11 Consistent Superstring Theories – part 2

11.1 Idea of the heterotic string (fermionic formulation)

- Left-moving & right-moving sectors of closed string are (almost) completely decoupled.
- WS-SUSY with GSO projection exclude the tachyon.
- Forbidding the tachyon on one side is sufficient because of level matching.

$$\Downarrow$$

We could try to combine r.-mov. superstring with l.-mov. bosonic string:

1) $X_L^\mu(\sigma^+)$ (l.m.)

2) $X_R^\mu(\sigma^-)$; $\psi_-^\mu(\sigma^-)$ (r.m.)

$\mu \in \{0, \cdots, D-1\}$, $D = 10$ from superstring

<u>ghosts</u>:

1) bc-system, l.m. only

2) bc-system + $\beta\gamma$-system, both r.m. only

<u>Central charges</u>: $(\tilde{c}, c) = (\underset{\substack{\uparrow \\ X_L}}{10} \underset{\substack{\uparrow \\ bc_L}}{-26}, \underset{\substack{\uparrow \\ X_R}}{10} + \underset{\substack{\uparrow \\ \psi_-}}{\tfrac{1}{2}10} \underset{\substack{\uparrow \\ bc_R}}{-26} \underset{\substack{\uparrow \\ \beta\gamma_R}}{+11}) = (-16, 0)$

$\Rightarrow$ We need fields contributing $+16$ in the l.m. sector, e.g.,

$$\lambda^A = \lambda_+^A(\sigma^+) \; ; \quad A \in \{1, \cdots, 32\}$$

(32 l.m. Majorana-Weyl fermions)

- <u>Thus</u>: $S = -\frac{1}{2\pi}\int d^2\sigma \left[\sum_{\mu=0}^{9}\left(\partial_a X^\mu \partial^a X_\mu - 2i\psi_-^\mu \partial_+ \psi_{-,\mu}\right) - 2i\sum_{A=1}^{32} \lambda_+^A \partial_- \lambda_+^A\right]$

- This theory has only "(0,1)" SUSY:

$$\delta X^\mu = i\epsilon\psi_-^\mu \; ; \; \delta\psi_-^\mu = \epsilon\partial_- X^\mu \quad (\epsilon - \text{only one chirality})$$

- Quantization as usual.

(Note: no neg. signature & no ghosts related to the λ^A.)

11.2 The SO(32) theory (fermionic formulation)

(obtained by respecting the full symmetry of λ^A)

- Levels:

$$\tilde{N} = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n + \sum_{r=\nu}^{\infty} r\cdot\lambda^A_{-r}\lambda^A_r$$

μ-contraction

$$\nu = \begin{cases} 0 & - R \\ 1/2 & - NS \end{cases}$$

$$N = \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n + \sum_{r=\nu}^{\infty} r\,\psi_{-r}\cdot\psi_r$$

- Level matching & mass-shell conditions:
  - Copying from 10.2 ("type II superstrings"), we have:

$$m^2 = \frac{4}{\alpha'}(N-a) = \frac{4}{\alpha'}(\tilde{N}-\tilde{a})$$

- SUSY-side: $\tilde{a} = \tilde{\nu}$, $\tilde{\nu} = \begin{Bmatrix} 0 & - R \\ 1/2 & - NS \end{Bmatrix}$ as before

- non-SUSY-side: <u>a needs to be determined!</u>

Recall: Boson: 1/24

R-fermion: −1/24 (by SUSY!)

NS-fermion: 1/48

(The total a resulting from these rules were:

$$24 \cdot \frac{1}{24} = 1 \quad (\text{Bosonic string})$$

$$8\left(\frac{1}{24}-\frac{1}{24}\right)=0 \quad (\text{R-sector of sup. str.})$$

$$8\left(\frac{1}{24}+\frac{1}{48}\right)=\frac{1}{2} \quad (\text{NS-sector of sup. str.})$$

Recall also the general formula

- $a = \frac{1}{2}\sum_{n=1}^{\infty}(n-\theta) = \frac{1}{2}\left(\frac{1}{24}-\frac{(2\theta-1)^2}{8}\right) = -\frac{1}{24}+\frac{\theta(1-\theta)}{4}$
- for fermions, the sign flips
- the above-cited numbers for periodic bosons and periodic or antiperiodic fermions follow from this formula.

Thus, for the l.m. (non-SUSY) side of the het. SO(32) string we have:

$$\left\| \begin{array}{ll} R: & a = 8\,\frac{1}{24} - 32\,\frac{1}{24} = -1 \\ NS: & a = 8\,\frac{1}{24} + 32\,\frac{1}{48} = 1 \end{array} \right\|$$

GSO projection:

- r.m. side: as before - demand $(-1)^F = 1$

  $\Rightarrow$ keep NS+, R+
- l.m. side: could consider both $(-1)^{\tilde{F}} = \pm 1$

  but - uneven # of λ's $\Rightarrow$ half-integer $\tilde{N}$ $\Rightarrow$ level-matching impossible since, after GSO, on the sup.string-side the half-integer modes (e.g. NS-vacuum) are gone.

