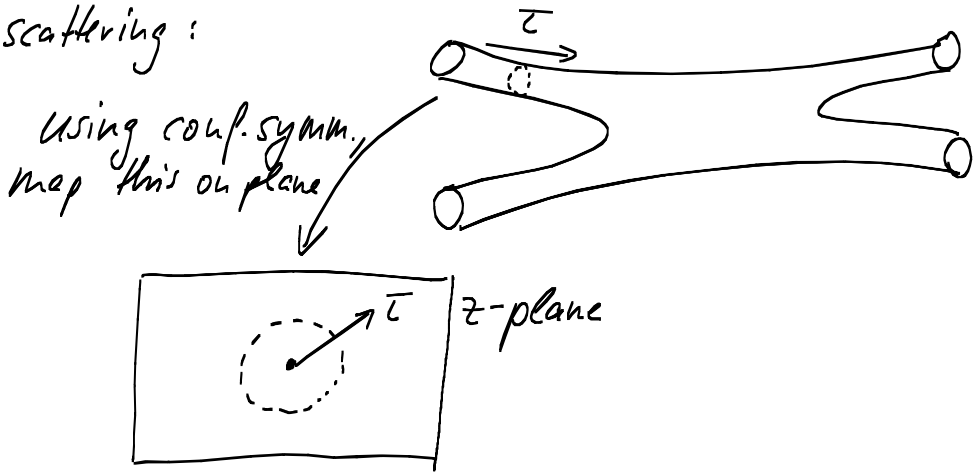
14 Filling some gaps & final remarks

14.1 Deriving Vertex operators

2 → 2 scattering:



- all information about incoming state (contained in inner part of the circle) can be described by wave functional:

$$\underset{\substack{\uparrow\\ \text{field at boundary}}}{\Psi[X_b]} = \underset{\substack{\uparrow\\ \text{over all fields inside circle}\\ \text{with correct boundary values}}}{\int [DX_i]_{X_b}} \exp\left[-\frac{1}{2\pi\alpha'} d^2z\, \partial X \bar{\partial} X\right] \cdot \underset{\substack{\uparrow\\ \text{some operator}\\ \text{at } z=0}}{O(X)}$$

- The above corresponds to separating a certain region of the WS in the process of doing the full path integral.
- Non-trivial fact: Any incoming state can be characterized by some local operator as shown above (the opposite is trivial by the above definition of Ψ).

$$\Rightarrow \left\| \text{ "State - Operator - Mapping" } \right\|$$

- let first $O(X) = 1$
- let $X_i = \underset{\substack{\uparrow\\ \text{should satisfy EOMs}}}{X_{cl}} + X_i'$ $\Rightarrow X_{cl} = X_0 + \sum_{n=1}^{\infty} \left(z^n X_n + \bar{z}^n X_{-n} \right)$

- $$\Psi_1[X_b] = \underbrace{e^{-S_{cl}}}_{\text{with bound. at } |z|=1} \underbrace{\int [DX_i']_{X_b'=0} \exp\left[-\frac{1}{2\pi\alpha'}\int d^2z\, \partial X \bar{\partial} X\right]}_{\text{indep. of } X_b}$$

Operator at $z=0$

$$S_{cl} = \frac{1}{\alpha'} \sum_{m=1}^{\infty} m X_m X_{-m} \longleftarrow \text{Fourier coeffs. of } X_b$$

- <u>Claim:</u> $\Psi_1[X_b] \sim \exp\left[-\frac{1}{\alpha'}\sum_{m=1}^{\infty} m X_m X_{-m}\right]$ describes the ground state of the 2d FT on S^1.

- <u>Check:</u> In Schrödinger picture

$$\alpha_n = -\frac{in}{\sqrt{2\alpha'}} X_{-n} - i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial X_n}$$
$$\tilde{\alpha}_n = -\frac{in}{\sqrt{2\alpha'}} X_n - i\sqrt{\frac{\alpha'}{2}} \frac{\partial}{\partial X_{-n}} .$$

Easy to check that $\alpha_n \Psi_1[X_b] = \tilde{\alpha}_n \Psi_1[X_b] = 0$.

