3. Old Covariant Quantization

3.1. Canonical quantization of the closed string

- We will quantize the Polyakov (not the Nambu-Goto) action. Note: The classical equivalence does not necessarily imply the quantum equivalence. The latter is a complicated issue...

- We work in flat or unit gauge:

$$S = -\frac{T}{2}\int d^2\sigma\, \partial^a X^\mu \partial_a X_\mu \quad ; \quad d^2\sigma = d\tau\, d\sigma$$

- coordinates: X^μ
- conjugate momenta: $\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}_\mu} = T\dot{X}^\mu$
- equal-time canonical commutation relations:

$$[\hat{\Pi}^\mu(\sigma,\tau), \hat{X}^\nu(\sigma',\tau)] = -i\delta(\sigma-\sigma')\eta^{\mu\nu} \; ; \quad \text{"}[\hat{X},\hat{X}] = [\hat{\Pi},\hat{\Pi}] = 0\text{"}$$

- We now return to our expression for X^μ (and the analogous expression for Π^μ) through $x^\mu, p^\mu, \alpha_n^\mu, \tilde{\alpha}_n^\mu$ (declare the operators), insert them in the commutators, and obtain the commutation relations for $x^\mu, \ldots, \tilde{\alpha}_n^\mu$:

$$[p^\mu, x^\nu] = i\eta^{\mu\nu}$$
$$[\alpha_m^\mu, \alpha_n^\nu] = m\,\delta_{m+n}\eta^{\mu\nu}$$
$$[\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\,\delta_{m+n}\eta^{\mu\nu}$$

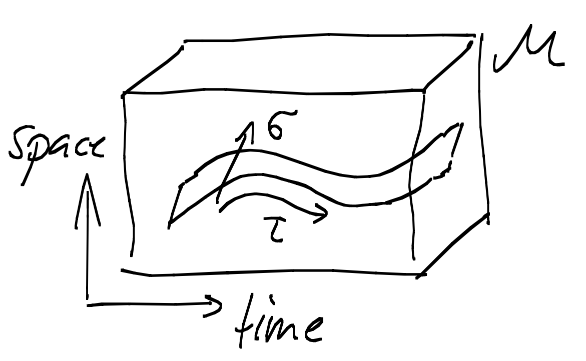
(after Fourier transform in σ & σ' on both sides of the equation)
(Note: $\delta_{m+n} \equiv \delta_{m+n,0}$)

Comment: Strictly speaking, we would have to start from Π^μ, X^ν at fixed τ; Fourier expand, determine the commutation relations of the coefficients, and finally determine the τ-dependence of the (Heisenberg-picture) operators Π^μ, X^ν from the Schrödinger equation. However, since we already know that in a free field theory the operators fulfill the class. EOMs, there is no harm in directly using the general classical solutions given above.

All of the above is (almost) exactly as in scalar FT in 2 dim. ($\to$ any QFT textbook). Differences are the constant x^μ and the term linear in τ (with coefficient p^μ). Especially the latter makes no sense in usual FT.

3.2 Canonical quantization of the open string

(So far, it was simpler to focus on the closed string to avoid discussing the boundaries. In the following, the operator algebra will be simpler for the open string. Therefore, we will proceed using mainly the open string.)



- WS Σ parameterized by τ and $\sigma \in (0, \pi)$.
- use again the flat gauge.

$$S = -\frac{T}{2}\int d^2\sigma\, (\partial^a X^\mu)(\partial_a X_\mu)$$

Variation:
$$\delta S = -T\int d^2\sigma\, (\partial^a X^\mu)(\partial_a \delta X_\mu)$$

$$= T\int d^2\sigma\, \underbrace{(\partial^2 X^\mu)}_{\Rightarrow \text{ standard EOM}}\delta X_\mu - T\underbrace{\int d\tau \int_0^\pi d\sigma\, \partial_\sigma(\partial^\sigma X^\mu\, \delta X_\mu)}_{\underbrace{\int d\tau\, (\partial_\sigma X^\mu)\, \delta X_\mu \Big|_{\sigma=0}^{\sigma=\pi}}_{=0,\ \text{if}}}$$

1) $\delta X_\mu = 0$, i.e. X_μ is fixed (Dirichlet Boundary condition). This breaks Poincare invariance and corresponds to introducing <u>D-branes</u> ("Dirichlet-branes")

2) $\partial_\sigma X^\mu = 0$ - This is the case with freely moving string end (without energy loss at the end). We will first mainly study this case (Neumann BCs).