  - even # of λ's is therefore preferred.

$\Rightarrow$ Choose $(-1)^{\tilde{F}} = +1$ (Both NS & R)

- Spectrum:

$$m^2 = \frac{4}{\alpha'}\underbrace{(N-a)} = \frac{4}{\alpha'}(\tilde{N}-\tilde{a})$$

Starts at $N-a=0$ because of SUSY & GSO

$\Downarrow$

$(8_v + 8)$ as in open superstring

$\swarrow$ $R\ (\tilde{a}=-1)$ not relevant for massless states

$\searrow$ $NS\ (\tilde{a}=1)$

(I) $\tilde{\alpha}^{\mu}_{-1}|0,k\rangle_L$

(II) $\lambda^A_{-1/2}\lambda^B_{-1/2}|0,k\rangle_L$

are massless

Together:

SO(32)-repr.

$$(8_v+8)\times(8_v,1) = \underset{\phi}{(1,1)} + \underset{B_2}{(28,1)} + \underset{G}{(35,1)} + \underset{\text{gravitino}}{(56,1)} + \underset{\text{dilatino}}{(8',1)}$$

$+$

$$(8_v+8)\times(1,496) = \underset{\text{gauge}}{(8_v,496)} + \underset{\text{gaugino}}{(8,496)}$$

$\uparrow$ adjoint of SO(32)

($\lambda^A_{-1/2}\lambda^B_{-1/2}|0,k\rangle$ is automatically antisymmetric).

- Thus, we have found an <u>N=1 supergravity multiplet</u> and a <u>SYM multiplet for group SO(32)</u>.

$\Rightarrow$ || SO(32) gauge-theory coupled to SUGRA. ||

11.3 The $E_8 \times E_8$ theory (fermionic formulation)

- We can restrict our symm. principle for the λ^As from $SO(32)$ to $SO(n) \times SO(32-n)$, i.e. allow for an indep. of NS & R BCs to the first n and the following $(32-n)$ fermions.
- Thus, just on the l.m. side, we have the sectors

$$\begin{array}{c} (NS, NS') \\ (R, NS') \\ (NS, R') \\ (R, R') \end{array}$$

$$\underset{\text{for } \lambda^1 \cdots \lambda^n}{\uparrow} \qquad \underset{\text{for } \lambda^{n+1} \cdots \lambda^{32}.}{\uparrow}$$

(This is to be combined with a NS or R sector of the l.m. side.)

- One finds: For generic n $(0 < n < 32)$ no new consistent theory arises since
  - a) level matching impossible (→ GSW); (→ problems)
    [This argument fails for $n = 8$ (or $n = 24$), where a new theory emerges, that turns out to be anomalous.]
  - b) modular invariance violated (→ Pol.)
- The only successful choice is $\underline{\underline{n = 16}}$.
- Consider the massless spectrum of this (naively $SO(16) \times SO(16)'$) theory:

→ <u>r.m. side</u>: with GSO-proj. $(-1)^F = 1$ one finds $(8_v + 8)$ as before.

→ <u>l.m. side</u>: with GSO-proj. $\underset{\uparrow}{(-1)^{\tilde F}} = \underset{\uparrow}{(-1)^{\tilde F'}} = 1$ one finds:

$[A \in \{1 \cdots 16\}, \; A \in \{17 \cdots 32\}]$

<u>NS-NS'</u>: $a = 1$ as before, first level massless ⇒

massless states: $\alpha^\mu_{-1} |0,k\rangle$

$\lambda^A_{-1/2} \lambda^B_{-1/2} |0,k\rangle$ with $A, B \in \{1 \cdots 16\}$

<u>or</u> $A, B \in \{17 \cdots 32\}$.

(The "mixed choice" $A \in \{1 \cdots 16\}$ & $B \in \{17 \cdots 32\}$ or vice versa is killed by the <u>separate</u> conditions $(-1)^{\tilde F} = 1$ & $(-1)^{\tilde F'} = 1$.)

<u>R-NS'</u>: $a = 8 \cdot \frac{1}{24} - 16 \frac{1}{24} + 16 \frac{1}{48} = 0$ ⇒

massless states: $\underset{\substack{\uparrow \\ \text{spinor of } SO(16) \\ 2^8 = 256}}{(\text{R-vacuum})} \times \underset{\substack{\uparrow \\ \text{total singlet}}}{(\text{NS-vacuum})'}$

GSO-proj. ↓

$128 + \cancel{128'}$ (as for the $16 = 8 + \cancel{8'}$ of type II superstring)

<u>NS-R'</u>: analogously ---> 128 of $SO(16)'$

<u>R-R'</u>: $a = -1$ as before ⇒ no massless states.

