the algebra is simpler. In particular, $T_\alpha = 0$.

Algebra of $\mathcal{D}$'s in Einstein-SUGRA:

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot\alpha}\} = -2i\,\mathcal{D}_{\alpha\dot\alpha}$$

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = -4R\,M_{\alpha\beta} \qquad \& \text{ h.c.}$$

$$[\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot\beta}] = i\varepsilon_{\alpha\beta}\{\bar{R}\bar{\mathcal{D}}_{\dot\beta} + G^\gamma{}_{\dot\beta}\mathcal{D}_\gamma - (\mathcal{D}^\gamma G^\delta{}_{\dot\beta})M_{\gamma\delta} + 2W_{\dot\beta}{}^{\dot\gamma\dot\delta}\bar{M}_{\dot\gamma\dot\delta}\} + i\bar{\mathcal{D}}_{\dot\beta}\bar{R}M_{\alpha\beta} \qquad \& \text{ h.c.}$$

$[\mathcal{D}_{\alpha\dot\alpha}, \mathcal{D}_{\beta\dot\beta}]$ follows from expressing $\mathcal{D}_{\alpha\dot\alpha}$ by the first relation and applying Jacobi identity.

- Without proof, we give the following important relation:

$$\boxed{\bar{\mathcal{D}}_{\dot\alpha}(\bar{\mathcal{D}}^2 - 4R)\phi = 0}$$

for any scalar SF ϕ. Thus, $\bar{\mathcal{D}} - 4R$ is a <u>"chiral projector"</u>.

11 Prepotentials

11.1 General idea

We now know how to express E, Ω using only E_α.
From E, Ω, we can derive the whole algebra of D's.
The latter can be given in terms of $R, C_\alpha, W_{\alpha\beta\gamma}, T_\alpha$.
In this procedure, all constraints are incorporated, except for those affecting our basic input quantities, the E_α's. Due to the repres- preserving constraints, they satisfy

$$\{E_\alpha, E_\beta\} = C_{\alpha\beta}{}^\gamma E_\gamma .$$

(The non-trivial statement of this relation is that $E_{\dot\alpha}$ and $E_{\alpha\dot\alpha}$ do not appear on the r.h. side.)

It is very important to express the E_α's through some <u>unconstrained</u> input data.

- To understand the name <u>prepotentials</u> recall the following theories:

Electrodynamics: $A_\mu \longrightarrow F_{\mu\nu}$

General Relativity: $\underbrace{g_{\mu\nu}}_{\substack{\text{unconstrained} \\ \text{("potentials")}}} \longrightarrow \underbrace{(\Gamma_{\mu\nu}{}^\rho \to T_{\mu\nu\rho}, R_{\mu\nu\rho\sigma})}_{\substack{\text{Constrained} \\ \text{(by Bianchi - Id.} \\ \text{and by } T=0\text{)}}}$

- Compare this to Supergravity:

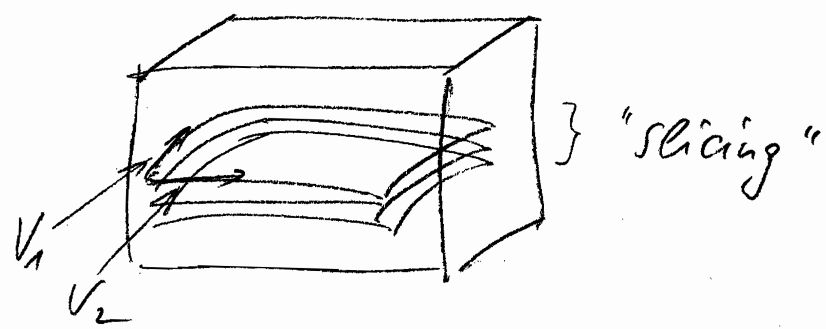
"prepotentials" $\longrightarrow$ E_α $\longrightarrow$ $(E \to \Omega \to T, R)$
?
(unconstrained) (constrained by repres. pres. constraint) (constrained by the other constraints and by Bianchi-Id.)

- We need to understand the geometric meaning of the statement that "the algebra of the E_α's closes".

11.2 Applying the Frobenius theorem

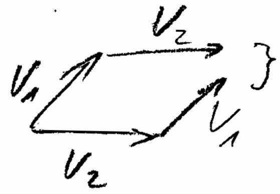
Frobenius theorem: A set of q linearly independent vector fields V_i on a p-dimensional manifold $(p > q)$ define a set q-dim. submanifolds if the algebra of V_i's closes ($[V_i, V_j] = c_{ij}{}^k V_k$).