<u>General solution:</u> (Neumann)

$$X^\mu = x^\mu + \ell^2 p^\mu \tau + i\ell \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma$$

$\Rightarrow$ We are basically dealing with "half" the d.o.f.s of the closed string case.

Canon. quantiz. $\Rightarrow$ $[p^\mu, x^\nu] = -i\eta^{\mu\nu}$

$$[\alpha_m^\mu, \alpha_n^\nu] = m\,\delta_{m+n}\,\eta^{\mu\nu}$$

($\to$ problems)

With the convenient definition

$\alpha_0^\mu = \ell p^\mu$, one also finds $H = \frac{1}{2}\sum_{-\infty}^{\infty} \alpha_{-n}\cdot\alpha_n$

(derivation $\to$ problems) "$\cdot$" stands for contraction of the "μ's".

(In the closed string case, one can define $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{\ell p^\mu}{2}$ and write $H = \frac{1}{2}\sum_{-\infty}^{\infty}\left(\alpha_{-n}\cdot\alpha_n + \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n\right)$.)

3.3 The Fock space – basic concepts

(focussing on the open string to minimize writing)

We have:
- set of oscillators ($\alpha_m^\mu/\sqrt{m}$ & $(\alpha_m^\mu/\sqrt{m})^+$ in standard normalization)
- hermitian operators p^μ & x^μ (each of them commuting with all oscillator creation/annihilation operators).

We write: Full space $=$ $\oplus$ eigenspaces of $\hat{p}^\mu$ with Eigenvalues p^μ

(This prefers p^μ relative to x^μ, which is what we are used to from QFT.)

- Focus on one particular eigenspace characterized by a certain 4-vector p^μ (here we really mean 4 real numbers).
- Call the state with no oscillator excited $|0;p^\mu\rangle$.
  (i.e. $\alpha_m^\mu |0;p^\nu\rangle = 0$ for $m>0$)
- Construct Fock space as $\alpha_m^\mu \alpha_n^\nu \cdots |0;p^\rho\rangle$ etc. $(m,n<0)$.
- <u>Immediate difficulty</u>: wrong sign of α_m^0-commutator ("-1" of $\eta^{\mu\nu}$)

$$\left([a,a^+] = -1 \;\Rightarrow\; |a^+|0\rangle|^2 = \langle 0|aa^+|0\rangle = \langle 0|(-1+a^+a)|0\rangle = -\langle 0|0\rangle \;;\; \text{negat. norm} \;\lightning \right)$$

<u>Aside</u>: In QED, we have already encountered & solved such a problem in the context of <u>Gupta-Bleuler-quantization</u>:

$$\mathcal{L} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2}_{} \;;\quad \text{constraint: } \partial_\mu A^\mu = 0$$

$$\Downarrow$$

set of oscillators; some with wrong sign; neg. norm states (exactly as above)

<u>Way out</u>: impose a quantum-mechanical version of the classical constraint on the Hilbert space:

$$(\partial_\mu A^\mu)^a_{\uparrow} |\psi\rangle = 0 \quad :\Leftrightarrow \quad |\psi\rangle \in \mathcal{H}_1 \subset \mathcal{H}$$

"annihilator part" — physical space

<u>Here</u>: Similar situation, since the classical dynamics is governed by the gauge-fixed action + EOM from h (which, of course, is not derivable from this action any more). Thus, we need to impose $T_{ab} = 0$ at the quantum level:

- 3 components: T_{00}, T_{11}, T_{10} ($T_{01} = T_{10}$, obviously)
- in light-cone basis: T_{++}, T_{+-}, T_{--}

(recall: $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$; $T_{+a} = \frac{1}{2}(T_{0a} + T_{1a})$;

$$T_{++} = \frac{1}{2}\left(\frac{1}{2}(T_{00} + T_{10}) + \frac{1}{2}(T_{01} + T_{11})\right) = \cdots)$$

