13 Calabi-Yau compactifications

13.1 Compactifications

- Assume background to respect 4d Poincare symm.
  Most general metric:

$$G_{MN}(x,y) = \left(\begin{array}{c|c} f(y)\eta_{\mu\nu} & \\ \hline & G_{mn}(y) \end{array}\right)$$

$x^\mu = (x^0 \dots x^3)$ $\quad y^m = (y^4 \dots y^9)$

- Topology: $\mathcal{M} = \mathbb{R}^4 \times_w K^6 \leftarrow$ compact 6d manifold

  w: to remind us of the "warped" metric ansatz

- Focus on $f(y) = 1$ and $\mathcal{M} = \mathbb{R}^4 \times K^6$ (non-warped case)
- For a scalar field φ:

$$\mathcal{L} \supset \varphi D^M \partial_M \varphi \equiv \varphi \Delta_{10} \varphi = \varphi \Delta_4 \varphi + \varphi \Delta_6 \varphi$$

- Let $\varphi = \sum_n \varphi_n(x) \underbrace{\tilde{\varphi}_n(y)}_{\text{eigenfunctions of } \Delta_6}$

  (e.g., for $K^6 = T^6$, $n = (n^1, \dots, n^6)$ and

$$\tilde{\varphi}_n = \exp\left[\sum_{m=4}^{9} \frac{n^m y^m}{R_m}\right]$$ with $\{R_m\}$ a set of radii)

- $S_{4d,\text{eff}} = \text{Vol}(K^6) \int d^4x \sum_n \varphi_n(x) \left(\partial_\mu \partial^\mu - m_n^2\right) \varphi_n(x)$

  $\varphi_n(x)$: set of 4d fields (Kaluza-Klein-modes)

  m_n^2: (in our example: $= \sum_{m=4}^{9} \left(\frac{n^m}{R_m}\right)^2$)

- Most important for us: 4d fields with $m_n = 0$
  ($\hat{=}$ zero modes of Δ_6)

- Simplest case: $\mathbb{R}^4 \times K^6$ solves EOMs with all background fields [except dilaton ($\varphi = $ const) and metric (see above)] set to zero.
- $T_{MN} = 0 \xRightarrow[\text{(Einstein eq.)}]{} R_{MN} = 0 \Rightarrow R_{mn} = 0$ ("Ricci flatness")
- It is convenient (though not strictly necessary) to insist on one surviving SUSY ($N=1$ SUSY in 4d).
- Recall: $\delta_\epsilon \psi_M = \nabla_M \epsilon$ + field-dependent terms (generically present in 10d SUGRAs)

  in general: Maj.-Weyl spinor in $d=10$
- Our vacuum should be invariant under (at least) one specific $\epsilon = \epsilon(y)$.
- Note: ϵ is (locally) from the 16 of $SO(1,9)$

$$SO(1,9) \supset SO(1,3) \times SO(6) \quad (\text{use } SO(6) \equiv SU(4))$$
$$16 = (2,4) + \underbrace{(\bar{2},\bar{4})}$$

because of the Majorana constraint, this will be fixed by the (2,4)-part

- 16 SUSY generators in 10d $\Longrightarrow$ maximum of 4×4 SUSY generators in 4d

  "(2, 4)

  compl. Weyl spinor 4 copies thereof
- We want one of these 4 4d-$N=1$-SUSYs to survive
- For this, we need only one solution to $\nabla_M \epsilon(y) = 0$ (The 3 other real generators will be guaranteed by $SO(1,3)$-symm.)

- Write $\epsilon_{10d}(y) = \epsilon_{4d} \times \epsilon_{6d}(y)$ with $\underline{\underline{\nabla_m \epsilon_{6d}(y) = 0}}$

(our manifold should have a covariantly constant spinor)

- This implies: $[\nabla_m, \nabla_n]\, \epsilon_{6d}(y) = \frac{1}{4} \underbrace{R_{mnpq} \Gamma^{pq}} \epsilon_{6d}(y) = 0$

This infinitesimal $SO(6)$ rotation should leave one spinor invariant. (This extends to finite parallel transports along the whole manifold.)

$\Rightarrow$ 

all such parallel transports should leave one spinor (one direction of the 4 of $SU(4)$) invariant.

$\Rightarrow$ The holonomy is reduced from $SO(6) = SU(4)$ (the generic case) to $SU(3)$.

$\left[\begin{array}{l} SU(4) \text{ matrix} \\ \text{(generic)} \end{array} \overset{!}{=} \begin{pmatrix} SU(3)\text{-matrix} & \\ & 1 \end{pmatrix} \right.$ which, applied to $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$, acts trivially. $\Big]$

|| $\underline{\underline{\text{Thus}}}$: We are looking for Ricci-flat manifolds with a covariantly constant spinor (or $SU(3)$ holonomy). ||

13.2 Calabi-Yau manifolds

General setting:

- Consider 3d-complex (rather than 6d real) manifolds. (An even-dim. real manifold is complex if the transition fcts. are holomorphic.)