- Illustration for $p = 3$, $q = 2$:



- Rough argument for the validity of the theorem: (again for $p = 3$, $q = 2$):

Try to construct a submanifold defined by V_1, V_2 infinitesimally.



} if this mismatch (characterized by $[V_1, V_2] \neq 0$) requires a step in a third, linearly independent direction, a 2-dim. submanifold "containing" V_1, V_2 can not be given. Thus, we need

$$[V_1, V_2] = \underset{\substack{\uparrow}}{\alpha} V_1 + \underset{\substack{\uparrow}}{\beta} V_2$$

scalar fcts. on manifold

(for more details see, e.g.,
- Choquet-Bruhat, DeWitt-Morelle, Dillard-Bleick: "Analysis on Manifolds & Physics.)

- The application of the Frobenius theorem to our case is non-trivial since the E_α's come together with the $\bar{E}_{\dot\alpha}$'s, but combined algebra of $E_\alpha, \bar{E}_{\dot\alpha}$ does <u>not</u> close.

- To be able to treat just E_α's $\left(\frac{\partial}{\partial \theta^\mu}\right)$ (independently of $\bar{E}_{\dot\alpha}$) $\left(\frac{\partial}{\partial \bar\theta^\mu}\right)$

we need to go from $\mathbb{R}^{4|4}$ $(x, \theta, \bar\theta)$

to $\mathbb{C}^{4|4}$ $(y, \theta, \bar\xi)$

y: complex; $\bar\xi$: $\neq \bar\theta$ in general

($\mathbb{R}^{4|4}$ is a subspace defined by $y = \bar{y}$ and $\bar\theta = \bar\xi$)

- For convenience, we will keep the notation $z^M = (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$, keeping in mind that $x^m = y^m$ is in general complex and $\bar{\theta}_{\dot{\mu}} \neq (\theta_\mu)^*$ in general.
- Thanks to the Frobenius theorem, there exist coordinates z' such that θ'^μ parametrize the submanifolds defined by E_α. (The other directions are parametrized by x', $\bar{\theta}'$)
- There exist non-singular matrices such that

$$E_\alpha = A_\alpha{}^\mu(z)\,\hat{E}_\mu \quad \text{with} \quad \hat{E}_\mu \equiv \frac{\partial}{\partial \theta'^\mu}.$$

- The transition from z to z' is a simple reparametrization. Infinitesimally:

$$z'^M = z^M + \delta z^M = z^M + w^M(z)$$

$$= z^M + w^N(z)\,\partial_N z^M = \underbrace{(\mathbb{1} + w)}_{\text{infinites. vector field.}} z^M$$

- The finite version is

$$\boxed{z'^M = \underbrace{e^w}_{\text{diff. operator}} z^M}$$

(recall that $e^{a^i \hat{p}_i}$ generates finite shifts in quantum mechanics)

- For vector fields we have $\boxed{\partial'_M = e^w \partial_M e^{-w}}$

(proof: $\partial'_M z'^N = e^w \partial_M e^{-w} e^w z^N = e^w \delta_M{}^N = \delta_M{}^N$)

Summary: $\exists$ (non-unique) W & A such that

$$E_\alpha = A_\alpha{}^\mu \hat{E}_\mu \quad ; \quad \hat{E}_\mu = e^W \partial_\mu e^{-W}$$

$$(W = W^\mu \partial_\mu)$$

It is convenient to write $A_\alpha{}^\mu = F \cdot N_\alpha{}^\mu$, where $\det(N) = 1$ and F is a scalar.

$\Rightarrow$ $\boxed{N, F, W \text{ are the prepotentials}}$

Given the above form of E_α (or, analogously, $\bar{E}_{\dot\alpha}$), it is particularly easy to characterize chiral SF:

$$\bar{D}_{\dot\alpha}\phi = \bar{E}_{\dot\alpha}\phi = 0 \iff \phi = e^{\bar{W}}\hat{\phi} \text{ with } \bar{\partial}_{\dot\mu}\hat{\phi} = 0$$

(proof: $\bar{E}_{\dot\alpha}\phi = -\bar{A}_{\dot\alpha}{}^{\dot\mu} e^{\bar{W}} \bar{\partial}_{\dot\mu} e^{-\bar{W}} e^{\bar{W}} \hat{\phi} = 0$)

