9 Quantization of the Superstring

9.1 Constraints

- bosonic gauge symms.: – diffeom. (2 d.o.f.)
  – local Lorentz (1 d.o.f.)
  – Weyl (1 d.o.f.)

  $\underbrace{}$

  sufficient to set $e^m_{\ a} = \delta^m_{\ a}$ as gauge condition
- fermionic gauge symms.: – local SUSY – δ_ϵ (2 d.o.f.)
  – super-Weyl – δ_η (2 d.o.f.)

  $\underbrace{}$

  sufficient to set $\chi_a = 0$ as gauge condition

(→ book of Lüst & Theisen for more details)

$\Rightarrow$ with this gauge choice, we arrive at the flat action

$$-\frac{1}{2\pi}\int d^2\sigma\left[(\partial_a X^\mu)(\partial^a X_\mu) - i\bar{\psi}^\mu\gamma^a\partial_a\psi_\mu\right]$$

Important: Need to take the EOMs of the eliminated fields into account as constraints.

- $\frac{\delta}{\delta e^m_{\ a}}$ is equivalent to $\frac{\delta}{\delta h_{ab}}$; we already know that the relevant constraint is $T_{ab} = 0$.

(where now $T_{ab} = (\partial_a X^\mu)(\partial_b X_\mu) + \frac{i}{2}\bar{\psi}^\mu\gamma_{(a}\partial_{b)}\psi_\mu - (\text{trace})$)

symmetrization ⟶ (points to the $(a\,b)$ indices)

$\sim \delta_{ab}\cdot$(trace of first 2 terms); normalized such that $T_a^{\ a} = 0$

- $$\frac{\delta}{\delta\chi^a}\Big(\underset{\substack{\uparrow\\ \text{no }\chi}}{S_2}+S_3+\underset{\substack{\uparrow\\ \text{involves 2 factors of }\chi}}{S_4}\Big)\Big|_{\chi=0} = \frac{\delta}{\delta\chi^a}S_3 \quad ;$$

$$\frac{\delta}{\delta\chi^a}S_3 = \frac{\delta}{\delta\chi^a}\left(-\frac{1}{\pi}\int d^2\sigma\, e\,\bar\chi_c\gamma^b\gamma^c\psi^\mu\partial_b X_\mu\right)$$

$$= -\frac{e}{\pi}\gamma^b\gamma_a\psi^\mu\partial_b X_\mu = -\frac{2e}{\pi}J_a$$

(The appearence of the supercurrent J_a is not surprizing since S_3 was introduced as $S_3 \sim \int \bar\chi^a J_a$ to cancel the variation of S_2 under δ_ϵ with $\epsilon=\epsilon(\sigma)$.)

<u>Finally</u>: The constraints are $\underline{\underline{T_{ab}=0}}$ & $\underline{\underline{(J_a)_\alpha = 0}}$

(to be used with the <u>flat</u> action.)

9.2 Mode expansion

$S = S_{\text{bosonic}} + S_{\text{fermionic}} = S_B + S_F$; Rewrite S_F using

$$\psi = \begin{pmatrix}\psi_-\\ \psi_+\end{pmatrix} \; ; \; \gamma^\pm = \gamma^0 \pm \gamma^1 = \begin{cases}\begin{pmatrix}0 & 0\\ 2i & 0\end{pmatrix} & \text{for "+"}\\[1ex] \begin{pmatrix}0 & -2i\\ 0 & 0\end{pmatrix} & \text{for "}-\text{"}\end{cases}$$

$$\Rightarrow S_F = \frac{i}{\pi}\int d^2\sigma\,\big(\underset{\uparrow}{\psi_-}\,\underset{\uparrow}{\partial_+}\psi_- + \underset{\uparrow}{\psi_+}\,\underset{\uparrow}{\partial_-}\psi_+\big) \quad \text{(index } \mu \text{ suppressed)}$$

right-movers   left-movers

- To perform a Fourier expansion, we first need to discuss <u>boundary conditions</u>.