- generic metric: $ds^2 = 2G_{i\bar{j}}\, dz^i d\bar{z}^{\bar{j}} + \underbrace{G_{ij}\, dz^i dz^j + G_{\bar{i}\bar{j}}\, d\bar{z}^{\bar{i}} d\bar{z}^{\bar{j}}}$

  if these vanish, G is called hermitian.

- A subclass of hermitian metrics are Kähler metrics:

  $G_{i\bar{j}}$ can be locally written as $G_{i\bar{j}} = \frac{\partial}{\partial z^i}\frac{\partial}{\partial \bar{z}^{\bar{j}}} \underbrace{K(z,\bar{z})}$

  "Kähler potential"

- Important for us: There exist many compact Kähler manifolds.

- Naive attempt: polynomial constraints in $\mathbb{C}^n$

  (e.g. $\mathbb{C}^2$ parameterized by (z^1, z^2); $K = z^1\bar{z}^1 + z^2\bar{z}^2$;

  submanifold defined by $(z^1)^2 + (z^2)^2 = 1$; ...

  ... easy to generalize to $n \geq 3$ and constraints making the submanifold 3-dimensional)

- Problem: All such submanifolds are non-compact!

- Better: Start with complex projective spaces $\mathbb{P}^n$ instead of $\mathbb{C}^n$. ($\mathbb{P}^n$ is the set of all vectors $(z^0, \ldots, z^n)$

  with $(z^0, \ldots, z^n) \underset{\uparrow}{\sim} (\lambda z^0, \ldots, \lambda z^n)$).

  equivalence relation

  The $\mathbb{P}^n$ are compact and have a natural Kähler metric. (other notation: $\mathbb{C}P^n$)

(in the patch where $z^j \neq 0$ choose coords. $\zeta^i = \frac{z^i}{z^j}$ $(i \neq j)$

& $K = \ln\left(1 + \sum_{i \neq j} |\zeta^i|^2\right)$ ($\Rightarrow$ "Fubini-Study-metric"))

<u>Theorem:</u> (Yau, conjectured by Calabi)

Given a Kähler manifold satisfying a certain topological restriction (vanishing 1st Chern class, $c_1 = 0$), then the Kähler metric can be deformed to a <u>unique</u> Ricci-flat metric (within a certain "Kähler class" (see below)).

$\Rightarrow$ <u>Calabi-Yau manifolds</u> (SU_3 holonomy & covar. constant spinor are then guaranteed.)

<u>Fact:</u> There exists a large class of submanifolds of $\mathbb{P}^n$ (and of products $\mathbb{P}^n \times \mathbb{P}^m \times \cdots$), defined by polynomial constraints, which have $c_1 = 0$ and thus provide examples of CY-manifolds. ["Complex Intersection CYs"]

(Although the Ricci-flat metric can, unfortunately, not be explicitly constructed, a lot is known about these spaces & the corresp. string-th. compactifications.)

<u>Simplest example:</u> Quintic in $\mathbb{P}^4$: Submanif. of $\mathbb{P}^4$ defined by $\underbrace{c_{i_1 \ldots i_5}} z^{i_1} \ldots z^{i_5} = 0$.

smoothly varying these coeff-s. corr. to deforming the submanifold.

- <u>To understand $c_1 = 0$ better</u>, we need some more technology:

1) <u>de Rham cohomology:</u>

$$\omega_p < \begin{array}{ll} \text{closed}: & d\omega_p = 0 \\ \text{exact}: & \omega_p = d\alpha_{p-1} \end{array} \quad ; \quad \text{exact} \Rightarrow \text{closed} \quad (\text{since } d^2 = 0)$$

- pth de Rham cohomology of K:

$$H^p(K) = \frac{\text{closed } p\text{-forms on } K}{\text{exact } p\text{-forms on } K} \quad ; \quad \text{Betti number: } b_p = \dim(H^p(K))$$

($\equiv$ lin. space of equivalence classes of closed forms; $\omega_p \sim \omega_p'$ if $\exists$ exact ω_p'' such that $\omega_p = \omega_p' + \omega_p''$)

- example: $- H^0(S^1) = \dfrac{\{\text{const. fctr.}\}}{\{0\}} = \text{const. fcts.} \cong \mathbb{R}$

(b_0 simply counts connected components)

$$- H^1(S^1) = \frac{\{\text{all 1-forms}\}}{\{\text{1-forms } f_1 dx^1 = (\partial_1 f)\cdot dx^1\}} \cong \mathbb{R}$$

(Locally, any f_1 can be written as $\partial_1 f$. Globally, this can fail if $\int_0^{2\pi} f_1 \cdot dx^1 = c \neq 0$. The constant c parameterizes the above equiv. classes. Thus $b_1(S^1) = 1$ is related to the presence of a non-contractible loop.)

$- H^2(S^1) = 0$ by dimensionality.

