6. Conformal field theory – I

for many more details see e.g.
Di Francesco / Mathieu / Senechal : CFT, Springer, '97

6.1 Conformal Trfs. for $d > 2$

- A conf. trf. is a diffeomorphism under which the metric changes only by an overall factor:

$$x \to x' \,;\; g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) \overset{!}{=} e^{2\omega(x)} g_{\mu\nu}(x)$$

- Such Trfs. preserve angles.
- Infinitesimally: Let $g_{\mu\nu} = \eta_{\mu\nu}$; $x \to x + \epsilon$

$$\Rightarrow \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \omega \cdot \eta_{\mu\nu} .$$

After some index reshuffling ($\to$ problems), this can be shown to imply $(2-d)\,\partial_\mu \partial_\nu \omega = \eta_{\mu\nu} \partial^2 \omega$ and $\partial^2 \omega = 0$.

$\Rightarrow$ ω is linear fct. of x^μ.

After some further algebra ($\to$ problems) it can then be shown that the ϵ^μ are quadratic fcts. of x^μ.

$\Rightarrow$ <u>The conf. group has finitely many parameters.</u>

The conformal trfs. are:

(1) translations : $x'^\mu = x^\mu + a^\mu$ — d parameters

(2) rotations : $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ — $\frac{d(d-1)}{2}$ params.

(3) dilation : $x'^\mu = \alpha x^\mu$ — 1 parameter

(4) special conf. trfs. : $x'^\mu = \dfrac{x^\mu - b^\mu (x^2)}{1 - 2(b\cdot x) + (b^2)(x^2)}$ — d params.

(The latter can be viewed as inversion + translation + inversion, where inversion means $x^\mu \to x^\mu / x^2$.)

Together: $\frac{d(d-1)}{2} + 2d + 1 = \frac{(d+2)(d+1)}{2}$ parameters.

It can be shown that, in the euclidean case ($\Lambda \in SO(d)$), the conf. group is $SO(1,d+1)$, while, in the Minkowski case ($\Lambda \in SO(1,d-1)$), the conf. group is $SO(2,d)$.

6.2 Conf. Trfs. in d=2

- Our above argument for the finiteness of the # of params. breaks down since $d-2=0$ such that $\partial_\mu \partial_\nu \omega \neq 0$ in general.
- $\partial_a \epsilon_b + \partial_b \epsilon_a = 2\omega \eta_{ab}$ $\Rightarrow$ $\partial_+ \epsilon^- = \partial_- \epsilon^+ = 0$

  ($\Rightarrow \epsilon^{+/-}$ depends on $\sigma^{+/-}$ only)

  $\Rightarrow \partial_+ \epsilon_- + \partial_- \epsilon_+ = \omega$ (this just defines ω)
- There are now infinitely many parameters encoded in the fcts. $\epsilon^+ = \epsilon^+(\sigma^+)$ & $\epsilon^- = \epsilon^-(\sigma^-)$
- In the euclidean version it is useful to write

  $$\sigma^1 + i\sigma^2 = z \,, \quad \sigma^1 - i\sigma^2 = \bar{z}$$

  The metric δ_{ab} corresponds to $h_{zz} = h_{\bar{z}\bar{z}} = 0$; $h_{z\bar{z}} = h_{\bar{z}z} = \frac{1}{2}$.

  (Check: $|z|^2 = (\sigma^1)^2 + (\sigma^2)^2 = h_{z\bar{z}} z\bar{z} + h_{\bar{z}z} \bar{z} z$)

- $\partial_a \epsilon_b + \partial_b \epsilon_a = 2\omega \delta_{ab} \quad \Longrightarrow \quad \partial_{\bar z} \epsilon^z = \partial_z \epsilon^{\bar z} = 0$

Thus
$$\epsilon^z(z) = \epsilon^1(z,\bar z) + i\,\epsilon^2(z,\bar z) \equiv \epsilon(z)$$
$$\epsilon^{\bar z}(\bar z) = \underbrace{\epsilon^1(z,\bar z)}_{} - i\,\underbrace{\epsilon^2(z,\bar z)}_{} \equiv \bar\epsilon(\bar z)$$
real fcts. of z and $\bar z$

$\Longrightarrow$ Conf. trfs. are holomorphic trfs. described infinitesimally by $z \to z + \epsilon(z)$ & $\bar z \to \bar z + \bar\epsilon(\bar z)$.