$T^a{}_a \equiv 0 \;\Rightarrow\; T_{+-} \equiv 0$ (h^{+-} & h^{-+} are the only non-zero metric components)

- Thus, we will try to impose

$$\underline{T_{++} = 0\,;\; T_{--} = 0} \quad \text{quantum-mechanically.}$$

A simple calculation shows

$$T_{++} = (\partial_+ X)\cdot(\partial_+ X) \quad ; \quad T_{--} = (\partial_- X)\cdot(\partial_- X).$$

This implies

$$T_{++} = (\partial_+ X_L)\cdot(\partial_+ X_L) \quad ; \quad T_{--} = (\partial_- X_R)\cdot(\partial_- X_R).$$

Consider <u>Fourier modes</u> (<u>closed string</u>):

$$L_m = \frac{T}{2}\int_0^\pi e^{-2im\sigma} T_{--}\, d\sigma = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n$$

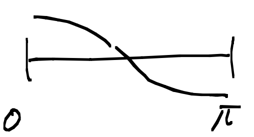
$$\tilde{L}_m = \frac{T}{2}\int_0^\pi e^{-2im\sigma} T_{++}\, d\sigma = \frac{1}{2}\sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n$$

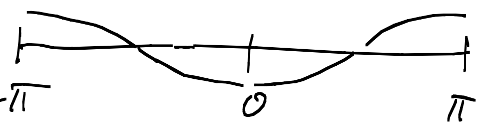
<u>Fourier modes</u> (<u>open string</u>):

slight complication ...

<u>Aside</u>: <u>Relation of X_L and X_R in open string</u>

X - fct. on $(0,\pi)$ with Neumann BC $\longleftrightarrow$ periodic fct. (period is 2π) with symmetry $\sigma \leftrightarrow -\sigma$:





<u>indeed</u>: (suppressing index μ)

$$X = x + \ell^2 p\tau + i\ell \sum_{n\neq 0} \frac{1}{n}\alpha_n e^{-in\tau}\cos n\sigma$$

can be rewritten as (easy calculation):

$$X = \frac{1}{2}x + \frac{\ell^2}{2}p\sigma_+ + \ell \sum_{n>0} \frac{1}{2n}\left\{ i(\alpha_n - \alpha_{-n})\cos n\sigma_+ + (\alpha_n + \alpha_{-n})\sin n\sigma_+ \right\}$$

$$+ \frac{1}{2}x + \frac{\ell^2}{2}p\sigma_- + \ell \sum_{n>0} \frac{1}{2n}\left\{ i(\alpha_n - \alpha_{-n})\cos n\sigma_- + (\alpha_n + \alpha_{-n})\sin n\sigma_- \right\}$$

$\Rightarrow$ X_L & X_R are the same functions (of σ_+ & σ_- respectively): $X_L(z) = X_R(z)$

$\uparrow$ some arbitrary variable

$\Rightarrow$ instead of giving X_L, X_R on $(0,\pi)$ with Neumann BC, one can give just a periodic X_L on $(-\pi,\pi)$.
X_R is then fixed by the above.

$\Rightarrow$ all information is contained just in T_{++} interpreted as a fct. on $(-\pi,\pi)$

relevant Fourier modes, e.g.,

$$L_m = T\int_{-\pi}^{\pi} e^{im\sigma}\, T_{++}\, d\sigma = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n$$

<u>Note in particular:</u> $H = L_0 + \tilde{L}_0$ (closed)

$H = L_0$ (open)

We will now again focus on the open case to avoid excessive writing.

