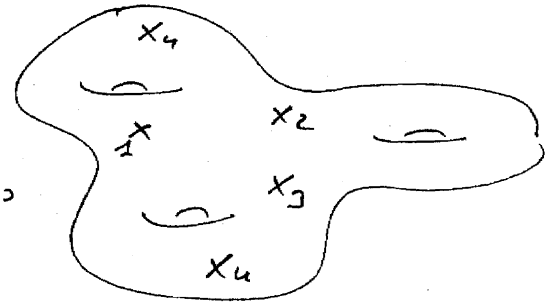
String Amplitudes

The interaction of n incoming/outgoing states can be written as

$A = \sum_{\text{topologies}}$ 

$$= \sum_{\text{topologies}} \int \frac{[dx][dg]}{V_{\text{diff} \times \text{Weyl}}} e^{-S_x} V_1 \ldots V_n$$

where the denominator $V_{d\times w}$ keeps into account the gauge invariance that must be modded out, and V_i is the vertex operator of the i-th in/out coming state.

We can avoid the denominator and properly quantize the theory by using the Faddeev Popov approach.

Given a "preferred" metric $\hat{g}$ (flat metric) we know that there exists always a transformation ξ in Diff × Weyl such that, for each possible metric g, $g = \hat{g}^{\xi} = \xi(\hat{g})$ locally

This allows us to write

$$1 = \Delta_{FP}(g) \int d\xi \; \delta(g - \hat{g}^{\xi})$$

and to replace it into the amplitude

$$A\Big|_{\substack{\text{given}\\\text{topology}}} = \int \frac{[dx\, dg\, d\xi]}{V_{d\times w}} \delta(g - \hat{g}^{\xi})\, e^{-S_x} V_1 - V_n \; \Delta_{FP}(g)$$

We can then integrate in dg

$$A_{g.t.} = \int \frac{[dx\, d\xi]}{V_{diff \times Weyl}} e^{-S_x[\hat{g}^\xi]} \, V_1 - V_n \, \Delta_{FP}(\hat{g}^\xi)$$

moreover, since S_x is ξ-invariant, and so is Δ_{FP} (check)

$$A_{g.t.} = \int [dx]\, e^{-S_x[\hat{g}]} \, V_1 - V_n \, \Delta_{FP}(\hat{g})$$

$$\times \underbrace{\int \frac{d\xi}{V_{diff \times Weyl}}}_{=1}$$

<u>Subtleties</u> In order that the "change" of variable $dg = d\xi \left(\frac{\partial g}{\partial \xi}\right)$, of which $\frac{\partial g}{\partial \xi} \sim \Delta_{FP}$, be well defined we need a one-to-one map, i.e.

$$\forall g \;\exists!\, \xi \mid \hat{g}^\xi = g \quad \underline{\text{globally}}$$

1) <u>Local $\to$ Global</u>

$\Rightarrow \exists$ in general $g, g' \mid$

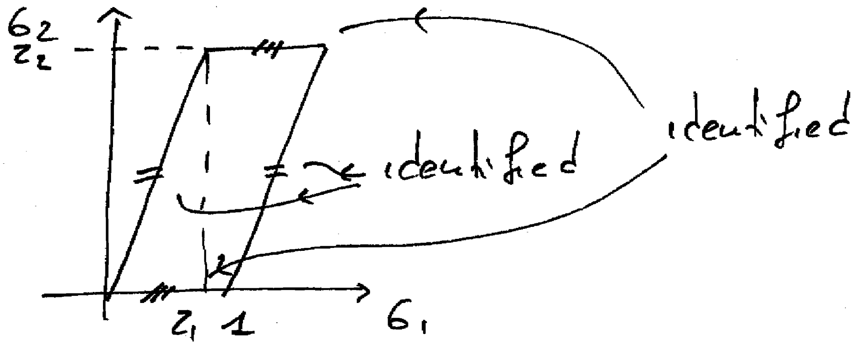
$\nexists\, \xi \mid g^\xi = g'$

2) $\underline{\underline{\exists !}}$ $\Rightarrow \exists$ in general $\xi \mid g^\xi = g$