$\hat{\phi}$ is called a "flat chiral" SF, $\hat{\phi} = \hat{\phi}(x, \theta)$

$\updownarrow e^{\bar{W}}$

ϕ is a covariantly chiral SF

analogous to the y, θ-form of chiral SF's in rigid SUSY

11.3 Gauge freedom of prepotentials

1) (super)local Lorentz group: $E'_A = \Lambda_A{}^B E_B$

with $E'_\alpha = N'_\alpha{}^\mu F' \hat{E}'_\mu$ and $E_\alpha = N_\alpha{}^\mu F \hat{E}_\mu$,

we can define $F' = F$, $\hat{E}' = \hat{E}$ and

choose Λ such that $\boxed{N' = \mathbb{1}}$

2) reparametrizations of $\mathbb{R}^{4|4}$

$$z' = e^{-K} z \qquad (K \text{ real !})$$

3) reparametrizations of $\mathbb{C}^{4|4}$

$$z' = e^{\bar{\Lambda}} z \qquad (\bar{\Lambda} \text{ complex } (\text{"}-\text{" is pure conv.})$$

2)+3): Recall that $\hat{E}_\mu = e^W \partial_\mu e^{-W}$

vielbein in $\mathbb{R}^{4|4}$ defined as subspace of $\mathbb{C}^{4|4}$

is submanif. of $\mathbb{C}^{4|4}$

Thus "2)" is the freedom to reparametrize $\mathbb{C}^{4|4}$ before applying e^W; "3)" is the freedom to reparametrize the physical $\mathbb{R}^{4|4}$ after applying e^W

$$\Rightarrow \boxed{e^W \xrightarrow{2),3)} e^{W'} = e^K e^W e^{-\bar{\Lambda}}}$$

This not enough freedom to gauge the complex W to zero since – K is real

– $\bar{\lambda}$ has to respect the closure of the ∂_μ-algebra.

($\Rightarrow$ without proof, we claim that

$\bar{\lambda}^\mu$ is arbitrary, $\partial_\mu \bar{\lambda}^\mu = 0$, $\partial_\mu \bar{\lambda}_{\dot{\nu}} = 0$.)

- Using this limited freedom for choosing K and $\bar{\lambda}$, we can achieve:

$$\boxed{W = W^m \partial_m \quad \text{with } \operatorname{Re} W = 0}$$

or, alternatively (allowing for some non-zero $\operatorname{Re} W$)

"gravitational superfield gauge"
$$\begin{cases} \boxed{e^{\bar{W}} x^m = e^{(\bar{W}^n \partial_n)} x^m = x^m + i\mathcal{H}^m(x,\theta,\bar{\theta})} \\ \boxed{\text{with } \mathcal{H}^m = \bar{\mathcal{H}}^m} \end{cases}$$

- There is a remaining (<u>residual</u>) set of K- and $\bar{\lambda}$-gauge-trfs. preserving the gauge $\boxed{e^{\bar{W}} x^m = x^m + i\mathcal{H}^m}$.

- A straightforward technical analysis shows that they allow to gauge the lowest components of $\mathcal{H}$ to zero:

<u>W-Z gauge:</u>
$$\boxed{\begin{aligned} \mathcal{H}^m = \theta\sigma^a\bar{\theta} e_a{}^m(x) + \left[i\bar{\theta}^2 \theta^\alpha \psi_\alpha{}^m(x) + h.c.\right] \\ + \theta^2\bar{\theta}^2 A^m(x) \end{aligned}}$$

$$\left.\begin{array}{l} e - \text{vielbein} \\ \psi - \text{gravitino} \\ A - \text{auxiliary vector} \end{array}\right\} \text{ of conf. SUGRA}$$

- We can now find an (even smaller) subset of dfs. preserving this WZ gauge:

$$\lambda^m(x,\theta) = b^m(x) + 2i\theta\sigma^a\bar\epsilon(x)\, e_a{}^m(x) - 2\theta^2\,\bar\epsilon\bar\psi^m(x)$$

$$\lambda^\alpha(x,\theta) = \epsilon^\alpha(x) + \tfrac{1}{2}(\sigma(x) + i\Omega(x))\theta^\alpha + K^\alpha{}_\beta(x)\theta^\beta + \theta^2\eta^\alpha(x)$$

with b, σ, Ω real ; $K_{\alpha\beta} = K_{\beta\alpha}$

<u>interpretation:</u>

$b \to$ reparametrizations
$K \to$ local Lorentz
$\epsilon \to$ local SUSY
$\eta \to$ "S-supersymmetry"
$\sigma, \Omega \to$ super-Weyl-symm.

(for all of them: calculate $\delta\mathcal{H}$, expand in e, ψ, A, find δe, $\delta\psi$, δA)

e.g. $\boxed{\delta_\sigma e_a{}^m = \sigma \cdot e_a{}^m}$ This shows that we have found <u>conf. SUGRA</u>, not "<u>Einstein SUGRA</u>".

