8 World Sheet Supersymmetry

Motivation: • need fermions (target space spinors)
• need to avoid tachyon

8.1 Global SUSY on the WS

• let us add fermions at the most fundamental level:

σ^a – WS coords. $\oplus$ θ_α – fermionic WS coords.
(Grassmann variables with spinor index α)

Recall basic facts about spinors, in particular in 2 dims.:

$\sigma^a \to \Lambda^a{}_b \sigma^b$ – WS Lorentz group ; vector repr.

$\theta_\alpha \to S_\alpha{}^\beta \theta_\beta$ – — " — ; spinor repr.

as usual: $\Lambda^a{}_b = \left(e^{i\varepsilon^{cd} J_{cd}}\right)^a{}_b$

$$S_\alpha{}^\beta = \left(e^{i\varepsilon^{cd}\{i[\gamma_c,\gamma_d]/4\}}\right)_\alpha{}^\beta ;$$

e.g. $\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $\gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ fulfil $\{\gamma^a, \gamma^b\} = -2\eta^{ab}$

$(\eta^{ab} = \mathrm{diag}(-1,1))$

• γ's imaginary $\Rightarrow$ S real $\Rightarrow$ can demand reality of θ, i.e. $\theta^* = \begin{pmatrix} \theta_- \\ \theta_+ \end{pmatrix}^* = \begin{pmatrix} \theta_- \\ \theta_+ \end{pmatrix} = \theta$

[This is just the particularly simple 2d-version of the familiar Majorana condition $\psi = \psi^c \equiv C\bar{\psi}^T$

- in short: We have added a 2d Majorana spinor θ_α to our bosonic coords. σ^a and thus introduced superspace.
- We also have to generalize our fields:

$$X^\mu(\sigma) \longrightarrow Y^\mu(\sigma, \theta) \quad (\text{"superfields"})$$

- Taylor expansion:

$$Y^\mu(\sigma, \theta) = X^\mu(\sigma) + \bar{\theta}\psi^\mu(\sigma) + \frac{1}{2}\bar{\theta}\theta B^\mu(\sigma)$$

(Note: – $\bar{\theta} \equiv \theta^+ \gamma^0$ – as in 4d

– e.g., a linear term in θ does not need to be introduced since $\theta^+ = \theta^T$.)

Symmetries of superspace:

- 2d Poincaré for σ^a
- could try to add translations in θ (generated by $\frac{\partial}{\partial \theta}$), but that would be "boring" since it doesn't mix θ & σ.
- better: $\frac{\partial}{\partial \bar{\theta}^\alpha} \longrightarrow \boxed{\frac{\partial}{\partial \bar{\theta}^\alpha} + i(\gamma^a \theta)_\alpha \partial_a \equiv Q_\alpha}$

"SUSY generator" (also a Majorana-spinor)

- the symm. algebra now includes the relation

$$\{Q_\alpha, \bar{Q}^\beta\} = -2i(\gamma^a)_\alpha{}^\beta \partial_a \qquad (*)$$

<u>Note:</u> $\{Q,Q\}$ & $\{\bar{Q},\bar{Q}\}$ are not independent since $(Q_\alpha)^* = Q^\alpha$.

To check (*), let us first determine $\bar{Q}^\alpha$:

1) $\overline{i(\gamma^a\theta)_\alpha \partial_a} = -i\,\overline{(\gamma^a\theta)}^\alpha \partial_a = -i\left(\theta^+\gamma^{a+}\gamma^0\right)^\alpha \partial_a =$

$= -i\left(\theta^+\gamma^0\gamma^0\gamma^{a+}\gamma^0\right)^\alpha \partial_a = -i\left(\bar{\theta}\gamma^a\right)^\alpha \partial_a$