More generally, one has $z \to \underbrace{w(z)}_{\text{arbitrary holom. fct.}}$ & $\bar z \to \bar w(\bar z)$

- Trf. of a scalar field: $\delta\phi = -\epsilon^a \partial_a \phi$

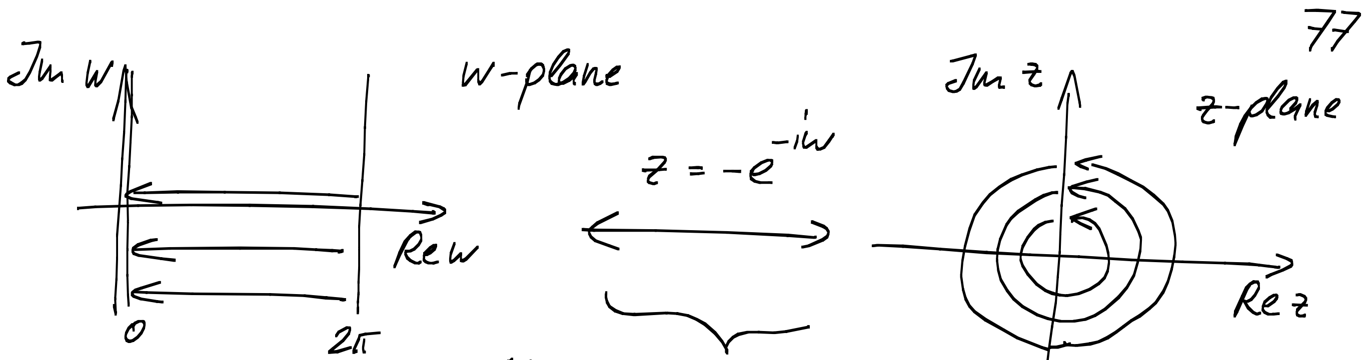
$$\to \quad \delta\phi = (-\epsilon^z \partial_z - \epsilon^{\bar z}\partial_{\bar z})\phi = (-\epsilon\partial - \bar\epsilon\bar\partial)\phi$$
$$\equiv \sum_{n=-\infty}^{\infty} (c_n l_n + \bar c_n \bar l_n)\phi ,$$

where $l_n = z^{n+1}\partial_z$ $(\bar l_n = \bar z^{n+1}\partial_{\bar z})$ and the c_n are the coeffs. of the Laurant expansion of ϵ.

Easy to check: l_n & $\bar l_n$ form two indep. Virasoro algebras

6.3 Relation to the string worldsheet

(It is convenient to focus on the closed string with periodicity 2π (rather than π).)



- Moving along the string, <u>Re w</u> changes or, equivalently, <u>the phase of z</u> changes.

- time $\Leftrightarrow$ radius

Now, consider "our" FT of the X^μ with $\mathcal{L} \sim (\partial X)^2$ and $h_{ab} = \delta_{ab}$ (euclidean!). This <u>is</u> a 2d CFT.

<u>Reason:</u>
- We started with a diff + Weyl-invariant FT with dynamical h_{ab}.
- After fixing $h_{ab} = \delta_{ab}$, we were left with a residual gauge symm. (a specific combination of diff. & Weyl) that corresponds exactly to the conf. trfs. discussed here.
  ($\partial_a \epsilon_b + \partial_b \epsilon_a \overset{!}{=} 2\omega \delta_{ab}$ in both cases!)
- <u>Thus:</u> Conf. invariance emerges as a result of diff & Weyl invariance.