A) Closed string:

- bosons: $X(\tau, \sigma+\pi) = X(\tau,\sigma)$ as before
- fermions: – for a Dirac fermion, a phase could be introduced: $\psi(\tau,\sigma+\pi) = e^{i\alpha}\psi(\tau,\sigma)$; since ψ is real (Majorana), we are, however, restricted to $e^{i\alpha} = \pm 1$.
  – since ψ_+ & ψ_- decouple, we can do this independently:

4 possibilities:

- $\psi_+(\tau,\sigma+\pi) = \psi_+(\tau,\sigma)$; $\psi_-(\tau,\sigma+\pi) = \psi_-(\sigma)$ – R-R
  (= "Ramond – Ramond")
- $\psi_+(\tau,\sigma+\pi) = \psi_+(\tau,\sigma)$; $\psi_-(\tau,\sigma+\pi) = -\psi_-(\sigma)$ – R-NS
  (= "Ramond – Neveu/Schwarz")
- --- analogously --- NS-R
- --- analogously --- NS-NS

The Fourier mode decomposition is now obvious. As an example, consider the R-NS sector:

$$\psi_+^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}} \tilde{\psi}_r^\mu \cdot e^{-2ir(\tau+\sigma)}$$

$$\psi_-^\mu(\tau,\sigma) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r^\mu \cdot e^{-2ir(\tau-\sigma)} .$$

(analogously for the othe three sectors)

B) Open string:

- Consider closed string on an S^1 with circumference 2π (rather than π) and reduce the theory to an intervall ($S^1/\mathbb{Z}_2$) by "modding out" the $\mathbb{Z}_2$-symm. $\sigma \to -\sigma$.
- This $\mathbb{Z}_2$-trf. also exchanges ψ_+ & ψ_-
  (This can be checked directly from the 2d-Lorentz group or read off from the action: $\psi_+\partial_-\psi_+ + \psi_-\partial_+\psi_-$; it is also intuitively clear that left- & right-movers have to be exchanged.)
- "modding out" means that we restrict the configur. space to fields invariant under this symmetry:
  - in the R-NS & NS-R sectors there are no such fields
  - from the R-R sector on the S^1 only those field configs. survive which fulfil:

$$\left.\begin{aligned} \psi_+^\mu(\tau,\sigma) &= \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}} \psi_r^\mu \cdot e^{-ir(\tau+\sigma)} \\ \psi_-^\mu(\tau,\sigma) &= \underset{\boxed{\text{just a convention}}\nearrow}{\frac{1}{\sqrt{2}}}\sum_{r\in\mathbb{Z}} \psi_r^\mu \cdot e^{-ir(\tau-\sigma)} \end{aligned}\right\Vert \text{ open-string-fermions on intervall } (0,\pi) \text{ with } \underline{\text{R-BCs}}.$$

  - from the NS-NS sector we get (in complete analogy):

$$\psi_\pm^\mu(\tau,\sigma) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r^\mu \cdot e^{-ir(\tau\pm\sigma)} \ \Big\Vert \ \text{open-string } \underline{\text{NS-BCs}}.$$

Note: Once again, the open string modes can be viewed as "half of" the closed string modes.

Note also: The above open-string mode-expansion could also have been found considering

$$\delta S_F = \{\sim \text{EOMs}\} + \underbrace{\{\text{surface terms}\}}_{\underbrace{\sim \psi_+\delta\psi_+ - \psi_-\delta\psi_-}_{\text{This vanishes if either:}}}$$

- $\psi_+(0) = \psi_-(0)$ & $\psi_+(\pi) = \psi_-(\pi)$ (or, equivalently, $\psi_+(0) = -\psi_-(0)$ & $\psi_+(\pi) = -\psi_-(\pi)$) → "Ramond"

or

- $\psi_+(0) = \psi_-(0)$ & $\psi_+(\pi) = -\psi_-(\pi)$ (or, equivalently, $\psi_+(0) = -\psi_-(0)$ & $\psi_+(\pi) = \psi_-(\pi)$) → "Neven-Schwarz"