2) To find $\overline{\left(\frac{\partial}{\partial\bar{\theta}}\right)}$, recall that Majorana spinors satisfy $\bar{\psi}\chi = \bar{\chi}\psi$ and try to enforce this on Q:

$$\bar{\epsilon} Q = \bar{Q}\epsilon \quad \text{or} \quad \bar{\epsilon}\cdot\left(\frac{\partial}{\partial\bar{\theta}}\right) = \overline{\left(\frac{\partial}{\partial\bar{\theta}}\right)}\cdot\epsilon\,;$$

calculate

a) $\left(\bar{\epsilon}^\alpha \frac{\partial}{\partial\bar{\theta}^\alpha}\right)\left(\bar{\theta}\psi\right) = \bar{\epsilon}^\alpha \delta_\alpha^\beta \psi_\beta = \bar{\epsilon}\psi$

b) $\left(\frac{\partial}{\partial\theta_\alpha}\epsilon_\alpha\right)\left(\bar{\psi}\theta\right) = \epsilon_\alpha \bar{\psi}^\beta \delta_\beta^\alpha = -\bar{\psi}\epsilon = -\bar{\epsilon}\psi$

$\Rightarrow$ We must define $\overline{\left(\frac{\partial}{\partial\bar{\theta}}\right)} = -\left(\frac{\partial}{\partial\theta}\right)$, or

$$\boxed{\bar{Q}^\alpha = -\frac{\partial}{\partial\theta_\alpha} - i\left(\bar{\theta}\gamma^a\right)^\alpha \partial_a}$$

- Now it is easy to check (*):

$\left\{\frac{\partial}{\partial\bar{\theta}^\alpha} + i(\gamma^a\theta)_\alpha \partial_a\,,\; -\frac{\partial}{\partial\theta_\beta} - i(\bar{\theta}\gamma^b)^\beta \partial_b\right\} =$

$= -i(\gamma^b)_\alpha{}^\beta \partial_b - i(\gamma^a)_\alpha{}^\beta \partial_a = -2i(\gamma^a)_\alpha{}^\beta \partial_a$ ✓

<u>Note</u>: The structure $\{Q,Q\} \sim P$ found above is (together with the usual P_μ, $J_{\mu\nu}$ - Poinc. alg. & the standard action of $J_{\mu\nu}$ on Q_α as appropriate for a spinor) the heart of the <u>SUSY algebra</u>.

The SUSY algebra is the unique (under certain conditions) extension of the Poinc. alg.

<u>Note</u>: This extension involves the generalization of the symmetry concept based on Lie algebras to a symmetry concept allowing for super Lie algebras (or super Lie groups). [both commutators & <u>anti-commutators</u> appear!]

- Q acts on superfields as a differential operator:
  - infinitesimal trf.: $\delta_\epsilon Y = (\bar{\epsilon}Q)\cdot Y(\sigma,\theta)$
  - finite version: $(\rightarrow$ problems$)$

$$e^{\bar{\epsilon}Q}\, Y = Y(\sigma^a + \underbrace{i\bar{\epsilon}\gamma^a\theta + \tfrac{1}{2}i\bar{\epsilon}\gamma^a\epsilon}, \theta + \epsilon)$$

Translation in θ and corresponding translation in σ

- Since Q is a diff. operator, it acts on products of superfields as prescribed by the Leibnitz-rule:

$$(\bar{\epsilon}Q)(Y_1 Y_2) = ((\bar{\epsilon}Q)Y_1)Y_2 + Y_1((\bar{\epsilon}Q)Y_2).$$

- To write down actions, we need supercovariant derivatives:

$$D_\alpha = \frac{\partial}{\partial \bar\theta^\alpha} - i(\gamma^a\theta)_\alpha \partial_a$$

$$\bar D^\alpha = -\frac{\partial}{\partial \theta_\alpha} + i(\bar\theta\gamma^a)^\alpha \partial_a .$$

- Crucial feature: $\{D_\alpha, Q_\beta\} = 0 \quad \Rightarrow$

|| If Y transforms as a superfield, then so does $D_\alpha Y$. ||

Actions: $S = \int d^2\sigma \, d^2\theta \, \mathcal{L}(Y_1, Y_2, \dots, DY_1, \dots)$

SUSY invariance is obvious since

$$\delta_\epsilon \mathcal{L} = (\bar\epsilon Q)\mathcal{L} = \text{total derivative}$$

Of course, we need to know that $\int d^2\theta \frac{\partial}{\partial\theta_\alpha} f(\theta) = 0$.
To see this, consider e.g.

$$\int d\theta_1 \frac{\partial}{\partial\theta_1} f(\theta_1) = \int d\theta_1 \frac{\partial}{\partial\theta_1}(c_0 + c_1\theta_1) = \int d\theta_1 \cdot c_1 = 0 .$$

This extends easily to many variables.