- Together:

$$(8_v \times 8) \times \left[(8_v,1,1) + (1,\underset{\substack{\uparrow \\ \text{e.g., adjoint of } SO(16).}}{120},1) + (1,1,120) + (1,128,1) + (1,1,128)\right]$$

Thus, we have a 10d $N=1$ SUGRA multiplet

+ 10d $N=1$ vector multiplet (vector + "gaugino")
with trf. properties

$$\boxed{(120,1) + (128,1) + (1,120) + (1,128) \text{ under } SO(16) \times SO(16)'}$$

- Interacting massless vectors can only be consistent in QFT if they transform in the adjoint repr. of a gauge group.
- The exceptional Lie group E_8 has the right properties:

$$E_8 \supset SO(16)$$

$$248 = 120 + 128 \qquad (248 \text{ is the adjoint of } E_8)$$

$\Rightarrow$ (correct) guess: $\boxed{\begin{array}{l}\text{10d } N=1 \text{ SUGRA} \\ + \text{ SYM theory with group } E_8 \times E_8'\end{array}}$

Some comments on group theory:

(for more see H. Georgi: "Lie algebras..." or Slansky's article in Physics Reports.)

- A Lie alg. $\mathfrak{g}$ is simple if it has no non-trivial invariant subalgebra (proper ideal), i.e. no $\mathfrak{h} \subset \mathfrak{g}$ such that $[X,Y] \in \mathfrak{h}$ for any $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$.
- All simple Lie algs. are known: (always complex)

A_n – $SU(n+1)$
B_n – $SO(2n+1)$
C_n – $Sp(2n)$
D_n – $SO(2n)$

G_2, F_4, E_6, E_7, E_8 – "exceptional"

(In the above, the index stands for the rank, i.e. the max. # of diagonal generators – for example
$\text{rank}(SU(2)) = 1$, $\text{rank}(SU(3)) = 2$.)

- For a better understanding of the above scheme:
  → "weights, roots, Dynkin-diagrams, ...".

A shortcut to E_8:

- $SO(16)$ algebra: $[J_{AB}, J_{CD}] = J_{AD}\,\delta_{BC} + \underbrace{\cdots}_{\text{3 similar terms}}$.
- Spinor representation:
  "$[J_{AB}, Q_\alpha]$" $= (\sigma_{AB})_\alpha{}^\beta Q_\beta$; $\sigma_{AB} \equiv \frac{1}{4}[\Gamma_A, \Gamma_B]$
- Choose Q to be Majorana–Weyl
- Define a Lie-alg.-product for the Qs:
  (respecting $SO(16)$, this is essentially unique)

$$[Q_\alpha, Q^\beta] = (\sigma_{AB})_\alpha{}^\beta J^{AB}$$

($Q^\beta = (Q_\alpha)^* = Q_\alpha$ since $Q^* = Q$; note also that $\overline{Q} = Q^{*T}$ since our group is euclidean – compare and contrast this to our discussion of $SO(1,1)$ in 8.1.)

- Need to check that Jacobi identity is fulfilled (→ GSW)

$\Downarrow$

|| J_{AB}, Q_α form a Lie alg. This is Lie(E_8)! ||

- To achieve a deeper understanding of the non-abelian gauge symm. of the heterotic string, the Bosonic formulation is useful. The main idea is

$$\lambda^A(\sigma^+)\,,\ A=1\cdots32 \longrightarrow X_L^{\mu}(\sigma^+)\,,\ \mu=10\cdots25$$

(in addition to $\mu=0\cdots9$ already present)

(16 Bosons have the same $\tilde{c}$ as 32 fermions)

Thus: We effectively combine the l.m. Bosonic with the r.m. superstring.

However: To construct the theory in detail in this approach, a number of new concepts will be needed:

- Bosonization / fermionization
- toroidal compactifications
- T duality
- Narain compactifications

11.4 Bosonization / Fermionization

- Recall some formulae for the free Boson: (with $X^\mu \to \phi$ for generality) $S=-\frac{1}{8\pi}\int d^2\sigma\, \partial_a\phi\,\partial^a\phi$

$$\langle\phi(\sigma)\phi(\sigma')\rangle = 4\pi\int\frac{d^2k}{(2\pi)^2}\cdot\frac{e^{ik\cdot(\sigma-\sigma')}}{k^2} = -\ln\left((\sigma-\sigma')^2\mu^2\right)$$ IR-cutoff