- Now we can finally write derive the oscillator algebra:
  - for the X^μ, everything is exactly as for the bosonic string
  - for the ψ^μ, we have

$$\Pi_+^\mu = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_{+\mu}} = \frac{i}{\pi}\psi_+^\mu \quad \left(\text{recall that } \partial_- = \tfrac{1}{2}(\partial_\tau - \partial_\sigma)\right)$$

canonical anti-commut.-relations: (at $\tau = 0$)

$$\{\psi_+^\mu(\sigma), \Pi_+^\nu(\sigma')\} = i\delta(\sigma-\sigma')\eta^{\mu\nu}$$

$$\Rightarrow \{\psi_+^\mu(\sigma), \psi_+^\nu(\sigma')\} = \pi\delta(\sigma-\sigma')\eta^{\mu\nu}$$

(and the same for ψ_-)

In terms of oscillators this reads
(focussing on the open string & including bosons):

$$\{\alpha^\mu_m, \alpha^\nu_n\} = m\,\delta_{m+n}\,\eta^{\mu\nu}$$

$$\{\psi^\mu_r, \psi^\nu_s\} = \delta_{r+s}\,\eta^{\mu\nu}$$ with either $r, s \in \mathbb{Z}$ (R)
or $r, s \in \mathbb{Z} + \frac{1}{2}$ (NS)

9.3 Old covariant quantization

Fourier-expand the constraints (i.e. T_{++} & J_+, focussing again on the open string)

$$L_m = \frac{1}{\pi}\int_{-\pi}^{\pi} d\sigma\, e^{im\sigma}\, T_{++}$$

$$G_r = \frac{\sqrt{2}}{\pi}\int_{-\pi}^{\pi} d\sigma\, e^{ir\sigma}\, J_+$$

(<u>Note</u>: $$L_m = \frac{1}{2} :\Big\{ \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot\alpha_{m+n} + \sum_{r\in\mathbb{Z}+\nu} \left(r + \frac{m}{2}\right)\psi_{-r}\cdot\psi_{m+r} \Big\}:$$

$$\uparrow$$

$$\left\{ \begin{matrix} \nu = 0 & - & R \\ \nu = 1/2 & - & NS \end{matrix} \right\}$$ – Polchinski's convention

and

$$G_r = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot\psi_{r+n}$$)

The above operators generate the superconformal symm. of the flat WS field theory (which is the remnant of the super-Weyl-symm. of the 2d SUGRA). They satisfy the <u>Super-Virasoro-algebra</u>:

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\,\delta_{m+n}$$
$$[L_m, G_r] = (\tfrac{1}{2}m - r)G_{m+r}$$
$$\{G_r, G_s\} = 2L_{r+s} + B(r)\,\delta_{r+s}$$

with anomalies:
$$\begin{aligned} A(m) &= \underbrace{\tfrac{1}{8}D(m^3-m)}_{} \quad \text{or} \quad \cdots = \tfrac{1}{8}Dm^3 \\ B(r) &= \underbrace{\tfrac{1}{2}D(r^2-\tfrac{1}{4})}_{NS\text{-case}} \qquad \cdots = \underbrace{\tfrac{1}{2}Dr^2}_{R\text{-case}} \end{aligned}$$

(One can also redefine L_0 by a constant to avoid this asymmetry between NS & R. This is done in "Polchinski" but not in "GSW".)

Note: In the bosonic case, we had $A(m) = \frac{1}{12}D(m^3-m)$. This has been increased by the fermionic contribution to the L's.