The natural action for the string WS reads:

$$\Big\| \quad S = \frac{i}{4\pi}\int d^2\sigma \, d^2\theta \, (\bar D^\alpha Y^\mu)(D_\alpha Y_\mu) . \quad \Big\|$$

Taylor-expanding in θ and performing the $d^2\theta$-integration one finds the "component action": ($\rightarrow$ problems)

$$S = -\frac{1}{2\pi}\int d^2\sigma \left(\partial_a X^\mu \partial^a X_\mu - i\bar{\psi}^\mu \underbrace{\not{\partial}}_{\equiv \gamma^a\partial_a} \psi_\mu - \underbrace{B^\mu B_\mu}_{=0} \right)$$

$=0$, since $B_\mu = 0$ by EOMs.

$\Downarrow$

B is an "auxiliary" field.

- <u>In short:</u> We have added a fermionic ψ^μ (Maj. spinor) for every X^μ under the requirement of WS SUSY.

- All of this could have been done <u>without SFs</u> (= "<u>component formulation</u>"). To see this, consider first how the component fields transform under SUSY:
  - $Y^\mu(\sigma,\theta) = X^\mu(\sigma) + \bar{\theta}\psi^\mu(\sigma) + \frac{1}{2}\bar{\theta}\theta B^\mu(\sigma)$
  - Calculate $\delta_\epsilon Y$ and expand in θ:

    $$\delta_\epsilon Y^\mu(\sigma,\theta) \equiv \delta X^\mu(\sigma) + \bar{\theta}\,\delta\psi^\mu(\sigma) + \frac{1}{2}\bar{\theta}\theta\,\delta B^\mu(\sigma)$$

    This <u>defines</u> the quantities $\delta X, \delta\psi, \delta B$ on the r.h.side.
  - We obtain ($\to$ problems)

$$\boxed{\begin{aligned} \delta X^\mu &= \bar{\epsilon}\psi^\mu \\ \delta\psi^\mu &= -i(\gamma^a\epsilon)\partial_a X^\mu + B^\mu\epsilon \\ \delta B^\mu &= -i\bar{\epsilon}\gamma^a\partial_a\psi^\mu \end{aligned}}$$

"off-shell" component formulation of SUSY

- Using EOMs ($B^\mu = 0$ & $\partial\!\!\!/ \psi^\mu = 0$) we obtain the "on-shell" formulation:

$$\underbrace{\left\| \begin{aligned} \delta X^\mu &= \bar\epsilon \psi^\mu \\ \delta\psi^\mu &= -i(\gamma^a \epsilon)\partial_a X^\mu \end{aligned} \right\|}$$

From this (using EOMs), the SUSY algebra can be verified by working out the action of $[\delta_\epsilon, \delta_{\epsilon'}]$ on every field:

$$[\delta_\epsilon, \delta_{\epsilon'}] \underset{\uparrow}{=} 2i\bar\epsilon\gamma^a\epsilon'\partial_a$$

The different sign compared to the Q-algebra is obtained since the Q's act on <u>superspace</u> while the δ_ϵ defined above act on the <u>fields</u> (cf. active vs. passive description of symmetries).

<u>Thus:</u> One can simply start with the above algebra, realize it on X, ψ by defining appropriate $\delta_\epsilon X$ & $\delta_\epsilon \psi$, and then construct an invariant Lagrangian ($\sim (\partial X)^2 - i\bar\psi\partial\!\!\!/\psi$). This is all we need. In this approach, neither θ nor B^μ have ever appeared.

<u>Next step:</u> need to make this action diff.-invariant while maintaining SUSY.