One way for solving this problem makes use of <u>a geometric interpretation of $\mathcal{H}^m$</u>:

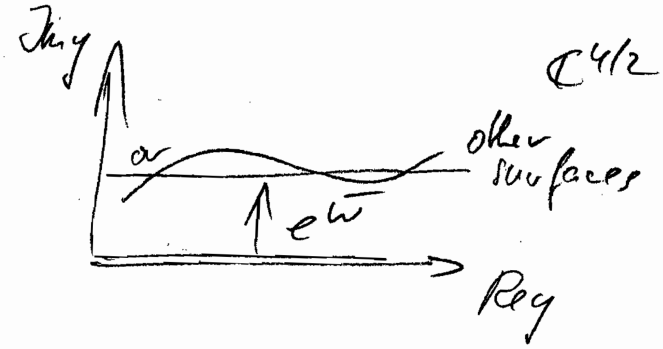
- leave y complex, but restrict to $\bar\theta = (\theta)^*$.

$\Rightarrow$ superspace $\mathbb{C}^{4|2}$

- $x \to e^{\bar w} x = x + i\mathcal{H}$ can be viewed as a diffeomorphism of $\mathbb{C}^{4|2}$

$$(x = \mathrm{Re}\,y, \theta, \bar\theta) \longrightarrow (x + i\mathcal{H}(x,\theta,\bar\theta), \theta, \bar\theta)$$

real surface in $\mathbb{C}^{4|2}$ — some other diffeomorphic surface



(Fact: This surface encodes all information of the original supergeometry (subj. to the constraints))

The residual gauge freedom (expressed just through λ^m, λ^μ with $\bar\lambda_{\dot\mu}$, K being dependent) can be viewed as diffeom. of $\mathbb{C}^{4|2}$:

$$y^m \to y'^m = y^m - \lambda^m(y,\theta)$$
$$\theta^\mu \to \theta'^\mu = \theta^\mu - \lambda^\mu(y,\theta).$$

It induces $\mathcal{H} \to \mathcal{H}'$ which, after going to WZ gauge, includes the σ-tf. (Weyl resc.) which we want to exclude. This can be done by

demanding
$$\mathrm{sdet}\left(\frac{\partial(y',\theta')}{\partial(y,\theta)}\right) = 1 .$$

(defined using the trf. of integration measure under reparameterizations).

This will prevent us from going to the WZ gauge (inconvenient!)

Better: - keep full reparam. invariance

- introduce a new chiral SF $\varphi(y,\theta)$ which, by definition, transforms as

$$\varphi \to \varphi'(y',\theta') = \mathrm{sdet}\left(\frac{\partial(y',\theta')}{\partial(y,\theta)}\right)^{1/3} \varphi(y,\theta)$$

(This is equivalent to the det = 1 constraint since we can always go to the gauge $\varphi = 1$, after which only det = 1 trfs. are allowed.)

- Our only physical d.o.f.s are now $\mathcal{H}^m$ (real) & φ (chiral).

Note: Strictly speaking F (of $A = NF$) is still around. It is discarded by defining, from the very beginning, the extra gauge freedom (super-Weyl)

can gauge $F=1$ $\Leftarrow \begin{cases} E_\alpha \to L E_\alpha \\ \bar{E}_{\dot\alpha} \to \bar{L} \bar{E}_{\dot\alpha} \\ E_a \to L\bar{L} E_a + \cdots \end{cases}$ (defined by algebra)

- Now the geometry is fully defined by $\mathcal{H}^m$
  ($\mathcal{H}^m \to W \to E_\alpha \to E, \Omega \to$ D-algebra);
  φ is needed to write non-conformal actions.
  This is Einstein-SUGRA seen as conf. SUGRA with conf. symm. spont. broken by $\langle\varphi\rangle \neq 0$.

A different (equivalent) formulation of Einstein-SUGRA starts by adding the constraint $T_\alpha = 0$.
($E_\alpha \to L E_\alpha$ is not a symm. of this theory.)
As before, we introduce prepotentials W, N, F.
Define $\boxed{\bar{\varphi}^{-3} \equiv E F^2 \operatorname{sdet}\left(\frac{\partial z'}{\partial z}\right)}$ (where $z'^M = e^W z^M$) $E = \operatorname{sdet}(E_A{}^M)$

One can show that φ satisfies $\bar{E}_{\dot\alpha}\varphi = 0$ and that, for $\varphi = e^W \hat{\varphi}$,

$$\hat{\varphi} \longrightarrow \operatorname{sdet}\left(\frac{\partial(y',\theta')}{\partial(y,\theta)}\right)^{1/3} \hat{\varphi}$$

under the K-λ-trfs. respecting the gravit. superfield gauge.