EOMs: $\partial_+\partial_-\phi = 0$

$$\left(\phi(\sigma^+,\sigma^-) = \phi^+(\sigma^+) + \phi^-(\sigma^-)\ ;\ \partial_-\phi^+ = \partial_+\phi^- = 0\right)$$

$$\Rightarrow \langle\phi^+(\sigma^+)\phi^+(\sigma'^+)\rangle = -\ln\left((\sigma^+ - \sigma'^+)\mu\right)$$

$$\langle\phi^-(\sigma^-)\phi^-(\sigma^-)\rangle = -\ln\left((\sigma^- - \sigma'^-)\mu\right)\ ;\ \underline{c = \tilde{c} = 1}$$

- Recall some formulae for the free fermion:

Majorana-Weyl fermion $\psi = \begin{pmatrix}\psi_-\\ \psi_+\end{pmatrix}$;

$$S = \frac{1}{2\pi}\int d^2\sigma\, i\bar{\psi}\gamma^a\partial_a\psi = \frac{i}{\pi}\int d^2\sigma\left(\psi_-\partial_+\psi_- - \psi_+\partial_-\psi_+\right)\ ;\ \underline{c = \tilde{c} = \frac{1}{2}}$$

EOMs: $\partial_+\psi_- = \partial_-\psi_+ = 0$

The similarity to the EOMs for $\phi^\pm$ leads one to hope for map between the two theories, but $c = \tilde{c} = 1/2$ is in the way. Thus, let's consider a pair ψ_+^1, ψ_+^2 instead of just ψ_+. (The discussion of ψ_- is analogous and will be suppressed.)

Simple rewriting:

$$i(\psi_+^1\partial_-\psi_+^1 \underset{\uparrow}{-} \psi_+^2\partial_-\psi_+^2) = i(\psi_+^1+\psi_+^2)\partial_-(\psi_+^1-\psi_+^2) \equiv \underline{\underline{i\chi^1\partial_-\chi^2}}$$

(This sign can be chosen at will.)

This last action is, in fact, more general than our derivation since, under $\sigma^+ \to \sigma^{+\prime} = f(\sigma^+)$, we can define

$$\chi^{1,2} \to \left(\frac{\partial\sigma'^+}{\partial\sigma^+}\right)^{-h_{1,2}}\cdot\chi^{1,2}$$ with $h_1 \neq h_2$ in general

(while $h_1 = h_2 = \frac{1}{2}$ in the ψ-case above).

All we need for conf. invariance is $h_1 + h_2 = 1$.

(Reason: Compensate the factor $\left(\frac{\partial\sigma'^+}{\partial\sigma^+}\right)$ from the integration measure $d\sigma'^+ = d\sigma^+ \cdot \left(\frac{\partial\sigma'^+}{\partial\sigma^+}\right)$.)

In particular, the bc-ghost-action

$$S = \frac{i}{\pi}\int d^2\sigma\, c^+ \partial_- b_{++}$$

falls into this class with $(h_b, h_c) = (2, -1)$.

Note: We could also have introduced the 2nd fermionic field by $i(\psi_+^1 \partial_- \psi_+^1 + \psi_+^2 \partial_- \psi_+^2) = i(\psi_+^1 - i\psi_+^2)\partial_-(\psi_+^1 + i\psi_+^2) \equiv \underline{\underline{i\bar{\chi}\partial_-\chi}}$, resulting in 1 complex rather than 2 real fermions.

Note: It would probably be more appropriate to discuss these issues on a euclidean WS with $\sigma^\pm \to z, \bar{z}$ (as done in Polchinski's Book).

To complete our fermionic formulae:

This derivation is careless! One should really properly think about distributions & inverse operators...

$$\langle \chi^1(\sigma)\chi^2(\sigma')\rangle = \frac{1}{2\pi}\int d^2k\, \frac{e^{ik\cdot(\sigma-\sigma')}}{k^+} + \cdots$$

$$= -\frac{i}{\pi}\partial_+ \int d^2k\, \frac{e^{ik\cdot(\sigma-\sigma')}}{k^2} + \cdots = -i\partial_+ \ln\left((\sigma-\sigma')^+(\sigma-\sigma')^-\right) + \cdots$$

$$= -\frac{i}{(\sigma-\sigma')^+} + \cdots$$

terms less singular as $\sigma^+ \to \sigma'^+$.

- This allows one to make the right guess:

The bosonic operators

$$D_1 = \mu^{1/2} :e^{i\phi^+(\sigma^+)}:$$
$$D_{-1} = \mu^{1/2} :e^{-i\phi^+(\sigma^+)}:$$

have $\langle D_1(\sigma^+) D_{-1}(\sigma'^+)\rangle = \frac{1}{(\sigma^+ - \sigma'^+)}$.