State-operator mapping: $1 \longleftrightarrow |0, k=0\rangle$

- Next example: Instead of "1" consider operator ":exp(ikX):".
- One finds analogously:

$$:\exp(ikX): \longleftrightarrow |0,k\rangle$$

(e.g. the tachyon for $k^2 = 4/\alpha'$)

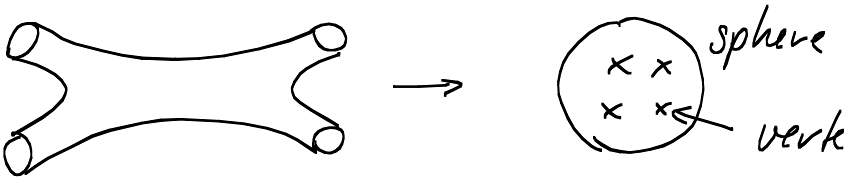
[Using again $X_{cl.} = X_0 + \sum_{n=1} (z^n X_n + \bar{z}^n X_{-n})$ and

$\exp(ikX(0)) \longrightarrow \exp ikX_0$ (X_0 - zero mode of X_b), we find that $\Psi_{\exp ikx}$ now contains a factor $\exp ikX_0$. Furthermore, $\hat{p} \sim \frac{\partial}{\partial X_0}$ in Schrödinger picture, such that $\hat{p}\Psi_{\exp ikx} = k\Psi_{\exp ikx}$.]

- *Note:* One should view this as a motivation rather than derivation. The crucial point is that, e.g., $:\exp ikX:$ has the correct trf. properties (e.g. under $X \to X+\Delta X$). Vertex operators are defined on the basis of such symm. requirements.)
- excited states: add prefactors ∂X, $\partial^2 X$ etc.

  (→ e.g. $:\partial X e^{ikX}:$ etc.)

14.2 Veneziano amplitude



amplitude: $A \sim \int \prod_i d^2z_i \int \mathcal{D}X \prod_i \exp(i\underset{\uparrow}{k_i}X_i)\, e^{-S[X]}$

momenta of incom. & outgoing particles

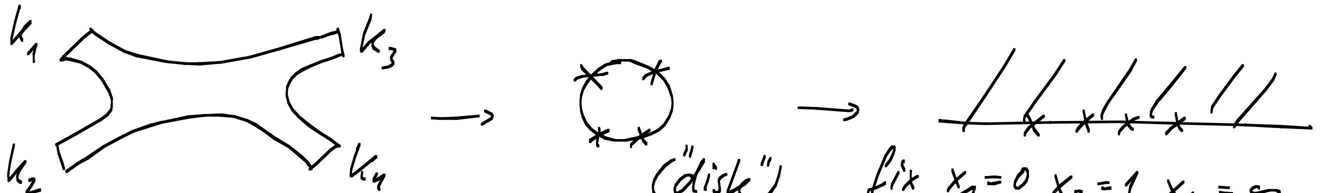
- Since $\int \mathcal{D}X e^{-S}\ldots$ describes a free FT, the result will be given by all possible contractions of the field ops.:

$$A \sim \int \prod_i d^2z_i \prod_{i<j} \exp\left\{\tfrac{1}{2} k_i\cdot k_j\, G(z_i,z_j)\right\}$$ where

$G(z,z') = \ln(\mu|z-z'|)$ is the Green fct. or propagator.

$$\Rightarrow A \sim \int \prod_i d^2z_i \prod_{i<j} |z_i-z_j|^{k_i\cdot k_j/2}$$ (→ Virasoro, Shapiro)

- *Simpler case:* Open strings (→ Veneziano)



fix $x_1=0$, $x_3=1$, $x_4=\infty$
using conf. inv., let $x_2=x$

- In analogy to the above (with $z_i \to x_i$), we have

$$A \sim \int_0^1 dx\, x^{k_1 \cdot k_2} (1-x)^{k_2 \cdot k_3} = B\left(-\frac{s}{2}-2, -\frac{t}{2}-2\right) \quad \text{(Euler } \beta \text{ fct.)}$$

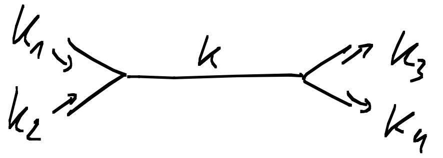
where $s = -(k_1+k_2)^2$, $t = -(k_2+k_3)^2$

$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$. Using features of Γ (or B) write

- $$A(s,t) = -\sum_{n=0}^{\infty} \frac{(\alpha(t)+1)(\alpha(t)+2)\cdots(\alpha(t)+n)}{n!\,(\alpha(s)-n)}$$

$\alpha(t) = \alpha' t + \alpha(0)$ ("Regge trajectory"; $\alpha(0) = 1$)

This can be interpreted as s-channel-exchange of particles with masses $M^2 = (n-\alpha(0))/\alpha'$ (i.e. of string excitations):