$\Rightarrow$ <u>φ is the chiral compensator</u>.

12 Actions

12.1 General form

1) $\int \underbrace{d^8z}_{d^4x\, d^2\theta\, d^2\bar\theta} \underbrace{E^{-1}}_{\text{analogue of } \sqrt{-g} \text{ of GR}} \underbrace{\mathcal{L}}_{\text{arbitrary real scalar super-field}}$

2) $\int \underbrace{d^6z}_{d^4x\, d^2\theta} \underbrace{\hat\varphi^3}_{e^{\bar W}\varphi} \underbrace{\hat{\mathcal{L}}_c}_{e^{-\bar W}\mathcal{L}_c}$

φ: covar. chiral form of chiral compensator

$\mathcal{L}_c$: covar. chiral SF

(Quantities with "^" are flat chiral)

- Both 1) & 2) are diff.-invariant.
- They roughly correspond to the $\int d^4\theta$ and $\int d^2\theta$ parts of the rigid susy action.
- They can be translated into each other:

$$\int d^8z\, E^{-1} \mathcal{L} = -\frac{1}{4} \int d^6z\, \hat\varphi^3 \hat{\mathcal{L}}_c$$

with $\mathcal{L}_c \equiv (\bar{\mathcal{D}}^2 - 4R)\mathcal{L}$

let us introduce a set of chiral SFs $\phi = (\phi_1 \cdots \phi_n)$.
Then the general action reads

$$S = \int d^8 z\, E^{-1} \underset{\substack{\Uparrow \\ \text{encodes Kähler potential}}}{\Omega(\phi, \bar{\phi})} + \left[\int d^6 z\, \hat{\varphi}^3 \underset{\substack{\Uparrow \\ \text{superpotential}}}{W}(\hat{\phi}) + \text{h.c.} \right]$$

gauge interactions are introduced through an extra term

$$\int d^6 z\, \hat{\varphi}^3 \underset{\substack{\uparrow \\ \text{gauge coupling}}}{\frac{1}{g^2}} W^\alpha W_\alpha \quad \text{\& extra factors } e^{2V} \text{ in } K$$

to get pure supergravity, let K & W be indep. of ϕ:

$$S_1 = \underbrace{\int d^8 z\, E^{-1}}_{\text{volume of superspace}} (-3 \bar{M}_p^2) \longrightarrow \int d^4 x \sqrt{-g}\, \frac{\bar{M}_p^2}{2} \mathcal{R}(g) + \cdots$$

$$S_2 = \int {}^6 z\, \hat{\varphi}^3 \bar{M}_p^{2} \mu + \text{h.c.} \longrightarrow \int d^4 x \sqrt{-g}\, \underbrace{3 \mu \bar{\mu} \bar{M}_p^{-2}}_{\text{neg. cosmol. constant}}$$

Simplest ϕ-dependent terms $(n=1)$

$$S_3 = \int d^8 z\, E^{-1} \phi \bar{\phi} \longrightarrow \text{chiral SF with canonical kinetic term}$$

$$S_4 = \int d^6 z\, \hat{\varphi}^3 m \hat{\phi}^2 + \text{h.c.} \longrightarrow \text{mass for } \phi$$

Explicit calculation simpler in d^6z - form:

$$S = \int d^6z \, \hat{\varphi}^3 \hat{\mathcal{L}}_c + h.c. \text{ with } \mathcal{L}_c = -\frac{1}{8}(\bar{\mathcal{D}}^2 - 4R) K(\phi, \bar{\phi}) + P(\phi)$$

SUGRA characterized by $\mathcal{H}$, φ (in comp.)

ϕ characterized by $\boxed{\hat{\phi}(x,\theta) = A_\phi + \sqrt{2}\,\theta\psi_\phi + \theta^2 F_\phi}$

go to W/Z gauge for $\mathcal{H}$:

$$\boxed{\mathcal{H}^m = \theta\sigma^a\bar{\theta} e_a{}^m + i\bar{\theta}^2\theta^\alpha \psi_\alpha{}^m + h.c. + \theta^2\bar{\theta}^2 A^m}$$

and choose for φ:

$$\boxed{\hat{\varphi} = e^{-1}\{1 - 2i\theta\sigma_a\bar{\psi}^a + \theta^2 B\}}$$

in principle, all is now said:

(recall $e^{\bar{W}}x = x + i\mathcal{H}$)

$\mathcal{H} \to W$; $W \to \hat{E}_\alpha$

$\varphi, \hat{E} \to F$; $W, F \to E_\alpha$;

$E_\alpha \to E_A, \Omega \to \mathcal{D}, R$ etc.