- As before, we can only impose part of the classical constraint quantum-mechanically:

$$\left\| \begin{array}{l} (L_m - a\,\delta_{m,0})\,|phys\rangle = 0 \quad \text{for} \quad m \geq 0 \\ G_r\,|phys\rangle = 0 \quad \text{for} \quad r \geq 0 \end{array} \right.$$
(no normal-ordering ambiguity $\Rightarrow$ no a)

- We now need to construct the Fock space and to determine the critical values of the
  - normal ordering constant a
  - space-time dimension D

- as in the bosonic case:
  - a_{critical} is the boundary value at which ghosts develop.
  - This boundary is defined by the presence of extra zero-norm states.
  - given a_{critical}, we then find D_{critical} by the same argument (extra zero-norm states).

- NS sector

- vacuum: $|0,k\rangle$
- $0 \overset{!}{=} (L_0 - a)|0,k\rangle = (\alpha' p^2 + N^{\alpha} + N^{\psi} - a)|0,k\rangle$

$$N^{\alpha} = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{-m}\cdot\alpha_m$$

$$N^{\psi} = \frac{1}{2}\sum_{r\in\mathbb{Z}+\nu} r\cdot\psi_{-r}\cdot\psi_r \;;\quad \nu = \frac{1}{2} \text{ for NS}$$

$\Rightarrow \alpha' M^2 = -\alpha' k^2 = -a$

- excited states are constructed, as usual, by applying the creation operators α^{μ}_{-n} & ψ^{μ}_{-r} $(n,r>0)$ to the vacuum.
- consider specifically $G_{-1/2} = \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot\psi_{-\frac{1}{2}+n}$ and the excited state

$$|\phi\rangle = G_{-1/2}|0,k\rangle = \alpha_0\cdot\psi_{-1/2}|0,k\rangle$$

$$(L_0 - a)|\phi\rangle = 0 \Rightarrow \alpha' M^2 = -\alpha' k^2 = \frac{1}{2} - a$$

$$\langle\phi|\phi\rangle = \langle 0,k|G_{1/2}G_{-1/2}|0,k\rangle \underset{\substack{\uparrow \\ \text{by Super-Virasoro algebra}}}{=} \langle 0,k|2L_0|0,k\rangle = 2\alpha' k^2$$

$\Rightarrow \langle\phi|\phi\rangle = 0$ for $\underline{\underline{a = 1/2}}$

- consider certain higher excitations:

$$|\phi\rangle = (G_{-3/2} + \lambda G_{-1/2} L_{-1}) \underset{\substack{\uparrow \\ \text{some phys. state}}}{|\tilde{\phi}\rangle}$$

demand that $|\phi\rangle$ is phys. & $\langle\phi|\phi\rangle = 0 \Rightarrow \lambda = 2$, $\underline{\underline{D = 10}}$.

- $\underline{R \text{ sector}}$

- here, the determination of a is even simpler:

Super-Virasoro algebra $\Rightarrow \{G_0, G_0\} = 2L_0 \Rightarrow L_0 = G_0^2$;

$G_0|\phi\rangle = 0 \Rightarrow L_0|\phi\rangle = 0$ "$\Rightarrow$" $\underline{\underline{a = 0}}$.

- $|\phi\rangle = G_0 G_{-1} \underset{\substack{\uparrow \\ \text{some phys. state}}}{|\tilde{\phi}\rangle}$ has zero norm for $\underline{\underline{D = 10}}$.

Very important point: $\underline{\underline{R\text{-vacuum}}}$

The oscillators ψ_0^μ play a special role:

1) $\{\psi_0^\mu, \psi_0^\nu\} = \eta^{\mu\nu}$ (Clifford algebra in D dims.)

2) $[\psi_0^\mu, M^2] = 0$

(since $[\psi_0^\mu, \alpha' M^2] = [\psi_0^\mu, N^\alpha + N^\psi - a] = [\psi_0^\mu, 0 \cdot \psi_{-0}^\nu \psi_0^\nu] = 0$)

$\Rightarrow$ at any mass level (including the string vacuum!), the states must form a representation of the Cliff.-alg.