Aside: L_m – the Virasoro operators

Classically, they satisfy the algebra

$$\{L_m, L_n\} = i(m-n)L_{m+n}$$

↑Poisson bracket

Possible interpretation:

- consider fcts. on S^1 ($\equiv$ fcts. with period 2π)
- consider infinit. reparameterizations of the argument θ according to $\theta \longrightarrow \theta + a(\theta)$
- these are generated by $D(a) = i a(\theta)\frac{d}{d\theta}$
- a basis is provided by

$$D_n = i e^{-in\theta}\frac{d}{d\theta} \qquad (n \in \mathbb{Z})$$

- they form a Virasoro algebra:

$$[D_m, D_n] f = \{D_m(ie^{-in\theta}) - D_n(ie^{-im\theta})\}\cdot f'(\theta) =$$

$$= e^{-i(m+n)\theta}(n-m) f' = i(m-n) D_{m+n} f$$

This algebra appears in the present context because of the reparameterization symmetry

$$X_L(\sigma^+) \longrightarrow X_L(\varphi(\sigma^+))$$

(and analogously for σ^-)

- quantum mechanically, L_0 (as well as H) are only well-defined if we specify the operator ordering:

$$\| \quad H \equiv L_0 \equiv \frac{1}{2}p^2 + \sum_{n>0} \alpha_{-n}\alpha_n \quad \| \quad \text{(normal order)}.$$

- this leads to the quantum version of the Virasoro algebra:

$$\| \quad [L_m, L_n] = (m-n)L_{m+n} + \underbrace{A(m)}_{\text{"anomaly term"}}\,\delta_{m+n} \quad \|$$

(technically, it arises since the sum of products of α_m^μ's that produces L_0 has to be normal ordered)

- after some algebra ($\to$ problems) one finds

$$\| \quad A(m) = \frac{1}{12} D\,(m^3 - m) \quad \|$$

D also known as the "central charge"

- in particular, we have

$$[L_m, L_{-m}] = 2mL_0 + \frac{D}{12}(m^3-m). \qquad (*)$$

$\Rightarrow$ We can not demand $L_m|\text{phys.}\rangle = 0$ for all m since $\langle\text{phys}|(*)|\text{phys}\rangle$ gives zero on the l.h. side and a non-trivial fct. of m on the r.h. side.

- it is, however, possible to demand

$$\| \quad (L_m - a\,\delta_m)|\text{phys}\rangle = 0 \quad \text{for } m \geq 0. \quad \|$$

($a\delta_m$ has been introduced to compensate, if necessary, for our arbitrary normal-ordered definition of L_0)

- indeed, the above weaker condition is sufficient from a physical perspective since for $m<0$ we have (using $L_{-m} = L_m^+$)

$$\langle phys | L_{-m} | phys \rangle = \langle phys | L_m | phys \rangle^* = 0$$

To summarize:

need to build space using $[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}$ on vacua $|0,p\rangle$ ($\alpha_m |0,p\rangle = 0$ for $m>0$) with constraint

$$(L_m - a\delta_m)|phys\rangle = 0 \text{ for } m \geq 0.$$

3.4 Fock space - physical states

① Start with vacuum: $|0,p\rangle$; $\hat{p}^\mu |0,p\rangle = p^\mu |0,p\rangle$

$$(L_0 - a)|0,p\rangle = 0 \Rightarrow \left\{ \frac{\ell^2}{2} p^2 + \sum_{m>0} \alpha_{-m}\alpha_m - a \right\} |0,p\rangle = 0$$

$$\Rightarrow p^2 = \frac{a}{\alpha'} \quad \text{or} \quad \underline{\underline{M^2 = \frac{-a}{\alpha'}}} \quad \text{(crucial result!)}$$

(all other constraints automatically ok since in the other L_m ($m>0$) each term $\alpha_{m-n}\alpha_n$ involves at least one annihilator.)

② <u>first excited level</u>

- states $\zeta_\mu \alpha_{-1}^\mu |0,p\rangle$ are characterized by a polarization vector ζ^μ.

- apply again the "$L_0 - a$" or "mass shell" condition:

$$0 = (L_0 - a)\zeta_\mu \alpha_{-1}^\mu |0,p\rangle = \{\alpha' p^2 + \alpha_{-1\nu}\alpha_1^\nu - a\}\zeta_\mu \alpha_{-1}^\mu |0,p\rangle$$

$$= \{\alpha' p^2 + 1 - a\}\zeta_\mu \alpha_{-1}^\mu |0,p\rangle \quad \Rightarrow \quad M^2 = \frac{1-a}{\alpha'}$$