$\hat{\phi} \xrightarrow{W} \phi$

all of the above can be explicitly worked out.

important: since $\mathcal{H} = \theta\bar{\theta}$ + higher,

$\mathcal{H}^3$ and all higher powers vanish

(simplification!)

11.2.

finally (pure SUGRA)

will play more imp. role in coupling to matter

$$S = M_p^2 \int d^4x\, e^{-1} \Big\{ \frac{1}{2} \mathcal{R}(\tilde{\nabla}) - \frac{1}{3} \bar{\mathbb{B}}\mathbb{B} + \frac{4}{3} \mathbb{A}^a \mathbb{A}_a$$

$$+ \Big[\frac{1}{4} \varepsilon^{abcd}\, \bar{\psi}_a \bar{\sigma}_b \left(\tilde{\nabla}_c \psi_d - \tilde{\nabla}_d \psi_c - \tilde{T}_{cd}{}^e \psi_e \right) + \text{h.c.} \Big] \Big\}$$

with $\tilde{\nabla}$ defined from ∇ by $\omega_{abc} = \tilde{\omega}_{abc}(e,\psi) - \frac{2}{3} \cdot \varepsilon_{abcd} \mathbb{A}^d$.

$\tilde{\nabla} \downarrow \tilde{\omega}_{abc}$, $\nabla \downarrow \omega_{abc}$

$\tilde{\omega}_{abc}(e,\psi)$: indep. of $\mathbb{A}$

12.3 General chiral supergravity model

Let $\Omega = -3e^{-K(\phi,\bar\phi)/3}$

$\uparrow$ Kähler potential

$$
\begin{aligned}
\mathcal{L} = & -\tfrac{1}{2}\mathcal{R}^{(+)} - g_{i\bar j}\,\partial_m A^i \partial^m \bar A^{\bar j} \\
& - i g_{i\bar j}\,\bar\chi^{\bar j}\bar\sigma^m D_m \chi^i + \varepsilon^{klmn}\bar\psi_k \bar\sigma_l D_m \psi_n \\
& - \tfrac{\sqrt2}{2} g_{i\bar j}\,\partial_n \bar A^{\bar j}\chi^i \sigma^m \bar\sigma^n \psi_m + h.c. \\
& + \tfrac{1}{4} g_{i\bar j}\left[i\varepsilon^{klmn}\psi_k\sigma_l\bar\psi_m + \psi_m\sigma^n\bar\psi^m\right]\chi^i\sigma_n\bar\chi^{\bar j} \\
& - \tfrac{1}{8}\left[g_{i\bar j}g_{k\bar l} - 2R_{i\bar j k\bar l}\right]\chi^i\chi^k\bar\chi^{\bar j}\bar\chi^{\bar l} \\
& - e^{K/2}\Big\{\bar W\psi_a\sigma^{ab}\psi_b + h.c. \\
& \qquad + \tfrac{i\sqrt2}{2}(D_i W)\chi^i\sigma^a\bar\psi_a + h.c. \\
& \qquad + \tfrac{1}{2}(D_i D_j W)\chi^i\chi^j + h.c.\Big\} \\
& - e^{K}\left(g^{i\bar j}(D_i W)\overline{(D_j W)} - 3|W|^2\right)
\end{aligned}
$$

Crucial new ingredient: Kähler geometry

Consider the ϕ^i as coordinates of a complex manifold. Metric: $g_{i\bar j} = \partial_i\partial_{\bar j} K$

$\uparrow$ Kähler potential

$(ds^2 = g_{ab}\,dx^a dx^b \longrightarrow ds^2 = g_{i\bar j}\,d\phi^i d\bar\phi^{\bar j})$

*) $R = R(\tilde{\nabla})$,

where $\tilde{\nabla}$ is defined like the Riemannian connection ∇ but with

$$\tilde{\omega}_{nml} = \omega_{nml} + \frac{1}{2}\left\{-\frac{i}{2} e_{la}\left(\psi_m \sigma^a \bar{\psi}_n - \psi_n \sigma^a \bar{\psi}_m\right) + \text{cycl.}\right\}$$

Riemann connection & curvature much simplified compared to real case because of Kähler metric.

$$\Gamma_{ij}{}^{k} = g^{k\bar{e}}\,\partial_i g_{j\bar{e}} \quad \& \text{ h.c.}$$