This equality follows from analysing the power series in $\ln(\sigma-\sigma')^+$ (coming from the $\phi^+\phi^+$-correlator), which sums to $\exp(-\ln(\sigma-\sigma')^+)$. (→ problems)

|| Thus, we propose to identify:

$$\chi^1(\sigma^+) \sim :e^{i\phi^+(\sigma^+)}: \ ; \ \chi^2(\sigma^+) \sim :e^{-i\phi^+(\sigma^+)}: \ ||$$

A highly non-trivial check is $\{D_1\underbrace{(\sigma,\tau)}, D_1(\sigma',\tau)\} = 0$

<u>beware</u>: We sometimes called this combination "σ" above.

(The derivation uses the commut. relations of the ϕ-FT and the Baker-Campbell-Hausdorff formula. → problems)

<u>Fermions on a circle – Bosons as "angular variables"</u>

- Let $\sigma \in (0, 2\pi)$ parameterize the closed string.

$$\chi^1(\sigma) = :e^{i\overset{\uparrow}{\phi^+(\sigma)}}: \quad (\text{with } \tau = 0, \text{ i.e. } (\tau,\sigma) = (0,\sigma))$$

$$\phi^+(\sigma) = \phi_0 + \sigma \cdot p_0 + i\sum_{n\neq 0} \frac{1}{n}\phi_n e^{-in\sigma}$$

- $\chi^1(\sigma) = \chi^1(\sigma+2\pi)$ ⇒ p_0 can take only discrete values because of the σ-linear term in the exponent

⇓

If, because of $[p_0, \phi_0] = -i$, we attempt to realize p_0 as $p_0 = -i\frac{\partial}{\partial\phi_0}$, the discreteness of p_0 forces us to conclude that ϕ_0 must be periodic.

More precisely: The wave function $\Psi = \Psi(\phi_0)$ (really: "wave functional") fulfils $\Psi(\phi_0 + 2\pi) = - \Psi(\phi_0)$

$\uparrow$ emerges from a careful discussion of the normal-ordering of ϕ_0 & p_0.

$\Rightarrow$ || Classically, ϕ_0 is to identified with $\phi_0 + 2\pi$. It is an "angular variable". ||

11.5 Toroidal compactifications & T duality

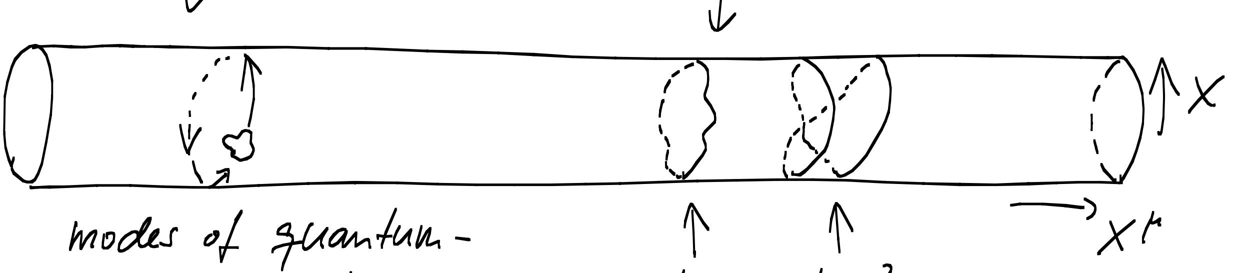
Consider Bosonic string with target space $\mathbb{R}^{25} \times S^1$ (X^μ with $\mu = 0 \dots 24$ & X with "$X \equiv X + 2\pi R$").

$$\Rightarrow X(\sigma,\tau) = x + p\tau + 2L\sigma + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\left[\alpha_n e^{-2in\sigma^-} + \tilde{\alpha}_n e^{-2in\sigma^+}\right]$$

with $p = \frac{m}{R}$ & $L = n \cdot R$ $(m, n \in \mathbb{Z})$

(cf. the periodicity vs. discreteness argument above)

(since X & $X + 2L\pi = X + 2\pi R \cdot n$ are identified, this extra term is allowed)



modes of quantum-mechanical particle on S^1 ("Kaluza-Klein modes")

$n=1$ $n=2$ ("Winding modes")

- write $p\tau + 2L\sigma = \frac{p}{2}(\sigma^+ + \sigma^-) + L(\sigma^+ - \sigma^-) = \left(\frac{p}{2} + L\right)\sigma^+ + \left(\frac{p}{2} - L\right)\sigma^-$.