8.2 Supergravity on the WS

- so far: flat WS with flat SUSY ; no metric
- could generalize our previous discussion to a "curved superspace" parameterized by σ & θ and find metric as a component field of a "superspace metric".
- simpler (and sufficient for our purposes):
  Write down diff.-inv. version of our action and extend it systematically to ensure invariance under (local) SUSY.
- diff.-inv. action:

$$S = -\frac{1}{2\pi}\int d^2\sigma \sqrt{h}\,\{h^{ab}\partial_a X^\mu \partial_b X_\mu - i\bar{\psi}^\mu \gamma^a \nabla_a \psi_\mu\}$$

- to understand $\nabla_a \psi$ we need a vielbein (to be general, in d dimensions):

  $e^m_{\ a}(\sigma)$ – set of d orthog. vectors at each point σ

$$e^m \cdot e^n = e^m_{\ a} e^n_{\ b} h^{ab} = \eta^{mn} \quad (\Rightarrow h_{ab} = \eta_{mn} e^m_{\ a} e^n_{\ b})$$

"curved" or "Einstein" indices

"frame" or "Lorentz" indices

- extra symm.: local Lorentz : $e^m_{\ a}(\sigma) \longrightarrow \Lambda^m_{\ n}(\sigma)\, e^m_{\ a}(\sigma)$

(Any vector v^a can be written with a Lorentz index: $v^m \equiv e^m_{\ a} v^a$. Then it transforms as $v \to \Lambda \cdot v$)

- <u>spinor</u>: local Lorentz: $\psi \to S(\sigma)\psi$ where $S(\sigma)$ belongs to $\Lambda(\sigma)$ as usual.
- as in a gauge theory, ∇_a is defined by demanding $\nabla_a v^m \to \Lambda^m{}_n \nabla_a v^n$ if $v^m \to \Lambda^m{}_n v^n$
- $\Rightarrow \nabla_a = \partial_a + \omega_a$ with $\omega_a \in \mathrm{Lie}(SO(1,d-1))$ in the appropriate representation (usually vector or spinor).
- demanding $\nabla_a e^m_b = 0$ fixes $\omega^m{}_n$ in terms of $\Gamma_{ab}{}^c$:

$$\left(0 \overset{!}{=} \nabla_a e^m_b = \partial_a e^m_b + (\omega_a)^m{}_n e^n_b - \Gamma_{ab}{}^c e^m_c\right)$$

- this is the natural <u>spin connection</u> ω coming with the vielbein e^m_a.
- the relevant explicit forms are:

$$(\omega_a)^m{}_n = \omega_a^{(pq)}(iJ_{(pq)})^m{}_n \quad \text{– vector}$$

$$\underbrace{(\omega_a)_\alpha{}^\beta}_{\text{this is the } \omega \text{ appearing in } \nabla_a\psi} = \omega_a^{(pq)}\left(i\tfrac{i}{4}[\gamma_p,\gamma_q]\right)_\alpha{}^\beta \quad \text{– spinor}$$

- note also: $\gamma^a \equiv e^a_m \gamma^m$

- Since ϵ links X and ψ ($\delta X = \bar\epsilon\psi$) and, under local Lorentz trfs., $\left\{\begin{matrix} X \to X \\ \psi \to S(\sigma)\psi \end{matrix}\right\}$, we must allow for a σ-dependence of ϵ: $\epsilon = \epsilon(\sigma)$ ("<u>local SUSY</u>").

- even if we treat $e_a^{\ m}$ just as a small perturbation around flat space ($e_a^{\ m} = \delta_a^{\ m}$), it needs a super-partner: $\delta e_a^{\ m} = -2i\bar{\epsilon}\gamma^m \chi_a$

"gravitino" or "Rarita-Schwinger field"

($\chi_a = (\chi_a)_\alpha$ is the smallest appropriate Lorentz-repres.)