- now also non-trivial:

$$L_1 \zeta_\mu \alpha_{-1}^\mu |0,p\rangle = 0 \qquad \Big(L_1 = \frac{1}{2}\sum_{-\infty}^{\infty} \alpha_{1-n}\alpha_n =$$

$$= \frac{1}{2}(\cdots \alpha_2 \alpha_{-1} + \alpha_1 \alpha_0 + \alpha_0 \alpha_1 + \cdots)\Big)$$

$$\Downarrow$$

relevant term!

$$p^\mu \zeta_\mu |0,p\rangle = 0$$

$$\Downarrow$$

$$p^\mu \zeta_\mu = 0$$

<u>but:</u> no need to worry about L_m with $m > 0$ at the presently considered "level 1".
(similarly at higher levels)

- the norm of our states is

$$\langle 0,p| (\zeta_\mu \alpha_{-1}^\mu)^\dagger (\zeta_\mu \alpha_{-1}^\mu)|0,p\rangle = \langle 0,p|0,p\rangle \zeta_\mu \zeta^\mu = \zeta_\mu \zeta^\mu$$

<u>We can now distinguish 3 cases:</u>

a) $a > 1$ $\Rightarrow$ $p^2 > 0$, $M^2 < 0$ $\Rightarrow$ $\exists$ time-like ζ with
(p space-like) $\qquad \zeta^\mu p_\mu = 0$

$\Rightarrow \exists$ state with neg. norm ($\xi \cdot \xi < 0$)
$\Rightarrow$ case a) excluded!

b) $a = 1$ $\Rightarrow$ $p^2 = 0$; $(D-1)$ independent ξ's

$$\begin{array}{cc} \swarrow & \searrow \\ D-2 \text{ transverse,} & 1 \text{ longitudinal,} \\ \text{positive-norm states} & \text{zero-norm states} \end{array}$$

... just like Gupta-Bleuler quantization of QED; expect consistent theory of massless vector particles ... → "critical string theory"

c) $a < 1$ $\Rightarrow$ $p^2 < 0$, $M^2 > 0$; $(D-1)$ indep. ξ's ; all states with positive norm; massive vectors

... could so far be ok, but will exhibit problems later on → "non-critical string theory"

③ second excited level

Now we have polarization tensors $\epsilon_{\mu\nu}$ and ϵ_μ and states

$$(\epsilon_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-1}^{\nu} + \epsilon_\mu \alpha_{-2}^{\mu})|0,p\rangle .$$

The mass shell condition gives, for states of this type $M^2 = \frac{2-a}{\alpha'}$ (hence the name "level 2").

We could go on and analyse all states of this mass in terms of irred. Lorentz-representations;

... determine the phys. polarizations, the norms of the states ...

<u>important</u>: at this (and higher) levels, D comes into the physical state condition:

- consider $a=1$ and

$$|\phi\rangle = \{ c_1 \alpha^\mu_{-1} \alpha_{-1,\mu} + c_2 p_\mu \alpha^\mu_{-2} + c_3 (p_\mu \alpha^\mu_{-1})^2 \} |0,p\rangle$$

- $(L_0 - 1)|\phi\rangle = L_1|\phi\rangle = L_2|\phi\rangle \Rightarrow c_2$ & c_3 can be expressed through c_1
- $\Rightarrow \langle\phi|\phi\rangle = \frac{2c_1^2}{25}(D-1)(26-D) \Rightarrow D \leq 26$ required!

more complete analysis of this type shows:

<u>no ghosts</u> only for $a = 1$; $D = 26$ ("critical")
$a \leq 1$; $D \leq 25$ ("non-critical")

<u>mini-summary for critic. open string</u>

phys. states are subject to

- mass-shell condition: $M^2 = -p^2 = \frac{1}{\alpha'}(N-1)$ with the "level" $N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n$
- L_m $(m \geq 1)$-constraints $(L_m |phys\rangle = 0)$
- in $D = 26$ there are no neg.-norm phys. states (ghosts).
- states that are orthogonal to all phys. states are called "spurious".