- write $X = X_L(\sigma^+) + X_R(\sigma^-)$ with

$$X_{L,R} = x^{\mu}_{L,R} + \left(\frac{p}{2} \pm L\right)\sigma_\pm + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}e^{-2in\sigma^\pm}$$

standard derivation - - - - - - - - - -> $\downarrow$
of Virasoro-alg.

$$L_0 = \frac{1}{8}\Big[\underbrace{p_\mu p^\mu}_{\mu=0\cdots 24} + (\underbrace{p-2L}_{26\text{th "momentum"}})^2\Big] + N$$

(+ same with $L_0, N, p-2L \longrightarrow \tilde{L}_0, \tilde{N}, p+2L$)

$\Rightarrow$ mass shell condition (restoring α'):

$$(\text{mass})^2 = \frac{2}{\alpha'}(N+\tilde{N}-2) + \underbrace{\frac{m^2}{R^2} + \frac{n^2R^2}{\alpha'^2}}_{(\text{from the } L^2 \text{ \& } p^2 \text{ terms})}$$

$\Rightarrow$ level matching:

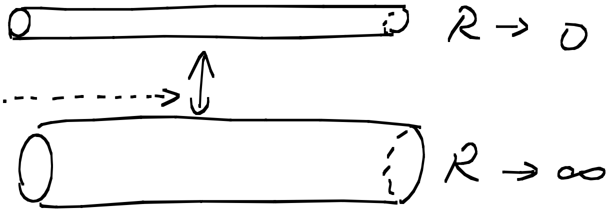
$$N - \tilde{N} = \underbrace{m\cdot n}_{(\text{from the } L\cdot p \text{ term})}$$

- <u>Observe</u>: spectrum symmetric under $m \leftrightarrow n$ & $R \to R' = \frac{\alpha'}{R}$.
- <u>Fact</u>: This symmetry extends to the interacting theory. This is a fundamental and general feature of string theory called <u>T-duality</u>.

("Duality" $\equiv$ $\exists$ two different string-theoretic descriptions of one and the same phys. situation.)

- <u>in particular</u>:

exchange of winding & KK modes



(→ In agreement with intuitive ideas about "quantum geometry" or an effective "minimal length", it is "not possible" to make the S^1 arbitrarily small: For $R \ll \sqrt{\alpha'}$, one finds that, effectively, a new large dimension has "opened up".)

- In addition to the usual massless states (with $m = n = 0$), new massless states appear at specific R:
- Let, e.g., $N - \tilde{N} = m \cdot n = 1$ (larger differences will not work!)

Let this be realized through $N = 1$ and $\tilde{N} = 0$.

$$\Rightarrow (\text{mass})^2 = \frac{2}{\alpha'}(\underbrace{N + \tilde{N} - 2}_{= -1}) + \frac{m^2}{R^2} + \frac{R^2}{m^2 \alpha'} \overset{!}{=} 0$$

$$\Rightarrow 2 = \frac{m^2 \alpha'}{R^2} + \frac{R^2}{m^2 \alpha'} \equiv x + \frac{1}{x} \quad \Rightarrow \quad x = 1$$

$\Rightarrow R^2 = \alpha'$; $m = n = \pm 1$ (no other options since we also have $m \cdot n = 1$)

("self-dual point")

$\Rightarrow$ **new states:** $\alpha_{-1}^{\mu} |\underset{\uparrow}{1}, \underset{\uparrow}{1}\rangle$; $\alpha_{-1}^{\mu} |-1,-1\rangle$; $\tilde{\alpha}_{-1}^{\mu} |1,-1\rangle$; $\tilde{\alpha}_{-1}^{\mu} |-1,1\rangle$

(with mass = 0)

m & n of string vacuum

Fact: Together with the massless vectors from $g_{\mu 25}$ & $B_{\mu 25}$ these realize gauge theory with group $SU(2)_L \times SU(2)_R$

11.6 Narain compactifications and the Bosonic formulation of the heterotic string

- For S^1 compactifications we have derived

$$(\text{mass})^2 = \frac{m^2}{R^2} + \frac{n^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2)$$

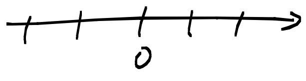
$$0 = m \cdot n + N - \tilde{N} .$$

- With $k_{L,R} = \frac{n}{R} \pm \frac{wR}{\alpha'}$, this can be rewritten as

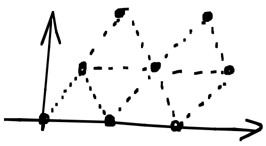
$$(\text{mass})^2 = \underset{\uparrow}{k_L^2} + \frac{4}{\alpha'}(N-1) = \underset{\uparrow}{k_R^2} + \frac{4}{\alpha'}(\tilde{N}-1)$$

Think of this as the (in general different) discrete momenta in the compact dimensions.

- If several dims. are compactified on a torus, k_L & k_R become vectors. Because of the periodicity, they come from a lattice Γ (with twice the dim. of the torus).
- Γ being a "lattice" implies $v, w \in \Gamma \Rightarrow n \cdot v + m w \in \Gamma$ for $n, m \in \mathbb{Z}$.
- Examples:



or



- In particular, we can consider indep. tori (→ indep. lattices) for k_L & k_R. (We will obviously need this for a bosonic description of the het. string.) Such a general framework is called "Narain compactification".

