Problem Sheet 1

(Due: October 24, 2012)

Problem 1.1:
Show that the "Nambu-Goto" action

$$S_{\mathrm{NG}} = -m \int \mathrm{d}\tau \sqrt{-\eta_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}, \quad \mu,\nu = 0,\dots,D-1 \tag{1}$$

of the relativistic particle is invariant under arbitrary reparametrizations $\tau \to \tau'(\tau)$.

Problem 1.2:
Derive the equations of motion $\ddot{X}^\mu = 0$ for the special parameter choice $\tau = s$, where s is the proper time ($\mathrm{d}s = \sqrt{-\eta_{\mu\nu}\mathrm{d}X^\mu \mathrm{d}X^\nu}$) of the paricle.

Problem 1.3:
Derive the non-relativistic limit

$$S_{\mathrm{NG}} \simeq \int \mathrm{d}t \left(\frac{m}{2}\vec{v}^2 - m\right).$$

Problem 1.4:
Demonstrate the reparametrization invariance ($\tau \to \tau'(\tau)$) of the "Polyakov" action

$$S_{\mathrm{P}} = -\frac{m}{2}\int \mathrm{d}\tau \sqrt{-h_{\tau\tau}}\left(h_{\tau\tau}^{-1}\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}X_\mu}{\mathrm{d}\tau} + 1\right). \tag{2}$$

Problem 1.5:
Derive the "Nambu-Goto" (1) from the "Polyakov" (2) form as sketched in the lecture, i.e. use the equations of motion for $h_{\tau\tau}$ to eliminate $h_{\tau\tau}$ in (2).

Problem 1.6:
Show that

$$\delta(\det A) = (\det A)\,\mathrm{tr}\left(A^{-1}\delta A\right)$$

for any matrix A with $\det A \neq 0$

(a) by using the identity $\ln(\det A) = \mathrm{tr}(\ln A)$,

(b) by using the explicit formulae for $\det A$ and A^{-1} with the Levi-Civita ε-symbol.

Problem 1.7:
The generalization of the "Polyakov" action (2) to a p-brane is given by

$$S_{\mathrm{P}}^p = -\frac{T_p}{2}\int_{\Sigma_{\mathrm{P}}} \mathrm{d}^{p+1}\xi \sqrt{-h}\, h^{ab}\partial_a X^\mu \partial_b X_\mu + \Lambda_p \int_{\Sigma_{\mathrm{P}}} \mathrm{d}^{p+1}\xi\sqrt{-h}, \quad a,b = 0,\dots,p. \tag{3}$$

(a) Show that (3) is diffeomorphism invariant.

(b) Derive the equations of motion for the world volume metric h_{ab}.

(c) Show that the ansatz

$$h_{ab} = \alpha_p \, \partial_a X^\mu \partial_b X_\mu$$

solves these equations and determine the constant α_p.

(d) Integrate out h_{ab} to recover a Nambu-Goto-like form for the action of a p-brane. Discuss the special case $p = 1$.

(e) What can you say about the behaviour of (3) under Weyl-rescalings? (Use the results of part (c).)

Problem 1.8:
Derive

$$T^{ab} = -2\pi T \left(G^{ab} - \frac{1}{2} h^{ab} h^{cd} G_{cd} \right), \quad a, b = 1, 2$$

from the Polyakov action of the bosonic string.

Problem 1.9:
Can you give a symmetry argument showing that the trace of T^{ab} must vanish identically?