(no "mixed" Γ-components)

$$R_{i\bar{j}k\bar{e}} = g_{m\bar{e}}\,\partial_{\bar{j}}\Gamma^{m}_{ik}$$ (all other R-comp. vanish)

In the above:

$$D_i W = W_i + K_i W \quad ; \quad g_{i\bar{j}} = K_{i\bar{j}}$$

partial derivatives w.r.t. ϕ^i

$$D_i D_j W = W_{ij} + K_{ij}W + K_i D_j W + K_j D_i W - K_i K_j W - \Gamma_{ij}{}^{k} D_k W$$

$$D_m \chi^i = \partial_m \chi^i + \omega_m \chi^i + \Gamma^{i}{}_{jk}\,\partial_m A^j \chi^k$$

usual "spin connection"

$$- \tfrac{1}{4}\left(K_j \partial_m A^j - K_{\bar{j}} \partial_m \bar{A}^{\bar{j}}\right)\chi^i$$

$$D_m \psi_n = \partial_m \psi_n + \omega_m \psi_n + \tfrac{1}{4}\left(K_j \partial_m A^j - K_{\bar{j}} \partial_m \bar{A}^{\bar{j}}\right)\psi_n$$

Kähler-geometry is inv. under "Kähler-tfs.":

$$K(\phi,\bar{\phi}) \to K(\phi,\bar{\phi}) + f(\phi) + \bar{f}(\bar{\phi})$$

This gives rise to an invariance of the SUGRA action, where, in addition, we have to replace

$$W(\phi) \to W(\phi)\, e^{-f(\phi)}.$$

The scalar potential reads:

$$\boxed{\begin{aligned} V &= e^{K}\left(K^{i\bar{j}}(D_i W)(D_{\bar{j}}\bar{W}) - 3|W|^2\right) \\ &= e^{G}\left(G^{i\bar{j}} G_i G_{\bar{j}} - 3\right) \\ &\qquad \text{with } G = K + \ln W + \ln \bar{W} \end{aligned}}$$

~~SUSY~~

gravitino mass requires $D_i W \neq 0$

<u>indeed:</u> ~~SUSY~~ $\Leftrightarrow D_i W \neq 0$
(F-term breaking)

We see $D_i W = 0 \Rightarrow$ cosm. const. ≤ 0
$(\sim |W|^2)$

<u>Simplest realistic model:</u> Polonyi model

$$W = c_1 + c_2 \phi$$

c_1 & c_2 can be fine tuned to realize $m_{3/2} \neq 0$ & $V_{vac} = 0$.

Polonyi model

$K = \phi\bar{\phi}$ ($\to$ canonical kinetic term for ϕ,
recall $\mathcal{L} \supset K_{i\bar{j}}(\partial_\mu A^i)(\partial^\mu \bar{A}^{\bar{j}})$)

$W = c_1 + c_2\phi$

$\Rightarrow D_\phi W \neq 0$ in vacuum; ~~SUSY~~ ; cosm. const. can be adjusted to be zero.

Derivation of the supergravity scalar potential

$$S = \int d^4x\, d^2\theta\, d^2\bar\theta\, E^{-1}\,\Omega(\phi,\bar\phi) + \int d^4x\, d^2\theta\, \hat\varphi^3 W(\hat\phi) + \text{h.c.}$$

[pure SUGRA: $\Omega = -3\bar M_p^2$ $\quad \left(\bar M_p = M_p/\sqrt{8\pi}\right.$,

$\left.\mathcal{L}_{ART} = \frac{1}{2}\bar M_p^2 R\right)$]

Using the relation between F, φ, $\hat E$ and G it can be shown that, in the flat background case

$$S = \int d^4x\, d^2\theta\, d^2\bar\theta\, \varphi\bar\varphi\,\Omega(\phi,\bar\phi) + \int d^4x\, d^2\theta\, \hat\varphi^3 W(\hat\phi) + \text{h.c.}$$

in more detail:

W/Z-gauge: $\mathcal{H}^m = \theta\sigma^a\bar\theta e_a{}^m + i\bar\theta^2\theta^\alpha\psi_\alpha{}^m + \text{h.c.} + \theta^2\bar\theta^2 A^m$

$$\hat\varphi = e^{-1}\{1 - 2i\theta\sigma_a\bar\psi^a + \theta^2 F_\varphi\}$$

flatness: $e_a{}^m = \delta_a{}^m$, $e^{-1} = 1$, $A^m = 0$

all fermions zero: $\psi_\alpha{}^m = 0$

$$\Rightarrow \mathcal{H}^m = \theta\sigma^a\bar\theta e_a{}^m \;;\; \hat\varphi = 1 + \theta^2 F_\varphi$$