$\Rightarrow \nexists$ unique vacuum $|0,k\rangle$ (as opposed to NS sector)

<u>instead</u>: – smallest repr. of Cliff. alg. – spinor
- at least as many vacuum states as indep. spinor components
- in D dims., the Dirac spinor has $2^{D/2}$ comps.;
  <u>here</u>: $D = 10 \Rightarrow 32$ comps.
- write vacuum states as $|\alpha,k\rangle$ with $\alpha = 1 \cdots 32$
  and
  $$\psi_0^\mu |\alpha,k\rangle = |\beta,k\rangle (\Gamma^\mu)_\beta{}^\alpha \frac{1}{i\sqrt{2}}$$
  with 32×32 matrices Γ^μ satisfying
  $$\{\Gamma^\mu, \Gamma^\nu\} = -2\eta^{\mu\nu}$$

Check that this really provides an explicit representation of the ψ_0^μ-algebra:

$$\{\psi_0^\mu, \psi_0^\nu\}|\alpha,k\rangle = \psi_0^\mu |\beta,k\rangle (\Gamma^\nu)_\beta{}^\alpha / i\sqrt{2} + (\mu \leftrightarrow \nu)$$

$$= |\gamma,k\rangle (\Gamma^\mu)_\gamma{}^\beta (\Gamma^\nu)_\beta{}^\nu / (-2) + (\mu \leftrightarrow \nu) = |\gamma,k\rangle (\mathbb{1})_\gamma{}^\alpha \eta^{\mu\nu}$$

as required by the ψ_0^μ-algebra

- <u>easy to check</u>: $a = 0 \Rightarrow k^2 = 0$ for R-vacuum $|\alpha,k\rangle$
- <u>also</u>: $G_0|\alpha,k\rangle = \sum_{n\in\mathbb{Z}} \alpha_{-n} \cdot \psi_n |\alpha,k\rangle = \alpha_0 \cdot \psi_0 |0,k\rangle$

$$\sim k_\mu \psi_0^\mu |\alpha,k\rangle \sim |\beta,k\rangle (\not{k})_\beta{}^\alpha$$

Then, a phys. state $|u_\alpha(k)\rangle = |\alpha, k\rangle \cdot u_\alpha(k)$ has to satisfy the phys. state condition

$$0 \overset{!}{=} G_0 |u_\alpha(k)\rangle = |\beta, k\rangle \underbrace{(\not{k})_\beta{}^\alpha u_\alpha(k)}_{\text{massless Dirac equ. !}}$$

<u>Mini summary:</u> (of what we have learned so far about the quantization of the (open) superstring)

<u>D = 10</u>

<u>NS-sector</u>: $|0,k\rangle$ - scalar, tachyon

$\psi^\mu_{-1/2}|0,k\rangle$ - massless vector (unphys. polariz. decouple as in the bosonic case)

⋮

massive states

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all of these are <u>target-space-bosons</u>

<u>R-sector</u>: $|\alpha, k\rangle$ - massless spinor

⋮

massive states

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all of these are <u>target-space-fermions</u>.

Before worrying about how to remove the tachyon and about the extension to the closed string, let us briefly review the other quantization procedures:

9.4 Light-cone quantization

Recall the Bosonic case:

- fix flat gauge
- use residual gauge symm. (≙ conf. invariance) to achieve $X^+(\tau,\sigma) = x^+ + p^+\tau$.

This still works for the superstring.

New: The fermionic part:

Recall that the trf. $\delta_{\epsilon,\eta}\chi_a = i\gamma_a\eta + \nabla_a\epsilon$ was used to set $\chi_a = 0$.

Residual symm.: $\delta_{\epsilon,\eta}$ with $i\gamma_a\eta + \nabla_a\epsilon = 0$.

This symm. can be used to set $\psi^+(\sigma,\tau) = 0$.

(This is consistent with fixing X^+ since, under global SUSY, $\delta_\epsilon X^+ = \bar{\epsilon}\psi^+$.)