- We will now use $\ell_{L,R} = k_{L,R}\sqrt{\alpha'/2}$ instead of $k_{L,R}$.

- Write $l = (l_L, l_R) \in \Gamma$.
- Consistency of the theory requires (→ problems):

$$\| \ \Gamma \text{ is even \& self-dual} \ \|$$

<u>even:</u> $l \cdot l \in 2\mathbb{Z}$ (with $l \cdot l' \equiv l_L \cdot l_L' \underset{\uparrow}{-} l_R \cdot l_R'$;
this is called "Lorentzian signature (k,k) for k compact dims.)

<u>self-dual:</u> $\Gamma = \Gamma^*$ ($\Gamma^* \equiv$ dual lattice)

<u>dual lattice:</u> all v with $v \cdot w \in \mathbb{Z}$ for all $w \in \Gamma$

<u>Note:</u> While this still allows any torus compactification where X_L & X_R are treated equally, it is very restrictive in the "non-geometric" case, where X_L & X_R are treated differently.

<u>Now return to the heterotic string:</u>

- Bosonic formulation: 26 l.-movers; 10 r.-movers.
- Could consider $d \leq 10$ non-comp. dims. and
  l_L^m, l_R^n with $d \leq m \leq 25$; $d \leq n \leq 9$
  where $l = (l_L, l_R) \in \Gamma$, Γ even, self-dual with Lorentzian signature $(26-d, 10-d)$.
  (This is possible and phenomenologically interesting!)
- However: We will focus on $d = 10$.
  ⇒ Need even self-dual lattice with dim. 16 & euclid. signature.

<u>Facts:</u> – Even self-dual eucl. lattices exist only for dim. $\in 8 \cdot \mathbb{N}$.

— For dim. = 8, there is only on such lattice:

Γ_8 : all points $(n_1, \cdots, n_8)$ & $(n_1 + \frac{1}{2}, \cdots, n_8 + \frac{1}{2})$
for n_i integers and $(\sum_i n_i)$ even.

— For dim. = 16, there are two such lattices:

① $\Gamma_8 \times \Gamma_8$

② Γ_{16} : all points $(n_1, \cdots, n_{16})$ & $(n_1 + \frac{1}{2}, \cdots, n_{16} + \frac{1}{2})$
for n_i integers and $(\sum_i n_i)$ even.

What are the massless modes?

recall: $$(\text{mass})^2 = \underbrace{\frac{2}{\alpha'} l_L^2 + \frac{4}{\alpha'}(N-1)} = \underbrace{\frac{2}{\alpha'} l_R^2}_{\equiv 0} + \underbrace{\frac{4}{\alpha'}(\tilde{N} - \tilde{\nu})}_{\text{usual } (8_v + 8) \text{ of superstring}}$$

either ① $l_L^2 = 0$ & one α^μ-excitation
or ② $l_L^2 = 0$ & one α^m-excitation
or ③ $l_L^2 = 2$ & no α-excitations

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① $\Rightarrow$ usual N=1 SUGRA multiplet

② & ③ $\Rightarrow$ a large number of N=1 vector multiplets
(These correspond precisely to the 248 generators of SO(32) ($\to \Gamma_{16}$) or $E_8 \times E_8$ ($\to \Gamma_8 \times \Gamma_8$).

This is very similar to the $SU(2)_L \times SU(2)_R$ of the Bosonic string on S^1 at special radius; recall that the generators of $U(1)_L \times U(1)_R$ arose "automatically" from $g_{\mu 25}$ & $B_{\mu 25}$ ($\hat{=}$ our case ②) while the others arose "in addition" because of the special radius ($\hat{=}$ our case ③).