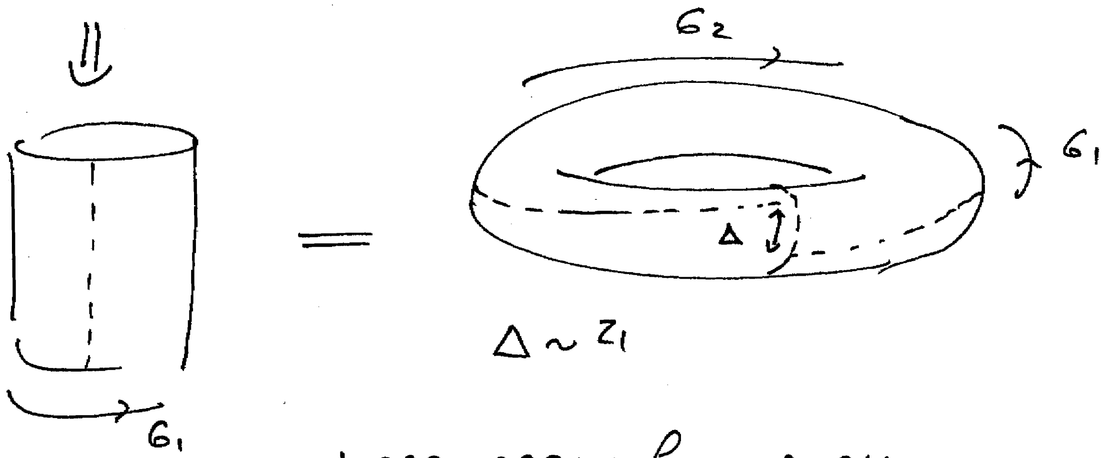
ξ is called conformal killing vector and is part of the " " group CKG

Torus case

(1) It is always possible to bring g to $\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ via Diff × Weyl, **but** the boundary conditions are then non trivial



$\Downarrow$



more precisely, given

$$z = \sigma_1 + i\sigma_2$$

$$z \sim z + (m + n\tau)$$

and the statement is that it is not possible to reabsorb τ via Diff × Weyl without a modification

$$\eta_{ab} \to g_{ab}(\tau)$$

<u>Choice</u> Take <u>twisted</u> b.c.

$$\Rightarrow (\sigma_1, \sigma_2) \sim (\sigma_1 + m + \tau_1 n, \sigma_2 + \tau_2 n)$$

let $\sigma_1' = \sigma_1 - \sigma_2 \frac{\tau_1}{\tau_2}$

$\sigma_2' = \frac{\sigma_2}{\tau_2}$

$$(\sigma_1', \sigma_2') \sim (\sigma_1' + m + \cancel{\tau_1 n} - \cancel{\tau_1 n}, \sigma_2' + n)$$

$\Rightarrow$ cyl

<u>but</u> $ds^2 = d\sigma_1^2 + d\sigma_2^2 = |d\sigma_1' + d\sigma_2' \tau|^2$

$= g_{ab} d\sigma'^a d\sigma'^b \qquad g = \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}$

<u>Consequences</u>

A <u>single</u> fiducial metric $\hat{g}$ is no more sufficient since $O_{\hat{g}} = \{ \hat{g}^{\xi}, \xi \in \text{diff} \ltimes \text{Weyl} \}$

$O_{\hat{g}} \neq$ set of all metric

<u>Statement</u> $\hat{g}(\tau)$ is indeed sufficient

$$\int d\tau \; O_{\hat{g}(\tau)} = \text{set of all metrics}$$

<u>Then</u> Replace $1 = \Delta_{FP}(g) \int d\xi \; \delta(g - \hat{g}^{\xi})$

with

$$1 = \Delta_{FP}(g) \int d\tau \; \delta(g - \hat{g}^{\xi}(\tau))$$

<u>Generalization</u> Torus $\to$ a single complex Teichmüller Moduli

Other manifolds $\to n_e$ real moduli

$$1 = \Delta_{FP}(g) \int \prod_{i=1}^{n_e} dt^i \; \delta(g - \hat{g}^{\xi}(t_i))$$

(2) $\xi : \sigma^a \longmapsto \sigma^a + \alpha^a$

is a perfectly well-defined Diff transformation

but

$$\xi : \eta_{\mu\nu} \longmapsto \eta_{\mu\nu}$$

Statement: on the torus only these two transformations, parametrized by α^1 and α^2, defines the continuous part of the CKG

Solution: insert a σ-dependence in Δ_{FP}

$$1 = \Delta_{FP}(\sigma, g) \prod_i \int dt^i \int d\xi \, \delta(g - \hat{g}^{\xi}(t^i)) \, \delta(\sigma_1 - \hat{\sigma}_1^{\xi}) \times \times \delta(\sigma_2 - \hat{\sigma}_2^{\xi})$$

the problem is solved since, given

$$\hat{\xi} \mid g = \hat{g}^{\hat{\xi}} , \ \exists ! \ \xi_{CKG} \mid \sigma = \hat{\sigma}^{\hat{\xi} \cdot \xi_{CKG}}$$