$\Rightarrow$ α^+, ψ^+ oscillators have been removed.

$$\Rightarrow \alpha^-_n = \frac{1}{2p^+}\Big[\sum_{i=1}^{D-2}\Big(\sum_m :\alpha^i_{n-m}\alpha^i_m: + \sum_r \big(r-\tfrac{n}{2}\big):\psi^i_{n-r}\psi^i_r:\Big) - a\delta_n\Big]$$

$$\psi^-_r = \frac{1}{p^+}\sum_{i=1}^{D-2}\sum_s \alpha^i_{r-s}\psi^i_s$$

Thus, only the transverse oscillators are independent.

Exactly as before, the Lorentz-alg. realized with these fundamental fields (using the oscillators and their algebra) fails unless

$$\underline{\underline{D=10}} \quad \& \quad \underline{\underline{a=1/2}} \text{ (NS) or } \underline{\underline{a=0}} \text{ (R)}.$$

As before, the Hilbert space is now built using only transverse creation operators.

(obvious for $\alpha_{-1}^i |0,k\rangle$; $\psi_{-1/2}^i |0,k\rangle$ etc. (NS sector); more thought required for R-vacuum with ψ_0^i-algebra realized on it ...)

as before: $\exists$ DDF operators & DDF states in OCQ which allow us to map

LCQ-Hilbert space on OCQ Hilbert space.

$\uparrow$

(Can think of this simply as all particles with physical (transverse rel. to its momentum p) polarization.)

9.5 Modern Covariant Quantization

- recall the essential step:

$$\int Dh \cdots \longrightarrow \int \underbrace{D\xi \, D\omega}_{} \det\left(\frac{\delta \hat{h}^{\xi,\omega}}{\delta \xi,\omega}\right) \cdots$$

gauge parameters (diff. + Weyl) linking h to the fiducial metric $\hat{h}$.

- ($\int \mathcal{D}\xi \mathcal{D}\omega$) is then dropped as a trivial overall factor; the Jacobian is expressed through the ghost path integral.

<u>We now need to extend this to the gravitino:</u>

$$\int \mathcal{D}\chi_a \cdots \longrightarrow \int \underbrace{\mathcal{D}\epsilon \mathcal{D}\eta}\, \det{}^{-1}\left(\frac{\delta \hat{\chi}^{\epsilon,\eta}}{\delta \epsilon,\eta}\right) \cdots$$

gauge parameters (local SUSY + super-Weyl) linking χ_a to the fiducial value $\hat{\chi}_a \equiv 0$.

(<u>Crucial point</u>: We need "$\det^{-1}$" rather than "det" because of the <u>fermionic</u> path integral.)

$$\Big\| \text{explicitly: } \chi_a = \hat{\chi}_a^{\epsilon,\eta} = i\gamma_a \eta + \nabla_a \epsilon \Big\|$$

- Let us analyse the Lorentz trfs. of this eq. using Euclidean conventions ($SO(2)$), where different spins correspond simply to different multiplicative phases.
- We have:
  - spinor: ± 1/2 (corresponding to $\psi_{+1/2} \to e^{i\alpha/2}\psi_{+1/2}$; $\psi_{-1/2} \to e^{-i\alpha/2}\psi_{-1/2}$)
  - vector: ± 1
  - gravitino or "vector-spinor": $\pm 3/2$, $\pm 1/2$

  --- etc. ---
- Our expression for χ can thus be decomposed as:

$$\chi_{-3/2} = \nabla_{-1}\epsilon_{-1/2}$$

$$\left.\begin{aligned}\chi_{-1/2} &= \nabla_{-1}\epsilon_{1/2} + \eta_{-1/2}\\ \chi_{1/2} &= \nabla_{1}\epsilon_{-1/2} + \eta_{1/2}\end{aligned}\right\} \mathcal{D}\chi_{\pm 1/2} \to \mathcal{D}\eta_{\pm 1/2} \text{ is just a shift of the integration variable}$$