<u>Note:</u> Conf. symm. is a symm. of a flat-space (i.e. $h_{ab} = \delta_{ab}$) field theory. The metric does not vary.

- Explicit check of the invariance:

$$(\partial X)^2 \sim (\partial_z X)(\partial_{\bar z} X) \quad ; \quad d^2\sigma \sim dz\, d\bar z$$

Under $z \to w(z)$: $\quad dz = dw \left(\frac{\partial z}{\partial w}\right)$ ⇐ These factors

$\qquad \partial_z X = \partial_w X \left(\frac{\partial w}{\partial z}\right)$ ↙ cancel each other.

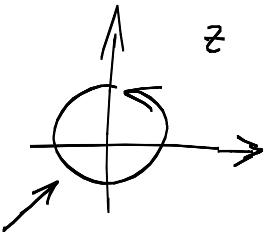
- As before, we can consider the energy-momentum tensor T as a fct. of X and its derivatives.
  - tracelessness $\Rightarrow T_{z\bar z} = 0$
  - covariant conservation $\Rightarrow \partial_{\bar z} T_{zz} + \partial_z T_{\bar z z} = 0 \Rightarrow \partial_{\bar z} T_{zz} = 0$

$$(\& \ \partial_z T_{\bar z \bar z} = 0)$$

(It will be convenient to write $T_{zz} = T$ & $T_{\bar z \bar z} = \bar T$.)

- The Virasoro generators, defined previously as the Fourier modes of T (in the "w-frame"), are now given by (in the "z-frame"):

$$L_m = \frac{1}{2\pi i} \oint dz\, z^{m+1}\, T(z)$$



This integration contour corresponds to integrating across the string WS.

- These are the operator realizations of the conf. trfs. (= holomorphic reparametrizations) generated by the l_m introduced in this section.

- As opposed to the l_m, the L_n-algebra has an anomaly: $[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12}(m^3 - m)\,\delta_{m+n}$;

  c is the <u>central charge</u> of the CFT.

- $[L_m, L_n]$ can be seen as a conf. trf. acting on the Laurant-coeffs. of T.

- It is thus related to the conf. trf. of $T(z)$, which classically reads: $z \to z + \epsilon(z)$

$$\delta T(z)\Big|_{\text{classic.}} = -\epsilon(z)\partial_z T(z) - 2\left(\partial_z \epsilon(z)\right) T(z)$$

<u>In $\sigma^{a,b}$-coordinates:</u> $T'_{ab}(\sigma') = T_{cd}(\sigma) \frac{\partial \sigma^c}{\partial \sigma'^a} \frac{\partial \sigma^d}{\partial \sigma'^b}$

$$\Rightarrow T_{ab}(\sigma) + \delta_\epsilon T_{ab}(\sigma) + \epsilon^c \partial_c T_{ab}(\sigma) = T_{ab}(\sigma) - \frac{\partial \epsilon^c}{\partial \sigma^a} T_{cb}(\sigma) - \frac{\partial \epsilon^d}{\partial \sigma^b} T_{ad}(\sigma)$$

The minus-signs arise from the change $(\partial\sigma/\partial\sigma') \leftrightarrow (\partial\sigma'/\partial\sigma)$.

$$\Rightarrow \delta_\epsilon T_{ab}(\sigma) = -\epsilon^c \partial_c T_{ab} - \frac{\partial \epsilon^c}{\partial \sigma^a} T_{cb} - \frac{\partial \epsilon^d}{\partial \sigma^b} T_{ad}$$

This straightforwardly translates into the above formula with $\epsilon(z)$.

The modification of the Virasoro-alg. by c given above corresponds to a "non-tensor" trf. rule for T.