- our previous action S is not inv. under such an extended symmetry. It is just the "quadratic approximation" S_2:

$$S_2 = -\frac{1}{2\pi}\int d^2\sigma \underbrace{e}_{= \det(e_a^{\ m}) = \sqrt{-\det(h_{ab})}} \left\{ h^{ab}\partial_a X^\mu \partial_b X_\mu - i\bar{\psi}^\mu \gamma^a \nabla_a \psi \right\}$$

- We construct the full S by the <u>Noether method</u>:

- calculate $\delta_\epsilon S_2$ at quadratic order in the fields:

$$\delta_\epsilon S_2 \sim \int (\nabla_a \bar{\epsilon}) J^a \quad \text{with} \quad \underbrace{J^a = \frac{1}{2}\gamma^b \gamma^a \psi^\mu \partial_b X_\mu}_{\text{"supercurrent"}}$$

(= Noether current of global SUSY trf.)

<u>Note</u>: – The calculation is simplified by the fact that $\int \bar{\psi}\gamma^a \nabla_a \psi = \int \bar{\psi}\gamma^a \partial_a \psi$ in $d=2$.

Note also: – at this order, we do not need to take δe_a^m into account.

- next, we try to make the action invariant by adding a term involving the gravitino χ_a with $\delta\chi_a = \nabla_a \epsilon$:

$$S_3 = -\frac{1}{\pi}\int d^2\sigma\, e\, \bar{\chi}_a \underbrace{\gamma^b\gamma^a\psi^\mu\partial_b X_\mu}_{\sim J^a \text{ as derived above}}$$

- calculate $\delta_\epsilon(S_2+S_3)$ to cubic order in the fields:
  - δS_2 from $\delta\psi$ & δX – compensated by δS_3 from $\delta\chi$
  - δS_3 from $\delta\psi$ & δX
  - δS_2 from δe_a^m

  } can be compensated by by
  1) modified trf. of ψ:
  $$\delta\psi = -i\gamma^a\epsilon(\partial_a X - \bar{\psi}\chi_a)$$
  2) extra term:
  $$S_4 = -\frac{1}{4\pi}\int d^2\sigma\, e\,(\bar{\psi}\psi)(\bar{\chi}_a\gamma^b\gamma^a\chi_b)$$

- finally: $S = S_2 + S_3 + S_4$ is inv. under:

$$\delta X = \bar{\epsilon}\psi\ ;\ \delta\psi = -i\gamma^a\epsilon(\partial_a X - \bar{\psi}\chi_a);\ \delta e_a^m = -2i\,\bar{\epsilon}\gamma^a\chi_a\ ;\ \delta\chi_a = \nabla_a\epsilon$$

- Note: Neither e_a^m nor χ_a have kinetic terms

  (eR is total derivative and the natura kinetic term for χ, $\bar{\chi}_a\gamma^{[a}\gamma^b\gamma^{c]}\nabla_b\chi_c$, vanishes since there is no 3rd-rank antisymm. tensor in $d=2$)

Note: This is different in $d=4$, where the symm.

$$\delta e^m_\mu = -\frac{i}{2}\bar{\epsilon}\gamma^m \chi_\mu \; , \quad \delta \chi_\mu = \nabla_\mu \epsilon$$

is respected by the non-trivial action

$$S = \int d^4x \, e \left\{ -\frac{1}{2} R - \frac{i}{2} \bar{\chi}_\mu \gamma^{\mu\nu\rho} \nabla_\nu \chi_\rho \right\}$$

"real" fermion with 2 on-shell d.o.f's

$\sim \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$

(antisymmetrized)

This 4d-supergravity action can be extended to couple to a 4d supersymm. theory of scalars & spinors (like X & ψ above), but this is much more complicated...

- returning to 2 dims., we find that S is inv. under

$$\delta X = 0 \, , \quad \delta e^m_a = \omega e^m_a \, , \quad \delta\psi = -\frac{1}{2}\omega\psi \, , \quad \delta\chi_a = \frac{1}{2}\omega\chi_a$$

(Weyl rescalings)

and their fermionic counterpart:

$$\delta X = \delta e = \delta \psi = 0 \, , \quad \delta \chi_a = i\gamma_a \eta$$

(η – infinitesimal Majorana spinor)

(problem: check that this is a symmetry!)

This makes our theory "Super-Weyl-invariant".

(This is often called "superconformal" which, however, clashes with our previous use of the term conformal for flat-space field theories only.)