$$\chi_{3/2} = \nabla_1 \epsilon_{1/2} \qquad \Rightarrow \text{No Jacobian!}$$

- Thus, the relevant Jacobian is

$$\det{}^{-1}\left(\frac{\delta\chi_{\pm 3/2}}{\delta\epsilon_{\pm 1/2}}\right) = \left[\det{}^{-1}\left(\nabla_1^{1/2\to 3/2}\right)\right]\cdot\left[\det{}^{-1}\left(\nabla_{-1}^{-1/2\to -3/2}\right)\right]$$

$$\sim \int \mathcal{D}\gamma_{1/2}\,\mathcal{D}\beta_{-3/2}\,\exp\left[-\int d^2\sigma\,\beta_{-3/2}\nabla_1\gamma_{1/2}\right]$$

$$\cdot\int \mathcal{D}\gamma_{-1/2}\,\mathcal{D}\beta_{3/2}\,\exp\left[-\int d^2\sigma\,\beta_{3/2}\nabla_{-1}\gamma_{-1/2}\right]$$

$$\sim \int \mathcal{D}\underset{\underset{\text{spinor}}{\uparrow}}{\gamma}\,\mathcal{D}\underset{\underset{\text{vector-spinor}}{\uparrow}}{\beta_c}\,\exp\left[-\frac{i}{2\pi}\int d^2\sigma\, e\, h^{ab}\,\bar{\gamma}\,\nabla_a\beta_b\right]$$

(both bosonic)

Note: To ensure that β_a, which in general has components $-3/2, -1/2, 1/2, 3/2$, has only components $-3/2$ & $3/2$, we impose $\underset{\uparrow}{(\gamma^a)}_\alpha{}^\beta\beta_\beta \equiv 0$.

(This is just a γ-matrix, not the bosonic ghost γ introduced above!)

- In total:

$$S = -\frac{1}{2\pi}\int d^2\sigma\, e\left\{(\partial_a X^\mu)(\partial^a X_\mu) - i\bar{\psi}^\mu\gamma^a\nabla_a\psi_\mu\right\}$$

$$-\frac{i}{2\pi}\int d^2\sigma\, e\left\{\underbrace{h^{ab}c^c\nabla_a b_{bc}}_{\substack{\text{"bc-system"}\\ \text{(fermionic ghosts)}}} + \underbrace{h^{ab}\bar{\gamma}\nabla_a\beta_b}_{\substack{\text{"}\beta\gamma\text{-system"}\\ \text{(bosonic ghosts)}}}\right\}$$

- Central charges:

X : D
ψ : $1/2\, D$
bc : -26
$\beta\gamma$: $+11$

(recall the anomalies $1/12$ and $1/8 = 1/12 + 1/2\,(1/12)$ for the X and X, ψ - theories encountered above)

Thus, for the superstring we need $0 \overset{!}{=} \frac{3}{2} D - 15 \Rightarrow \underline{\underline{D = 10}}$.

(Recall: $c = 0 \Rightarrow$ no Weyl anomaly $\Rightarrow$ formal path integral manipulations justified.)

- There is, in general, also an anomaly in the "{fermion, fermion} ~ boson"-sector of of the super-Virasoro algebra:

$$\{ G_r, G_s \} = 2 L_{r+s} + B(r)\, \delta_{r+s} .$$

The contributions to B are:

- from $\alpha \cdot \psi$: $B(r) = \frac{1}{2} D (r^2 - 1/4)$ (NS) ; $\dots = \frac{1}{2} D r^2$ (R)
- from ghosts : $+\frac{1}{4} - 5r^2$ (NS) ; $-5r^2$ (R)

$\Rightarrow$ NS : $B(r) = -1$ for $D = 10$ (to be absorbed in shift of L_0 by $1/2$ corresponding to normal ordering const. a)
R : $B(r) = 0$ for $D = 10$ (OK without shift)

- This vanishing of $B(r)$ is necessary to ensure that Q (the BRST-op. ≡ generalized gauge trf., incl. local SUSY & super-Weyl) satisfies the crucial relation $Q^2 = 0$.