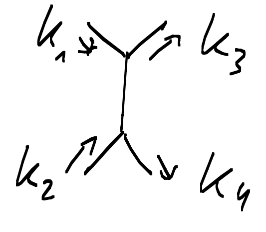


; $s = -k^2$

- Alternatively, A can be written as

$$A(s,t) = -\sum_{n=0}^{\infty} \frac{(\alpha(s)+1)(\alpha(s)+2)\cdots(\alpha(s)+n)}{n!\,(\alpha(t)-n)}$$

$\Rightarrow$ t-channel-exchange:



(hence the name "dual models")

- Such amplitudes were first constructed on the basis of these field-theoretic consistency requirements (in particular $A(s,t) = A(t,s)$). String theory was then found to supply the microscopic foundation.

14.3 Target space theory from Weyl invariance

Strings in curved space:

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}\left[\left(h^{ab}G_{\mu\nu}(X) + \underset{\uparrow}{i}\,\epsilon^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu \partial_b X^\nu + \alpha' R\phi(X)\right]$$

from Euclidean continuation

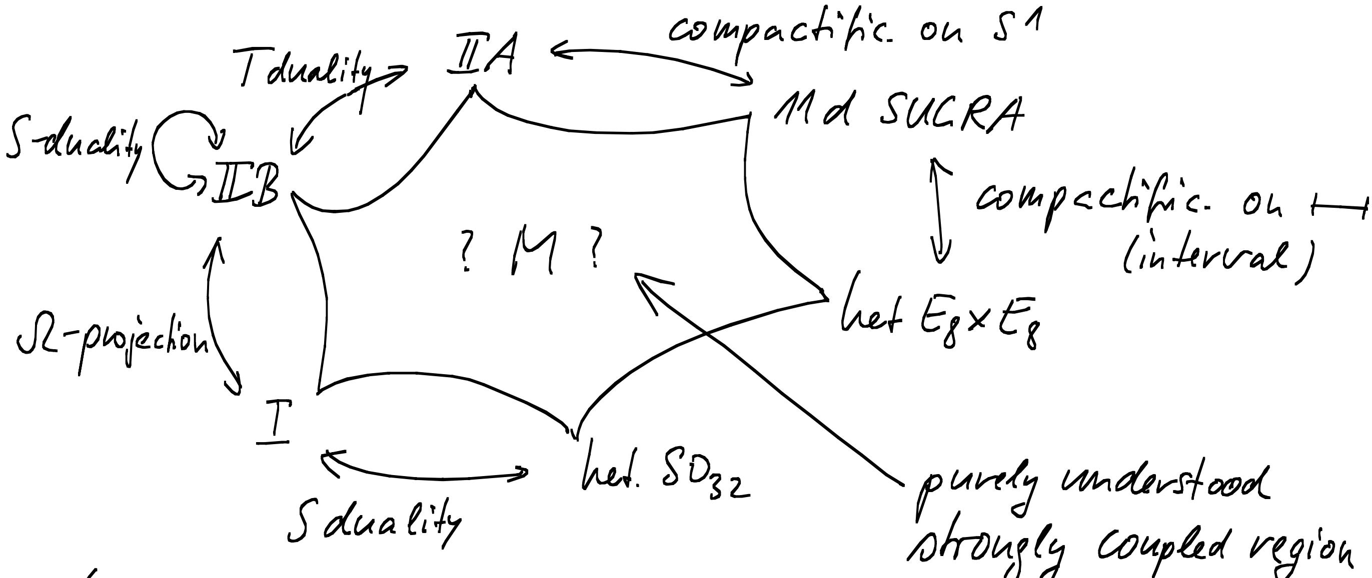
$$T^a{}_a = -\frac{1}{2\alpha'}\underset{\uparrow}{\beta^G_{\mu\nu}}\, h^{ab}\partial_a X^\mu \partial_b X^\nu + \underbrace{\cdots}_{\text{contains further } \beta\text{-fcts.}}$$

β-fct. of the 2d FT (arising from need of regularization of quantum level)

$$\beta^G_{\mu\nu} = \underbrace{\alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\phi - \frac{\alpha'}{2}H_{\mu\lambda\sigma}H_\nu{}^{\lambda\sigma} + \cdots}_{}$$

$= 0 \Rightarrow$ EOMs of target space theory

14.4 M theory



(+ many more connections between the different weakly coupled "corners")

The End

of this lecture and the beginning of much more to be learned for the audience...