Recall that $\Omega = -3e^{-K/3}$ and that we have the freedom of Kähler-trfs:

$$K \to K + f + \bar f \qquad \text{(with } f \text{ a holo-}$$
$$W \to W e^{-f} \qquad \text{morphic fct. of } \phi\text{)}$$

in terms of Ω and W, it reads

$$\Omega \to \Omega e^{-f/3 - \bar{f}/3}$$
$$W \to \Omega e^{-f}$$

or, simply $\boxed{\varphi \to \varphi e^{-f/3}}$.

We can use this freedom to set $W = 1$.
(Which corresponds to $\Omega = -3e^{-G/3}$, see above.)
We find: (setting all gradients to zero)

$$\mathcal{L} = |F_\varphi|^2 \Omega + F_\varphi \bar{F}_\phi \Omega_{\bar\phi} + \bar{F}_\varphi F_\phi \Omega_\phi + |F_\phi|^2 \Omega_{\phi\bar\phi}$$
$$+ 3F_\varphi + \text{h.c.}$$

$\bar{F}_\varphi$-EOM: $\quad F_\varphi \Omega + F_\phi \Omega_\phi + 3 = 0$

$\bar{F}_\phi$-EOM: $\quad F_\varphi \Omega_{\bar\phi} + F_\phi \Omega_{\phi\bar\phi} = 0$

This can be easily solved for F_φ, F_ϕ, giving
($\mathcal{L} = -V$ in this case)

$$V_{BD}(\phi, \bar\phi) = \frac{9\,\Omega_{\phi\bar\phi}}{\Omega_{\phi\bar\phi}\Omega - |\Omega_\phi|^2}$$

This is to remind us
that the "Einstein-Hilbert-term" reads (BD stands for "Brans-Dicke frame")

$$\mathcal{L} = \frac{\Omega/3}{2} R + \cdots$$

(Since $\int d^8z\, E^{-1} 3$ gives $\frac{1}{2}R$
and we have $\int d^8z\, E^{-1}\Omega$)

To correct this, let $g_{\mu\nu} \to g_{\mu\nu}/(\Omega/3)$

Then: $\sqrt{g} \to \sqrt{g}\,/(\Omega/3)^2$

$\Gamma_{\mu\nu}{}^{\rho} \to \Gamma_{\mu\nu}{}^{\rho}$ (since it involves $g_{\mu\nu}$ & $g^{\mu\nu}$)

$R_{\mu\nu} \to R_{\mu\nu}$ (since it corresponds to one contraction of $\partial_\mu \Gamma_{\nu\rho}{}^{\sigma} + \cdots$)

$R \longrightarrow R\cdot(\Omega/3)$ (since $R = R_{\mu\nu} g^{\mu\nu}$)

$\sqrt{g}\,\frac{\Omega}{3} R \to \sqrt{g}\, R$

$\sqrt{g}\, V \longrightarrow \sqrt{g}\, V/(\Omega/3)^2$

<u>Hence:</u> $V = V_{BD}/(\Omega/3)^2 = \left(\frac{3}{\Omega}\right)^2 \frac{9\,\Omega_{\phi\bar\phi}}{\Omega_{\phi\bar\phi}\Omega - |\Omega_\phi|^2}$

simple algebra, $\Omega = -3e^{-G/3}$

$$V = e^G\left((G_{\phi\bar\phi})^{-1}|G_\phi|^2 - 3\right)$$

(which is the 1-field-case of the general result

$$V = e^G\left(G^{i\bar j} G_i G_{\bar j} - 3\right)$$

No-scale model

Looking at $V_{BD} = \frac{9 \Omega_{\phi\bar\phi}}{\Omega_{\phi\bar\phi}\Omega - |\Omega_\phi|^2}$, it is obvious

that $V_{BD} \equiv 0$ for $\Omega = -3(\phi + \bar\phi)$ or,

equivalently, $K = -3 \ln(\phi + \bar\phi)$.

SUSY is broken since $D_\phi W = D_\phi 1 = K_\phi \neq 0$.
The Planck mass (or, equivalently, the gravitino mass at fixed Planck mass) shifts as ϕ shifts.
(ϕ is a flat direction) hence the name: "no-scale".

This appears to be the perfect solution to the cosmol. const. problem. However:

Terms $\sim \phi\bar\phi$ etc. in Ω are introduced by loops $\Rightarrow$ "the no-scale structure does not survive radiative corrections"