($\to$ problems)

- It reads: $\delta_\epsilon T(z) = -\frac{c}{12}\partial_z^3 \epsilon(z)$ + classical part

(Specifically: $c=1$ for 1 scalar, i.e. $c=D$ for Bosonic string)

- In complete analogy, the ghosts form a classically Weyl-inv. 2d FT: $S_g = -\frac{i}{2\pi}\int d^2\sigma \sqrt{h}\, h^{ab} c^c \nabla_a b_{bc}$

Weyl rescaling factors compensate eachother.

- This, in turn, gives rise to a 2d CFT with T^g and

$$\delta_\epsilon T^g = -\frac{c^g}{12}\partial_z^3 \epsilon + \dots \quad \text{where } c^g = -26.$$

(There is always an analogous $\tilde{c}$ for $\delta_\epsilon \bar{T}(\bar{z})$, which is not necessarily the same as c.)

6.4 Relation to Weyl anomaly

- Our path integral manipulations required Weyl invariance of the measure (which, as we explained, was highly questionable).
- Classically, Weyl-invariance means $T^a{}_a = 0$.
- Weyl-inv. of path integral measure $\Leftrightarrow$ $T^a{}_a$ vanishes also quantum-mechanically,

i.e. $\langle T^a{}_a \rangle = 0$

- In the flat case, $\langle T^a{}_a \rangle = 0$ is achieved simply by using a normal-ordered definition of T^{ab}.
- In curved space, we can find $\langle T^a{}_a \rangle = b \cdot \mathcal{R}$

  $\uparrow$ (path int. over X, b, c; fixed metric)

(<u>Note</u>: In usual FT conventions, where $[R] = 2$ and $[T] = \left[\frac{\text{energy}}{\text{volume}}\right] = 1 + (d-1) = d = 2$, the constant b is dimensionless.)

- $\mathcal{R}$ is the leading scalar term in a derivative expansion. Terms $\sim \mathcal{R}^2/\Lambda^2$ etc. are possible but vanish as the cutoff Λ is taken to infinity.
- <u>Now</u>: $\dashrightarrow$ $T_{z\bar{z}} = \frac{b}{2} h_{z\bar{z}} R \Rightarrow \nabla^{\bar{z}} T_{\bar{z}z} = \frac{b}{2} \nabla^{\bar{z}}(h_{z\bar{z}} R)$

  $\Rightarrow -\nabla^z T_{zz} = \frac{b}{2} \partial_z R$ (using covariant conservation of T)
- Perform a Weyl trf.: (by infinit. ω)

– r.h. side: $\delta R = -2\nabla^2 \omega$; $\delta\left(\frac{b}{2}\partial_z R\right) = -4b\,\partial_z^2 \partial_{\bar{z}}\,\omega$

– l.h. side: Weyl = conf. trf. "minus" diffeom.

$$\delta_\omega T = -\frac{c}{12}\partial_z^3 \epsilon(z) \underbrace{- 2(\partial_z \epsilon(z))T - \epsilon\,\partial_z T(z)}_{\text{precisely cancel eachother}} \underbrace{\text{"}-\text{" diffeom.}}$$

- Using $2\omega = \partial\epsilon + \bar\partial\bar\epsilon$, we thus find

$$\delta_\omega T = -\frac{c}{6}\partial_z^2\omega \quad \text{or} \quad \delta_\omega \partial^{\bar z} T = -\frac{c}{6}\partial^{\bar z}\partial_z^2\omega$$
$$= -\frac{c}{3}\partial_{\bar z}\partial_z^2\omega$$

$$\Rightarrow \quad \frac{c}{3}\partial_{\bar z}\partial_z^2\omega = -4b\,\partial_z^2\partial_{\bar z}\omega \quad \Rightarrow \quad b = -\frac{c}{12}$$

$$\Rightarrow \quad \boxed{T^a{}_a = -\frac{c}{12}\mathcal{R}}$$

We need a vanishing central charge of the WS CFT to ensure the Weyl invariance (which we used in the path integral). Given $c^g = -26$, we need $c^X = +26$, i.e. <u>26 dimensions</u>.