- states that are both "physical" and "spurious" are called "null" (cf. the longit. polarized vector encountered above).
- the "real" physical Hilbert space is the space of equivalence classes $\mathcal{H}_{OCQ} = \mathcal{H}_{phys.} / \mathcal{H}_{null}$
- so far we explicitly know: $N = 0, 1, 2, \ldots$

  $N=0$: scalar tachyon; $N=1$: massless vector

3.5 Closed string Fock space (critical case)

- everything (except p) doubled: $L_m \to L_m, \tilde{L}_m$

  $\alpha_m \to \alpha_m, \tilde{\alpha}_m \quad (m \neq 0)$

- instead of $(L_0 - a)|phys\rangle = 0$ & $(\tilde{L}_0 - a)|phys\rangle = 0$, it is convenient to consider the sum & difference of these constraints:

  1) $(N - \tilde{N})|phys\rangle = 0$ "level matching"

  2) $(-\frac{\alpha'}{2}M^2 + N + \tilde{N} - 2a)|phys\rangle = 0$, i.e.

  $$M^2 = \frac{1}{\alpha'}(2(N+\tilde{N}) - 4a) \quad (a = 1\,!)$$

Proceeding as before, we find:

① level 0 $\quad |0,p\rangle$; $M^2 = -4/\alpha'$ (tachyon)

② <u>level 1</u> level matching $\Rightarrow$ need both α_{-1} & $\tilde{\alpha}_{-1}$

states: $\xi_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0,p\rangle$; $M^2 = \frac{4}{\alpha'}(1-a) = 0$ for $a=1$.

- the $L_1, \tilde{L}_1$ - constraints give, in complete analogy to the open string case, $p^{\mu}\xi_{\mu\nu} = 0$ & $\xi_{\mu\nu}p^{\nu} = 0$
- this corresponds to $2D-1$ conditions (since the second set of eqs. implies that one linear combination of the first set of eqs. vanishes — $p^{\mu}\xi_{\mu\nu}p^{\nu} = 0$)

  $\Rightarrow$ only $D^2 - (2D-1)$ indep. phys. states left
- using $a=1$, $p^2=0$ we can chose $p \sim (-1, 1, \underbrace{0, \cdots, 0}_{D-2 \text{ entries}})$
- there remains some gauge freedom:

  $$\xi_{\mu\nu} \quad \& \quad \xi'_{\mu\nu} = \xi_{\mu\nu} + \alpha_{\mu}p_{\nu} + p_{\mu}\beta_{\nu}$$

  (where α, β fulfill $\alpha\cdot p = \beta\cdot p = 0$)

  are equivalent (they differ by a null state).

  This subtracts further $2(D-1)-1$ states from the "real" physical space.

  since $\alpha\cdot p = \beta\cdot p = 0$

  since $\alpha_{\mu}p_{\nu}$ & $p_{\mu}\beta_{\nu}$ coincide for $\alpha = \beta = p$.

- Now, we have $D^2 - (2D-1) - (2D-3) = (D-2)^2$ independent phys. states left.
- They are precisely described by the transverse $\xi_{\mu\nu}$:

$$\xi_{\mu\nu} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & & \ast & \\ 0 & 0 & & \end{pmatrix}.$$

Counting them in terms of representations of the transverse Lorentz group $SO(D-2)$ gives

$$(D-2)^2 = \underbrace{\tfrac{1}{2}(D-1)(D-2)-1}_{\substack{\text{symmetric, traceless} \\ \to \text{graviton } G_{\mu\nu}}} + \underbrace{\tfrac{1}{2}(D-2)(D-3)}_{\substack{\text{antisymm.} \\ B_{\mu\nu}}} + \underbrace{1}_{\substack{\text{scalar} \\ \text{(dilaton)}}}$$

$B_{\mu\nu}$: just like A_μ in QED: $H = dB \sim \partial_\mu B_{\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho$ is the corresponding field strength

<u>Brief summary of critical closed string:</u>

- level matching: $N = \tilde{N}$
- mass shell: $M^2 = -p^2 = \frac{1}{\alpha'}(2(N+\tilde{N}) - 4a) = \frac{4}{\alpha'}(N-1)$
- levels: $N = (0, 1, 2, \dots)$

0: tachyon; 1: $G_{\mu\nu}, B_{\mu\nu}, \phi$

ϕ: dilaton, determines g_s