Generalization

It is possible to reabsorb the ambiguity of a CKG of real dimension n_K by introducing n_K Dirac deltas $\delta(\sigma_i - \hat{\sigma}_i^{\xi})$ in the definition of Δ_{FP}

$$1 = \Delta_{FP}(\sigma_i, g) \prod_{i=1}^{n_t} \int_F dt^i \int d\xi \, \delta(g - \hat{g}^{\xi}) \prod_{i=1}^{n_K} \delta(\sigma_i - \hat{\sigma}_i^{\xi})$$

Integration region for the moduli

$\Rightarrow$ Torus case

In principle $F \subset \mathbb{R}^2$. Let's see the details

take the parametrization

$$(\sigma_1, \sigma_2) \sim (\sigma_1 + m + \tau_1 n, \sigma_2 + \tau_2 n)$$

$$ds^2 = d\sigma_1^2 + d\sigma_2^2$$

def $\quad \omega = \sigma_1 + i\sigma_2$

$\Rightarrow \quad ds^2 = d\omega \, d\bar{\omega}$

$$\omega \sim \omega + m + \tau n$$

Now (1) replace τ with $\tau + 1$ $\quad (T)$

$\Rightarrow$ same b.c. in $m' = m + n$, $n' = n$

(2) replace τ with $-1/\tau$ $\quad (S)$

$$\omega \sim \omega + m - \frac{n}{\tau}$$

redefine $\omega' = \tau \omega$

$\rightarrow \quad \omega' \sim \omega' + m\tau - n \Rightarrow$ same as before

$(ds^2 = d\omega' d\bar{\omega}'$ via Weyl resc.$)$

$\Rightarrow F = \mathbb{R}^2 /$ Group Generated by T and S

Statement (1) T, S generate (span)

$SL(2, \mathbb{Z})$ of transformation

$$\tau' = \frac{a\tau + b}{c\tau + d} \qquad a, b, c, d \in \mathbb{Z}$$

$$ad - bc = 1$$

(2) $F \Rightarrow \quad \mathrm{Re}\,\tau \in [-\tfrac{1}{2}, \tfrac{1}{2}]$

$|\tau| \geq 1 \qquad (\tau_2 > 0)$

$$1 = \Delta_{FP}(g, \sigma) \prod_{i=1}^{n_e} \int dt^i \int d\xi \; \delta(g - \hat{g}(t^i)^\xi) \prod_{a=1}^{n_u} \delta(\sigma^a - \hat{\sigma}^{a\xi})$$

$$\Delta_{FP}^{-1}(g, \sigma) = \prod_{i=1}^{n_e} \int dt \int d\xi \; \delta(g - \hat{g}(t)^\xi) \prod_{a=1}^{n_u} \delta(\sigma^a - \hat{\sigma}^{a\xi})$$

now $\exists!$ zero but (last subtlety)

notice that we fixed all continuous symmetries; $\exists$ a last discrete ck symmetry

TORUS $\quad \sigma^a \to -\sigma^a$

To fix it as it is enough to remember that there are $n_R = 2$ zeros and to expand around just one of them

$$\Delta_{FP}^{-1}(g, \sigma) = n_R \prod_{i=1}^{n_e} \int dt^i d\xi \; \delta(g - \hat{g}(t)^\xi) \prod_{a=1}^{n_u} \delta(\sigma^a - \hat{\sigma}^{a\xi})$$

$$d\xi = d\omega \, d\epsilon$$

$$g - \hat{g}(t)^\xi = \delta g \Big|_{\sim \text{zero}} = (2\omega - \nabla_c \epsilon^c)\hat{g}_{ab} - (\nabla_a \epsilon_b + \nabla_b \epsilon_a - \nabla \epsilon^c)\hat{g}_{ab} + t^i \partial_{t^i} \hat{g}_{ab}$$

$$\Delta_{FP}^{-1}(g, \sigma) = n_R \prod_{i=1}^{n_e} \int dt \, d\omega \, d\epsilon \; \delta\left((2\omega - \nabla_c \epsilon^c)\hat{g}_{ab} - 2P\epsilon_{ab} + t\partial_t \hat{g}_{ab}\right) \prod_{a=1}^{2} \delta(\epsilon^c) \Leftarrow \text{Specific for the Torus}$$

exponentiate the δ's

$$\Delta_{FP}^{-1}(\hat{g}, \hat{\sigma}) = n_R \prod_{i=1}^{n_e} \int dt^i d\omega \, d\epsilon^a \, d\beta^{ab} \, dx_a$$