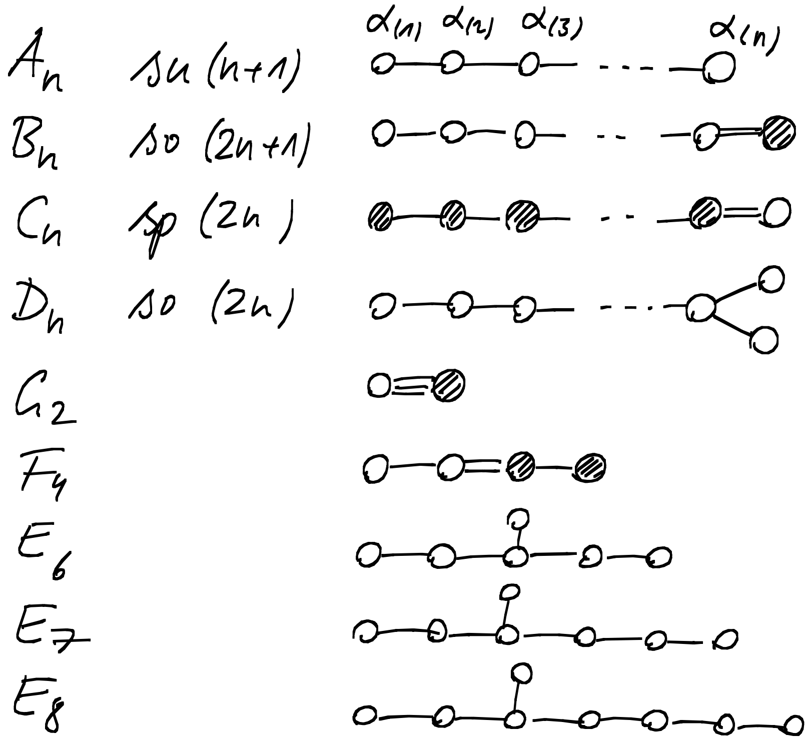
How are Γ_{16} & $\Gamma_8 \times \Gamma_8$ related to gauge groups?

- Consider some Lie alg. and work in the adj. repr.
- For "rank r", r generators can be simultaneously diagonalized: H_i ; $i = 1 \cdots r$ (Cartan subalg.)
- Choose them such that $\mathrm{tr}_{adj.}(H_i H_j) = \lambda \delta_{ij}$ (with λ arbitrary but fixed)
- Choose the other generators such that $[H_i, E_\alpha] = \alpha_i E_\alpha$.
  (E_α – "root"; α – "root vector" ≡ vector in $\mathbb{R}^r$ labelling the non-Cartan generators, i.e. the roots)
- The Lie-alg. is fully defined by adding the commutators

$$[E_\alpha, E_\beta] = \underbrace{N_{\alpha,\beta}}_{\text{some (easily calculable) numbers}} E_{\alpha+\beta} \quad ; \quad [E_\alpha, E_{-\alpha}] = \alpha_i H_i \, .$$

- All sets of vectors α for which this defines a simple Lie alg. have been classified (by simple geometric methods, → Book by Georgi).
- Focus on a minimal set of α's from which all the others follow by addition ("simple roots").
- Facts: – They come in at most two different lengths (symbols: ○ – long, ● – short).
  – They only form angles
    90° – symbol: "   "
    120° – symbol: ——
    135° – symbol: ═══
    150° – symbol: ≡
- This allows for a complete description in terms of ...

Dynkin diagrams:



Fact: The roots of SO(32) and $E_8 \times E_8$ come from Γ_{16} and $\Gamma_8 \times \Gamma_8$ described above (properly choosing λ).

How does the gauge symm. (the actual Lie-alg.) enter string theory? (very roughly)

- With our massless states come vertex ops.

$e^{il_L\cdot X_L}\cdots$ — for the roots

(as for our bosonic vacuum above)

$\partial X_L^{m}\cdot e^{i\underset{\uparrow\,=0}{l_L} X_L}\cdots$ — for the Cartan generators

(because of the α^m-excitation) $(m = 10 \cdots 25)$

- Such holomorphic (or anti-holomorphic) expressions give rise to (conserved) currents $j^A(z)$

(Simply since $\partial_a (j^A)^a = 0$ for $(j^A)^a = (j_z^A(z), 0)$ in the $(z,\bar z)$-basis and $\partial_a (j^A)^a = \partial_{\bar z}\, j_z^A(z) = 0$.)

- Laurant-expansion: $j^A(z) = \sum_{m\in\mathbb{Z}} \frac{j_m^A}{z^{m+1}}$

$$[j_m^A, j_n^B] = m k^{AB} \delta_{m,-n} + i f^{ABC} j_{m+n}^C$$

(current-alg., affine Lie-alg., Kac-Moody-alg.)

- The $m=0$ modes form a <u>Lie-alg.</u> (f^{ABC} are the structure constants.)

- <u>In</u> our case:

$$e^{il_L X_L} \cdots \text{ with } e^{il_L' X_L} \cdots \quad \text{--}\text{--}\text{--}\!\!> \quad e^{i(l_L + l_L')X_L}$$

(cf. $[E_\alpha, E_\beta] \sim E_{\alpha+\beta}$)

$$\partial X_L^m \cdots \text{ with } e^{il_L X_L} \cdots \quad \text{--}\text{--}\text{--}\!\!> \quad l_L^m e^{il_L X_L}$$

(cf. $[H_i, E_\alpha] \sim \alpha_i E_\alpha$)

This is (very roughly) how the Lie alg. relations in terms of roots and Cartan generators find their way into the current algebra and thus into string theory.