2) Dolbeault cohomology:

- in the complex case: $d = \partial + \bar\partial$ with $\partial = dz^i \partial_i$ & $\bar\partial = d\bar z^{\bar i} \partial_{\bar i}$

$\partial^2 = \bar\partial^2 = 0$

- using the concept of p,q-forms ($\omega_{p,q} = (\omega_{p,q})_{i_1 \cdots i_p, \bar i_1 \cdots \bar i_q} \cdot dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar z^{\bar i_1} \wedge \cdots \wedge d\bar z^{\bar i_q}$)

define

$$H^{p,q}_{\bar\partial}(K) = \frac{\bar\partial\text{-closed } (p,q)\text{-forms}}{\bar\partial\text{-exact } (p,q)\text{-forms}}$$

- For Kähler manifolds $H_{\bar\partial}^{p,q} = H_{\partial}^{p,q} \equiv H^{p,q}$

  $\dim\left(H^{p,q}\right) = h^{p,q}$ (Hodge numbers)

Back to our main interest: <u>CYs</u>

- Kähler metric $\Rightarrow$ Ricci tensor has only non-zero components of type $R_{i\bar{j}}$.
- Ricci-form: $\mathcal{R}_{1,1} = R_{i\bar{j}}\, dz^i \wedge dz^{\bar{j}}$; $d\mathcal{R}_{1,1} = 0$
- $\mathcal{R}_{1,1}/(2\pi)$ defines element in $H^{1,1}$. This element is the 1st Chern class c_1. $c_1 = 0$ means $\mathcal{R}_{1,1}$ is exact.

(Conditions on constraint polynomials in products of $\mathbb{P}^n$ are known which ensure $c_1 = 0$.)

- Furthermore, for a CY-3-fold:

$$
\left.\begin{array}{ccccccc}
 & & & h^{3,3} & & & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array}\right\} \text{by symm.}
\quad = \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

(Euler #: $\chi = 2\left(h^{1,1} - h^{2,1}\right)$)

- particularly important entries:
  - volume form ; unique holom. 3-form Ω (with $\bar{\Omega}$);
  - $h^{1,1} > 0$ (since metric-form $\in H^{1,1}$)

- A cohomology class H^p contains closed forms ω_p differing by exact forms.
- Demanding $d*\omega_p = 0$ (in addition to $d\omega_p = 0$) one finds a <u>unique</u> harmonic form ω_p in each H^p

$$\left(\Delta \omega_p = (d + *d*)^2 \omega_p = 0 \right)$$

- This carries over to the Dolbeault-cohomology of Kähler manifolds.
- A metric fluctuation $G_{i\bar{j}} \to G_{i\bar{j}} + g_{i\bar{j}}$ can be described by a (1,1)-form $g = g_{i\bar{j}}\, dz^i \wedge d\bar{z}^{\bar{j}}$. It is massless if $\Delta g = 0 \Rightarrow h^{1,1}$ counts # of these "<u>Kähler moduli</u>".

(${G_{i\bar{j}}\, dz^i \wedge d\bar{z}^{\bar{j}} \in H^{1,1}}$ specifies the Kähler class, which is changed by this fluctuation.)

- A metric fluctuation $G_{ij} = 0 \to G_{ij} = 0 + g_{ij}$ can be described by a (1,2)-form $\omega_{i\bar{e}\bar{m}} = g_{ij}\, G^{j\bar{k}}\, \Omega_{\bar{k}\bar{e}\bar{m}}$. It is massless if $\Delta\omega = 0 \Rightarrow h^{1,2}$ counts # of "<u>complex structure moduli</u>"

(After the metric is modified by g_{ij}, the "complex structure" has to be changed to make the manifold again "Kähler".)

- So far, we have been discussing generic CY-compactifications (this is relevant for all 10d superstring theories).
- more specifically: <u>heterotic case</u>:

$\tilde{H}_3 = 0 \Rightarrow$ SUGRA EOMs require $tr(R_2 \wedge R_2) - tr_v(F_2 \wedge F_2) = 0$

- simplest solution: Identify gauge connection ($SU_3 \subset E_8$) with spin connection ($SU_3 \subset SO_6$)
- use the "standard embedding" of SU_3 in E_8:

$$E_8 \supset E_6 \times SU_3 \quad \text{(max. rank subgroup)}$$

- One finds fields charged under the "surviving" 4d gauge group E_6:
  - $h^{2,1}$ chiral superfields (in particular 4d Weyl fermions) in 27 of E_6
  - $h^{1,1}$ chiral superfields in $\overline{27}$ of E_6
- Note:

$$E_6 \supset SO_{10} \supset SU_3 \times SU_2 \times U_1$$

$$27 = \begin{array}{l} 16 \\ +10 \\ +1 \end{array}$$

16 = one complete family (incl. singlet ν) of the standard model

- $\Rightarrow$ # of families: $h^{2,1} - h^{1,1} \equiv -\frac{\chi}{2}$ Euler #.
- for recent work on models of this type (in fact, much more involved...) $\rightarrow$ B. Ovrut et al.