$$\exp\left\{2\pi i \int d^2\sigma \sqrt{g} \, \beta^{ab} \left[-(2\omega - \nabla_c \epsilon^c) h_{ab} + 2(P\epsilon)_{ab} + t^i \partial_{t^i} g_{ab}\right]\right\}$$

$$\exp\left\{2\pi i \epsilon^a x_a\right\}$$

Integrate $d\omega$ → β^{ab} traceless $(\beta^a{}_a = 0)$

$$\Delta_{FP}^{-1}(\hat{g},\hat{\sigma}) = n_R \prod_{i=1}^{n_c} \int dt^i \, d\epsilon^a \, d\beta^{ab} \, dx_a$$

$$\exp\left\{2\pi i \int d^2\sigma \sqrt{g}\, \beta^{ab}\left[(2P\epsilon)_{ab} - t^i \partial_{t^i} g\right]\right\}$$

$$\exp\{2\pi i \epsilon \cdot x\}$$

→ It is just a determinant of a new modified operator

⇒ Pass to Grassmann variables to "invert" it

$$\epsilon^a \to c^a$$
$$\beta_{ab} \to b_{ab}$$
$$x \to \eta$$
$$t \to \xi$$

$$\Delta_{FP}(\hat{g},\hat{\sigma}) = \frac{1}{n_R} \int [db\,dc]\, d\xi^i d\eta_a$$

$$e^{-\int d^2\sigma \sqrt{g}\left[b^{ab}\left((Pc)_{ab} + \xi^i \partial_{t^i} g\right)\right]}$$

$$e^{\eta_a c^a(\hat{\sigma})}$$

integrate in ξ and η

$$\Delta_{FP}(\hat{g},\hat{\sigma}) = \frac{1}{n_R} \int [db\,dc]\, e^{-S_{ghost}} \times \prod_{i=1}^{n_c}\left[\underline{\int d^2\sigma \sqrt{g}\, b^{ab} \partial_{t^i} g_{ab}}\right]$$

$$\times \underline{\prod_{a=1}^{2} c^a(\hat{\sigma})}$$

Torus amplitude

$$A_{torus} = \frac{\int [dx][dg]}{V_{diff \times Weyl}} \int_F d\tau_1 d\tau_2 \, \Delta_{FP}(\sigma, g)$$

$$\times \int d\zeta \, \delta(g - \hat{g}^{\zeta}(\tau)) \, \delta(\sigma_1 - \hat{\sigma}_1^{\zeta}) \, \delta(\sigma_2 - \hat{\sigma}_2^{\zeta}) \, e^{-S_x[g]} \, V_1 \dots V_n$$

This amplitude is meaningless unless an explicit σ_1, σ_2 integration is explicitated.

In the torus case, since the volume of the CKG is finite and equal to the volume of the torus itself, it is possible to introduce an integration of the form
$$1 = \frac{\int d^2\sigma}{V_{TORUS}}$$

In other words, one fixes the CKG problem by just dividing away $V_{CKG} = V_{TORUS}$?
No $\rightarrow$ B-insertions in Δ_{FP} !

In the general case, knowing $V_i = \int d^2\sigma \sqrt{g} \, \mathcal{V}_i(\sigma)$, we can identify the σ-dependence in Δ_{FP} with the σ-dep. in $n_{K/2}$ vertex operators.
The CKG symmetry is fixed by fixing points on the manifold + including insertions in the path integral.

In the torus case, replacing the Δ_{FP}

$$A_{torus} = \frac{1}{n_R} \int \frac{dz_1\, dz_2}{V_{TORUS}} \int [dx]\,[db\,dc] \overbrace{\int d^2\sigma \sqrt{\hat{g}}\, (b^{ab} \partial_{z_1} g_{ab})}^{B_1} \times$$

$$\times \overbrace{\int d^2\sigma \sqrt{\hat{g}}\, (b^{ab} \partial_{z_2} g_{ab})}^{B_2}\; c^1(\hat{\sigma})\, c^2(\hat{\sigma})\; e^{-[S_x + S_{ghosts}]}\; V_1 - V_n$$

$n_R = 2$

$= z_2$

Partition function

$$Z_{torus} = \frac{1}{2} \int_F \frac{dz_1\, dz_2}{z_2} \quad \langle B_1 B_2 c^1 c^2 \rangle_{ghosts} \times \langle 1 \rangle_{x\text{-fields}}$$

$\Rightarrow$ Relation with Coleman-Weinberg formula

$\Rightarrow$ Role of the ghost partition function

$\Rightarrow$ Self-regulation of UV divergences